REFLECTION OF LIGHT BY A PERIODICALLY STRATIFIED MEDIUM

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1. *Introduction*

The problem of the reflection of electromagnetic waves by a periodically stratified medium is of interest in many branches of physics, *e.g.*, the diffraction of X-rays by crystals and the scattering of light in solids and liquids. Although the theory of the phenomena has been elaborately investigated in the case of the reflection of X-rays by crystals, the optical analogues, which were known much earlier, have not been so thoroughly studied. The late Lord Rayleigh (1917) developed the theory of the reflection of light by a laminated structure, but he did not discuss its significance with regard to the spectral distribution of intensity in the reflected light. Recently, R. V. Subrahmanian (1941) has sought to rectify this omission and has computed the intensity distribution numerically from Rayleigh's formula in a number of cases.

In this paper an expression for the intensity of reflection from a stratified medium is derived in a very simple manner by a method somewhat similar to that employed by Darwin (1914) for the X-ray problem. A very general case is considered in which the medium is supposed to have a periodic structure, the nature of this being unspecified. The particular case of Rayleigh is then deduced from the general theory. The expressions so obtained are then discussed, having regard to the manner in which the functions appearing in it vary. This shows that it is not quite correct to say that the sharpness of the principal maxima increases in direct proportion to the number of stratifications, or that it is possible to deduce the number from the sharpness of the maxima, as has been assumed by various authors including the late Lord Rayleigh and R. W. Wood. It is found that the width of the primary maxima is not appreciably diminished by an increase in the number, except when the latter is small, and that the width is finite even when the number is infinite. This minimum width is found to depend only on the reflecting power of the stratifications. The discussion also shows that there exist a number of secondary maxima in the interval between the principal maxima, and that they are asymmetrically distributed as regards their intensity on either side of a principal maximum.

2. *Derivation of the Fundamental Formulæ*

We shall consider a non-absorbing medium in which there is a periodic variation of optical properties. Let O_1, O_2, $O_3, \ldots O_{n+1}$ be consecutive points in the medium at which the properties of the medium are identical. Then the distances O_1O_2, $O_2O_3, \ldots O_nO_{n+1}$ are all identical, and each is equal to the thickness of the stratification. It is supposed that there are n such stratifications.

Since there are variations in the optical properties, it is evident that there will be a large number of multiple reflections ; and consequently, there will be two streams of energy in the medium, one passing upwards, and the other downwards. Let T_1, R_1 ; T_2, R_2 ; etc., represent the electric vectors in the upward and downward streams of energy at the points O_1, O_2, etc., respectively.

According to our notation, T_1 is the amplitude of the incident wave, and let θ_1 be the angle made by T_1 with the common normal to the stratifications. Then, at each of the points O_2, O_3, etc., the two streams of energy will make the same angle θ_1 with the common normal. We shall now denote by r and t the complex reflection and transmission coefficients for a single stratification, on which light is incident at the angle under consideration. Here again, we refer the vibration to points O_1, O_2, etc., that is, t is the ratio of the electric vector in the transmitted wave at O_{s+1} to that at O_s, and r is the ratio of the reflected wave at O_s to that at O_s in the incident wave.

With these definitions, we see that R_s consists of two parts : (1) due to the transmission of $R_{s+1} = tR_{s+1}$ and (2) due to the reflection of $T_s = rT_s$. Hence,

$$R_s = rT_s + tR_{s+1} \quad (1)$$

Similarly, $$T_s = tT_{s-1} + rR_s \quad (2)$$

Our aim is now to find the amplitude T_{n+1} of the wave transmitted by the last stratification, and of that reflected out of the first, *viz.*, R_1. For this, eliminating the R's and the T's successively from (1) and (2), we get

$$T_{s-1} = \frac{1 + t^2 - r^2}{t} T_s - T_{s+1} \quad (3)$$

$$R_{s-1} = \frac{1 + t^2 - r^2}{t} R_s - R_{s+1} \quad (4)$$

For convenience, call the factor $\frac{1+t^2-r^2}{t}$ as y. It is evident that $R_{n+1}=O$, since no light is incident from below. From (4), therefore

$$R_{n-1}=yR_n$$

Now, applying relation (4) successively, we get

$$R_{n-2}=yR_{n-1}-R_n=(y^2-1)\,R_n, \text{ etc., giving}$$

$$R_1=\left[y^{n-1}-\frac{(n-2)}{\underline{|1}}\,y^{n-3}+\frac{(n-4)\,(n-3)}{\underline{|2}}\,y^{n-5}-\ldots\right]R_n=f_n(y)\,R_n\ (\text{say})$$

