THE EQUATIONS OF FIT IN GENERAL RELATIVITY

For an isolated mass of fluid having the boundary

$$f(x^{\mu}) = 0 \qquad (1)$$

the internal field is given by $g_{\mu\nu}$ satisfying the equations,

$$G_{\mu\nu} - \tfrac{1}{2} G g_{\mu\nu} = -KT_{\mu\nu} = -k(p+\rho) v_{\mu}v_{\nu} + kpg_{\mu\nu}, \qquad (2)$$

and the external field by $g'_{\mu\nu}$ which are subject to

$$G_{\mu\nu} = 0. \qquad (3)$$

The equations of fit are usually stated as

$$g_{\mu\nu} = g'_{\mu\nu}, \; p = 0. \qquad (4)$$

These boundary conditions are not usually sufficient. Consider, as an example, the external field,

$$ds^2 = -(1 + m/2r)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + \left(\frac{1 - m/2r}{1 + m/2r}\right)^2 dt^2 \qquad (5)$$

for $r > a$ and the internal field, for $r \leqslant a$,

$$ds^2 = -e^{\mu} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + e^{\nu} dt^2. \qquad (6)$$

μ and ν are functions of r and when the differential equations for them are solved four arbitrary constants appear in them. The equations of fit (4) provide only three conditions. We find now that there is a fourth condition, *viz.*,

$$\left[e^{\mu/2} r^2 \frac{\partial}{\partial r} (e^{\nu/2}) \right]_{r=a} = m, \qquad (7)$$

which has got to be satisfied if the distribution given by (6) is to behave like a particle of mass m at great distances. Using (4) and (7) we get

$$e^{\mu} = \frac{(1 + m/2a)^6}{(1 + mr^2/2a^3)^2},$$

$$e^{\nu} = \left(\frac{2 - m/2a}{1 + m/2a} - \frac{1}{1 + mr^2/2a^3}\right)^2, \qquad (8)$$

$$8\pi\rho = \frac{6m/a^3}{(1 + m/2a)^6},$$

$$8\pi p = \frac{(3m^2/a^4)(1 - r^2/a^2)}{(1 + m/2a)^6 (1 - m/a + mr^2/a^3 - m^2r^2/4a^4)}