$$\text{Then } \frac{R_1}{R_n}=f_n(y) \tag{5}$$

Again, putting $s=(n+1)$ and $R_{n+1}=O$ in (2),

$$T_n=T_{n+1}/t$$

Applying relation (3) successively,

$$T_{n-1}=yT_n-T_{n+1}=(y/t-1)\,T_{n+1}, \text{ etc., giving}$$

$$T_1=\left[\frac{1}{t}\,f_n(y)-f_{n-1}(y)\right]T_{n+1} \text{ or}$$

$$\frac{T_1}{T_{n+1}}=\frac{1}{t}\,f_n(y)-f_{n-1}(y) \tag{6}$$

Also, from relation (1), since $R_{n+1}=O$, $R_n=rT_n$ giving

$$R_n=\frac{r}{t}\,T_{n+1}, \text{ and from (5), we get}$$

$$\frac{R_1}{T_{n+1}}=\frac{r}{t}\,f_n(y) \tag{7}$$

We have thus evaluated both T_{n+1} and R_1 in terms of T_1. Now, the series appearing as $f_n(y)$ is well known, and is summable. Putting $y=2\cosh\beta$, we have, in fact, (Jolley, 1925)

$$f_n(y)=f_n(2\cosh\beta)=\frac{\sinh n\beta}{\sinh\beta}.$$

Also, the right-hand side of equations (6) and (7) can both be expressed in terms of hyperbolic functions by writing

$$\frac{r}{\sinh\beta}=\frac{t}{\sinh\alpha}=\frac{1}{K}.$$

(7) goes over into the form $\frac{R_1}{T_{n+1}}=\frac{\sinh n\beta}{\sinh\alpha}.$ (8)

We may evaluate K using the relation $\cosh\beta = (1 + t^2 - r^2)/2t$, and obtain $K = \sinh(\alpha + \beta)$. Hence,

$$\frac{r}{\sinh\beta} = \frac{t}{\sinh\alpha} = \frac{1}{\sinh(\alpha+\beta)} \tag{9}$$

and

$$\frac{T_1}{T_{n+1}} = \frac{1}{t} f_n(y) - f_{n-1}(y) = \frac{\sinh(\alpha + n\beta)}{\sinh\alpha} \tag{10}$$

Finally,

$$\frac{R_1}{\sinh n\beta} = \frac{T_{n+1}}{\sinh\alpha} = \frac{T_1}{\sinh(\alpha + n\beta)}. \tag{11}$$

The angles α and β can be expressed in terms of r and t. We have seen that

$$\cosh\beta = \frac{1 + t^2 - r^2}{2t}. \tag{12}$$

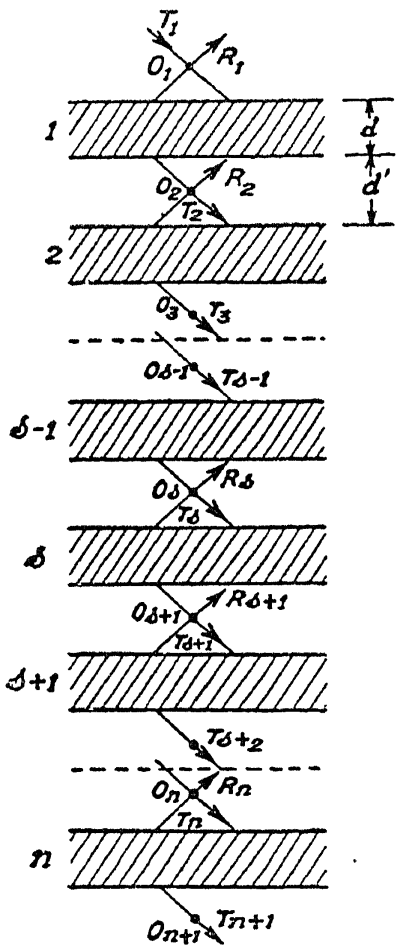
Similarly

$$\cosh\alpha = \frac{1 - t^2 + r^2}{2r}. \tag{13}$$

From these, it follows at once that

$$\sinh^2\alpha = \frac{\{(r+t)^2 - 1\}\{(r-t)^2 - 1\}}{4r^2} \tag{14}$$

$$\sinh^2\beta = \frac{\{(r+t)^2 - 1\}\{(r-t)^2 - 1\}}{4t^2} \tag{15}$$



So far, our theory has been quite general. We shall now consider a case treated by Lord Rayleigh, one in which the stratification consists of a set of n parallel plates, each of thickness d and of refractive index μ separated by empty gaps of thickness d'. Let the rays make an angle θ_2 with the common normal in the space within the plates and an angle θ_1 in the space between. For shortness, let us write the path retardations $2\mu d \cos \theta_2$ and $2d \cos \theta_1$ within the plates and in the spaces between, respectively, as δ and δ'. Also let η be the reflecting power of the surfaces of separation between the two media. Then, referring by O_1, O_2, etc., to the points bisecting the intervals between the plates, we can show from the electromagnetic theory of light that

$$r = \frac{-\eta\,(e^{ik\delta} - 1)}{(e^{ik\delta} - \eta^2)\, e^{\frac{1}{2}ik\delta'}}\ ; \quad t = \frac{(1-\eta^2)\, e^{\frac{1}{2}ik\delta}}{(e^{ik\delta} - \eta^2)\, e^{\frac{1}{2}ik\delta'}} \tag{16}$$

where $k = 2\pi/\lambda$.

Substituting these values, we get

$$\sinh^2 \alpha = \frac{\{\cos^2 \frac{1}{4}k\,(\delta+\delta') - \eta^2 \cos^2 \frac{1}{4}k\,(\delta-\delta')\}\{\sin^2 \frac{1}{4}k\,(\delta+\delta') - \eta^2 \sin^2 \frac{1}{4}k\,(\delta-\delta')\}}{\eta^2 \sin^2 \frac{1}{2}k\delta} \tag{17}$$

$$\sinh^2 \beta = \frac{-4}{(1-\eta^2)^2}\{\cos^2 \tfrac{1}{4}k\,(\delta+\delta') - \eta^2 \cos^2 \tfrac{1}{4}k\,(\delta-\delta')\}\{\sin^2 \tfrac{1}{4}k\,(\delta+\delta') - \eta^2 \sin^2 \tfrac{1}{4}k\,(\delta-\delta')\} \tag{18}$$

Since we shall have often to make use of these relations, we put $\frac{1}{4}k\,(\delta+\delta') = \phi$ and $(\delta-\delta')/(\delta+\delta') = c$, and rewrite them in a simpler form as

$$\sinh^2 \alpha = \frac{\{(\cos^2 \phi - \eta^2 \cos^2 c\phi)\,(\sin^2 \phi - \eta^2 \sin^2 c\phi)\}}{\eta^2 \sin^2 (1+c)\,\phi} \tag{19}$$

$$\sinh^2 \beta = \frac{-4}{(1-\eta^2)^2}\{\cos^2 \phi - \eta^2 \cos^2 c\phi\}\,\{\sin^2 \phi - \eta^2 \sin^2 c\phi\} \tag{20}$$

3. *The Two Cases*

We are now in a position to consider the intensity of the reflected beam. We need discuss only the reflected part, since the transmitted part will exhibit the complementary phenomena. Since r and t are complex, we should expect R_1 also to be complex, so that the intensity is represented by $|R_1|^2$. Let us now represent the ratio of the intensity $|R_1|^2$ of the reflected beam to that T_1^2 of the incident beam by R, which we may call the "reflecting power" of the stratified medium. Then

$$R = \frac{\sinh^2 n\beta}{\sinh^2 (\alpha + n\beta)}, \tag{21}$$

where $\sinh^2 \alpha$ and $\sinh^2 \beta$ have the values given in (19) and (20).

Before proceeding further, it is important to note that both α and β are complex. Hence, we write $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$, where α_1, β_1, α_2, β_2 are real. A consideration of equations (19) and (20) will show that $\sinh^2 \alpha$ may have positive or negative values, and that $\sinh^2 \beta$ will correspondingly have negative or positive values. Hence we must distinguish two distinct cases:

(i) $\sinh^2 \alpha$ is $-$ve, and consequently $\sinh^2 \beta$ is $+$ve. In this case $\sinh \alpha$ is imaginary and $\sinh \beta$ is real, so that α is pure imaginary $= i\alpha_2$ and β is real $= \beta_1$. Hence,

$$\frac{T_1}{R_1} = \cosh \alpha + \sinh \alpha \coth n\beta = \cos \alpha_2 + i \sin \alpha_2 \coth n\beta_1.$$

Therefore, $$\frac{1}{R} = \left| \frac{T_1}{R_1} \right|^2 = 1 + \frac{\sin^2 \alpha_2}{\sinh^2 n\beta_1}. \tag{22}$$

As β_1 is finite, $\sinh^2 n\beta_1$ steadily increases with n tending to the value ∞ as $n \rightarrow \infty$. Hence, for a particular value of α and β, R steadily increases with n, and $\rightarrow 1$ as $n \rightarrow \infty$.

(ii) $\sinh^2 \alpha$ is $+$ve and $\sinh^2 \beta$ is $-$ve, making $\alpha = \alpha_1$, and $\beta = i\beta_2$. This gives

$$\frac{1}{R} = 1 + \frac{\sinh^2 \alpha_1}{\sin^2 n\beta_2} \tag{23}$$

so that R fluctuates between O and a maximum as n is increased.

We may now consider the conditions under which these cases occur. From (19) and (20), it is easily seen that case (i) occurs when η^2 lies between

(*a*) $\dfrac{\cos^2 \phi}{\cos^2 c\phi}$ and (*b*) $\dfrac{\sin^2 \phi}{\sin^2 c\phi}$, and that case (ii)

occurs when it lies beyond these limits.

It is easy to see that if one of the two quantities (*a*) or (*b*) is less than unity, the other is greater. But η^2 has a range of values only between O and 1. Hence, the conditions of occurrence of the two cases may be rewritten as below :—

Case (i) occurs if $\eta^2 >$ either (*a*) or (*b*) whichever is < 1, and

case (ii) occurs if $\eta^2 <$ either (*a*) or (*b*) whichever is < 1.

4. *The Principal Maxima*

We shall now consider the nature of the reflection. At the very outset, we note that the rays from the corresponding parts in each stratification will reinforce each other, if the path retardation $\delta + \delta' = s\lambda$, where s is an integer. It then follows that the reflecting power R of the n plates must be a maximum, $R_{max.}$ for these values of λ. We shall call these maxima as the "primary maxima". In this case, $k(\delta + \delta') = 2\pi s$ and $\phi = s\pi/2$, which gives

$$\sinh^2 \alpha = \eta^2 \cos^2 \tfrac{1}{2}k\delta - 1 \tag{24}$$

Since η and $\cos \frac{1}{2}k\delta$ are always < 1, $\sinh^2 \alpha$ is always negative. This corresponds to case (i), R steadily increasing with n to the limiting value 1, as $n \to \infty$.

Now, this phenomenon of a steady increase of R with n can occur, as we have seen, only if η^2 is $> \dfrac{\sin^2 \phi}{\sin^2 c\phi}$ or $\dfrac{\cos^2 \phi}{\cos^2 c\phi}$, whichever is less than unity. Let us therefore find the solutions of the equations

$$\sin^2 \phi = \eta^2 \sin^2 c\phi \quad .. \quad \text{(A)}$$

$$\cos^2 \phi = \eta^2 \cos^2 c\phi \quad .. \quad \text{(B)}$$

This is easily done graphically. Fig. (2) shows the curves $y = \cos \phi$; $y = \pm \eta \cos c\phi$ and $y = \sin \phi$; $y = \pm \eta \sin c\phi$, drawn for two sets of

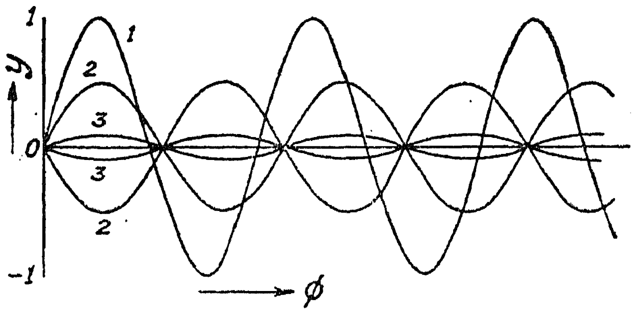


FIG. 2 *(a)*

1. $y = \sin \phi$ 2. $y = \pm \cdot 5 \sin \cdot 9\phi$ 3. $y = \pm \cdot 1 \sin \cdot 9\phi$

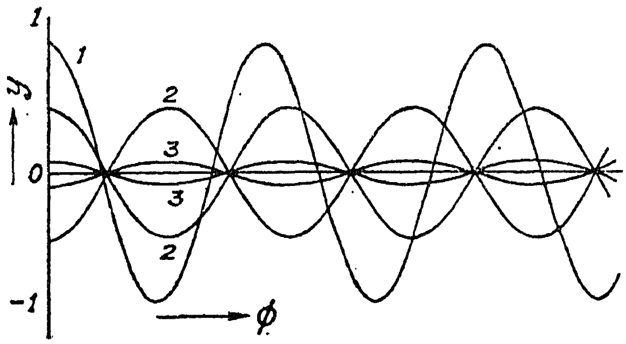


FIG. 2 *(b)*

1. $y = \cos \phi$ 2. $y = \pm \cdot 5 \cos \cdot 9\phi$ 3. $y = \pm \cdot 1 \cos \cdot 9\phi$

values of $\eta = \cdot 5$ and $\cdot 1$ and for $c = \cdot 9$. We see from these that, in general, the solutions of the equations (A) and (B), which are given by the values of ϕ corresponding to the intersection of the two curves, always occur in pairs on either side of $\phi = s\pi/2$. The pairs of solutions of A and B occur alternately. Let us call the two solutions on either side of $s\pi/2$ as ϕ_1 and ϕ_2.

5. *The Subsidiary Maxima*

It is easily seen that for values of ϕ situated in those intervals between ϕ_1 and ϕ_2 which contain $s\pi/2$, case (i) holds and the reflection steadily increases with n. Let us now consider the other region of values of ϕ. In this region, it is evident that case (ii) holds, so that

$$\frac{1}{\mathrm{R}} = 1 + \frac{\sinh^2 \alpha_1}{\sin^2 n\beta_2}. \tag{22}$$

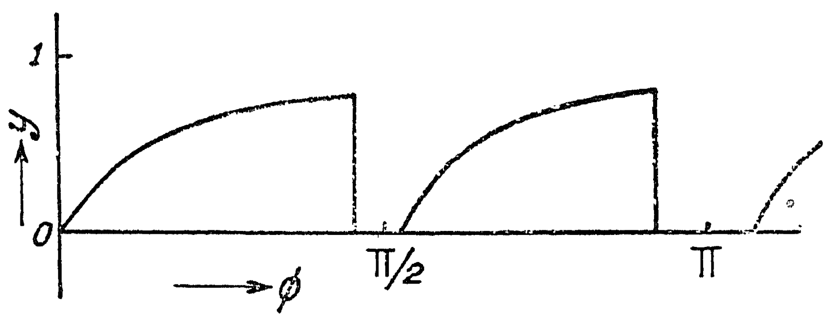
Taking the expression (20) for $\sinh^2 \beta$, it is easily seen that $\sinh^2 \beta = 0$, for the values of ϕ corresponding to the solutions of A and B and that it is negative in the region between, reaching a maximum negative value somewhere between. Hence, $|\beta_2|$ must also increase from O to a maximum, which is less than $\pi/2$ and then decrease to O. Then, if n is sufficiently large, $\sin^2 n\beta_2$ must undergo fluctuations between O and 1, and consequently, R also must fluctuate between zero and a maximum in this range. This gives rise to subsidiary maxima in the region between the primary maxima.

It is also easily seen that for the same range of values of β_2, the fluctuations increase in number as n is increased, so that the number of secondary maxima, and consequently, their sharpness increase as n increases.

Coming now to the intensity of the secondary maxima, it is influenced only by the values of $\sinh^2 \alpha_1$ since $\sin^2 n\beta_2 = 1$, for every one of them. Therefore, the secondary maxima must all lie on a curve whose equation is

$$\frac{1}{\mathrm{R}} = 1 + \sinh^2 \alpha_1. \tag{25}$$

The general form of this curve is shown in Fig. 3 for a typical value of c and of η. It may be noted that, in general, there is a marked asymmetry in the intensity of the secondary maxima about a primary maximum.



6. *The Width of the Primary Maxima*

We have already seen that in those regions between the solutions of A and B which contain $s\pi/2$, the intensity steadily increases with n, and that, in the other regions, it fluctuates with n. Now for a particular value of n, we note that, on passing out from the former region to the latter, the intensity becomes zero for values of ϕ corresponding to the first subsidiary minimum on either side. Thus, the width of a primary maximum is given by the distance between the first subsidiary minima on either side of $s\pi/2$.

Now, the first subsidiary minimum occurs for that value of ϕ for which $n\beta_2 = \pi$. Also, we know that β_2 increases from the value O at $\phi = \phi_1$ or ϕ_2 to a maximum and then diminishes. Hence, the larger the value of n the smaller will be the range of values of ϕ, outside $\phi_1\phi_2$ in which R falls to zero. Hence, as n increases, the width of the primary maximum must decrease.

However, it is easily seen that the width cannot go on diminishing indefinitely, since R can never be equal to zero in the range of values of ϕ from ϕ_1 to ϕ_2. Hence, the minimum width is given by the distance $\phi_1\phi_2$. We call this the " limiting width " for any order of primary maximum. We then see that, as n increases, the width steadily decreases, until at a fairly high value of n, it is very nearly equal to the limiting width. After this, the width does not undergo any appreciable diminution, however much n is increased.

7. *Variation of the Primary Maxima with Other Factors*

We shall now see how the intensity and width of the primary maxima vary with other factors when n is a constant.

Variation with η.

We have already seen that for the primary maxima

$$\sinh^2 \alpha = \eta^2 \cos^2 \tfrac{1}{2}k\delta - 1 = \{\eta^2 \cos^2 (1 + c)\, s\pi/2\} - 1 \qquad (26)$$

In the same way, $\sinh^2 \beta$ reduces to

$$\sinh^2 \beta = \frac{4\eta^2}{(1-\eta^2)^2} (\cos^2 cs\pi/2)\,(1 - \eta^2 \sin^2 cs\pi/2) \qquad (27)$$

Since, in this case, $\sinh^2 \alpha = (i \sin \alpha_2)^2$, we see from (26) that $\sin \alpha_2$ decreases as n increases. Also, from (27), we see that $\sinh^2 \beta_1 = \sinh^2 \beta$ increases, making β_1 and consequently $\sinh^2 n\beta_1$ increase. Hence, from relation (22) we find that R increases as η increases.

At the same time, as increase in η produces an increase in the limiting width as may be inferred from the curves in Fig. 2.

Variation with μ.

Let us suppose that the refractive index of the intervening medium is increased. Then μ is diminished, resulting in a decrease in the value of η.

The intensity of the primary maximum is diminished and its limiting width is also diminished.

Variation with Obliquity and the State of Polarization

Both these factors affect only the value of η and the effect is easily predictable. If the beam is polarised with the electric vector in the plane of incidence, the reflection coefficient η_1 is equal to unity at angle of incidence 90°, which rapidly diminishes with θ_1 to $\eta_1 = \text{O}$, at $\theta_1 = \tan^{-1}\mu$, the Brewsterian angle. As θ_1 is further diminished, η_1 rises to a small value. $R_{max.}$ also must show similar variation with θ_1.

If, however, the beam is polarised with the electric vector perpendicular to the plane of incidence, the reflection coefficient η_2 is equal to unity at $\theta_1 = 90°$ and slowly decreases to the same value as η_1 at $\theta_1 = \text{O}$. Hence, R steadily diminishes as θ_1 diminishes.

Corresponding to the variations in η, we must also expect a variation in the limiting width of the primary maxima as the angle of incidence is varied.

8. *A Particular Case*

It can be seen from Fig. 2 that the limiting width of the primary maxima varies in general, from order to order. However, for a special case, when $\delta = \delta'$, this is the same for all orders. We shall now consider this case in detail. Here, since $\delta = \delta'$, $c = \text{O}$, and the equations A and B reduce to

$$\cos^2\phi = \eta^2\cos^2 c\phi = \eta^2 \quad \ldots \quad (\text{A}') \text{ and}$$

$$\sin^2\phi = \eta^2\sin^2 c\phi = \text{O} \quad \ldots \quad (\text{B}')$$

the solutions of which are $\phi = s\pi$ and $\phi = s\pi \pm \cos^{-1}\eta$. We see that both the solutions of (B′) coincide at $s\pi$, so that we can have no principal maximum at $\phi = s\pi$. Hence, only odd orders of principal maxima exist in this case. This can also be explained physically, since for an even order, $\delta + \delta' = 2s\lambda$, and since $\delta = \delta'$, each is $= s\lambda$. Now the alternate reflecting surfaces are dissimilar, and therefore produce a relative phase change of π or path retardation of $\lambda/2$, so that the beams from each pair of consecutive surfaces destructively interfere, and completely annul each other, and there will be no reflection for that particular order. Also, for every one of these odd orders of reflection, we have from (25) $\sin^2\alpha_2 = 1$, and from (26) $\sinh^2\beta_1 = \dfrac{4\eta^2}{(1-\eta^2)^2}$, which is a constant independent of s. Hence, from (22) we see that the intensity of the primary maxima is the same for all orders. The limiting width also is given by $2\sin^{-1}\eta$, which is the same for all orders, and the maximum possible for a given η.

Again, taking the secondary maximum curve, $1/R = 1 + \sinh^2\alpha_1$, we get in this particular case.

$$\sinh^2 \alpha = \frac{(\cos^2\phi - \eta^2\cos^2 c\phi)\,(\sin^2\phi - \eta^2\sin^2 c\phi)}{\eta^2\sin^2(1+c)\,\phi}$$

$$= \frac{\cos^2\phi}{\eta^2} - 1, \text{ so that}$$

$$\frac{1}{R} = \frac{\cos^2\phi}{\eta^2}, \text{ and } R = \frac{\eta^2}{\cos^2\phi}. \tag{28}$$

This is symmetrical about $\phi = (2s + 1)\,\pi/2$, so that the secondary maxima are symmetrical about the primary maxima. Also R for these subsidiary maxima may have a value $= 1$ at $\phi = \cos^{-1} \pm \eta$, and it decreases to a value η^2, at $\phi = s\pi$, midway between the primary maxima. Thus the secondary maxima may have considerable intensities in the immediate neighbourhood of the primaries, and *may even have an intensity greater than them.*

9. *Stratifications of Vanishing Thickness*

Let us suppose that one of the media is very thin, *e.g.*, let δ' be very small compared with δ. In this case, $c = \frac{\delta - \delta'}{\delta + \delta'}$ will be very nearly equal to unity. Let it be $= 1 - x$, where x is very small. Then we may put $\sin x s\pi/2 = xs\pi/2$ and $\cos xs\pi/2 = 1$, approximately, so that the values of $\sinh^2 \alpha$ and $\sinh^2 \beta$ become

$$\sinh^2 \alpha = -\sin^2 \alpha_2 = -(1 - \eta^2) \tag{29}$$

$$\text{and } \sinh^2\beta = \sinh^2\beta_1 = \frac{4\eta^2}{(1-\eta^2)^2}\left(\frac{xs\pi}{2}\right)^2 \tag{30}$$

for the sth order. From these, it is easily seen that if $\delta' = O$, then $x = O$, and $\sinh^2\beta_1 = O$, making the reflection $R = O$. This is obvious, for the condition $\delta' = O$ means that there is no second medium, and no reflections can occur in this case.

Also, from (30) we see that if x increases, $\sinh^2\beta$ also increases with it, and from (22), it follows that R also increases. Thus, when the thickness of one of the layers is small, the intensity of the primary maxima is a direct function of the thickness.

A more detailed analysis, involving no approximations, shows that the intensity actually fluctuates as x is increased to larger values. The value of x is $2\delta'/(\delta + \delta')$, and an increase in x means an increase of δ' relative to δ. It is found that as x increases from O to $1/s$, the intensity of the sth order increases from zero to a maximum. Then it decreases to zero at $x = 2/s$. Thereafter, it undergoes fluctuations, having maximum values if $x = (2m + 1)/s$, and zero values if $x = 2m/s$, where m is an integer. Although it is not possible to vary x in any specimen and observe these fluctuations, this discussion is of interest because it enables us to arrive at a rough idea of the relative thicknesses of the two layers by means of a study of the relative intensities of the various orders of primary maxima. Thus, if $\delta = \delta'$, $x = \frac{1}{2}$, and all even orders must vanish, as we have already seen. If $\delta = 2\delta'$, $x = \frac{1}{3}$, and the 3rd, 6th, etc., orders vanish, and so on.

10. Graphical Illustration

In order to illustrate the preceding discussion, a particular case has been numerically computed, and the results are shown in Fig. 4. Here,

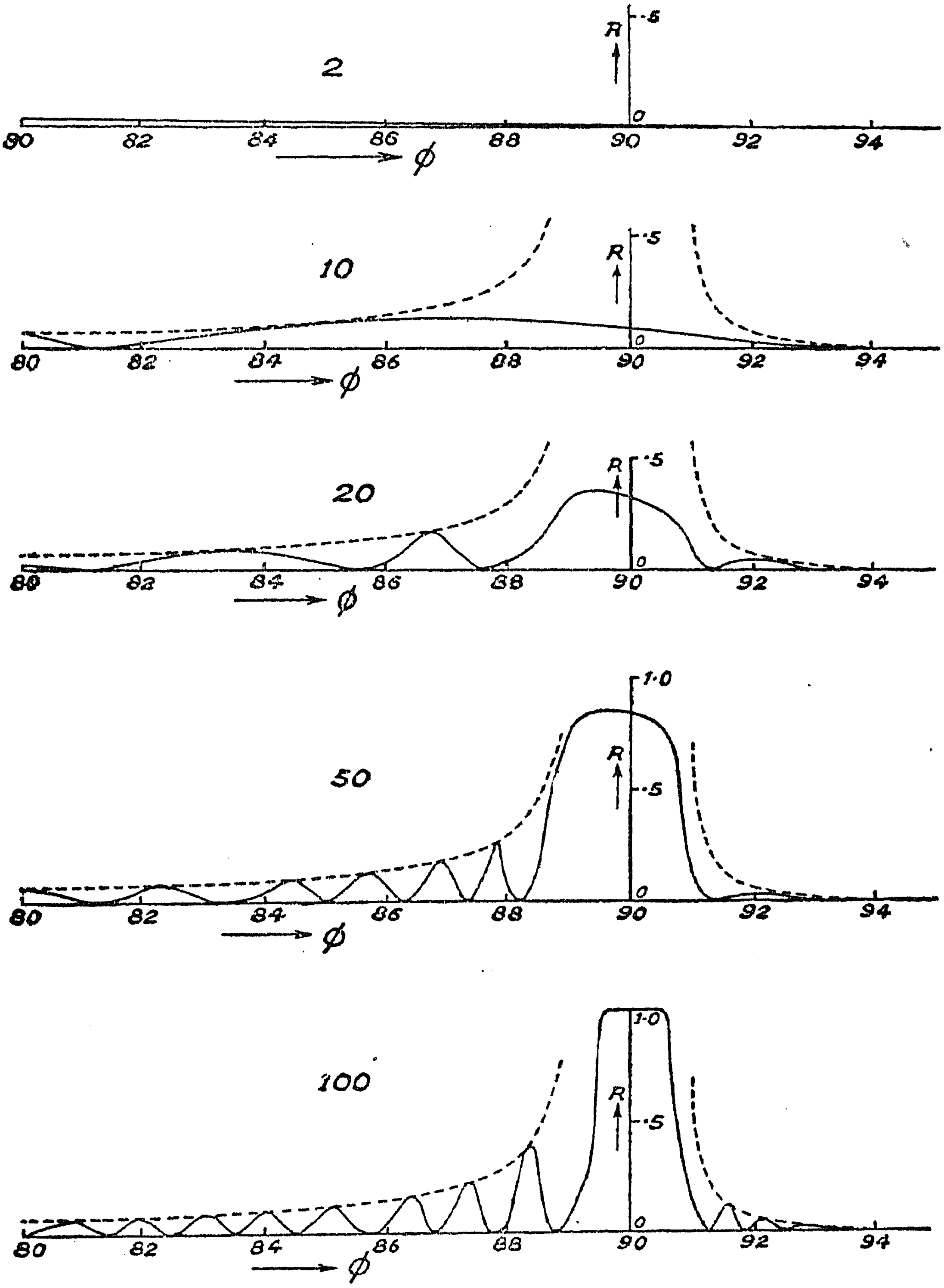


FIG. 4

the reflecting power η is supposed to be 0·1, and the value of c, 0·9. The spectral distribution of intensity in the neighbourhood of the first order principal maximum has been drawn for 2, 10, 20, 50 and 100 plates. The curves show the way in which the principal maximum is built up as the number of plates is increased. They also show that, when the number is increased above 20, the diminution in width is small.

The dotted line marks the curve on which the secondary maxima lie, and it shows how the intensity of these are widely different on either side of the principal maximum. It is also interesting to note how the number of the secondary maxima increases as n increases.

In conclusion, the author wishes to express his sincere gratitude to his professor, Sir C. V. Raman, for suggesting the method of attacking the problem, and also for the deep interest he took in the investigation.

11. *Summary*

An expression for the reflection from a medium consisting of n stratifications is derived in a simple manner, utilising Darwin's method in the analogous case of X-rays, modified to suit the optical case. The particular case of Lord Rayleigh is deduced from the general theory, and the expression so obtained is discussed in a general way by a consideration of the manner in which the functions vary. An important result that comes out of this is that the sharpness of the principal maxima cannot be indefinitely increased by increasing the number of stratifications, but that the width has a minimum limiting value, which depends only on the reflecting power of the stratifications. The mathematical discussion also shows that there exist a number of secondary maxima in the interval between the primary maxima, and that, in general, these show a marked difference in their intensities on either side of a principal maximum. Particular cases, when the paths in the two media are equal, and also when the path in one of them is small, are also investigated, and they yield some interesting results.

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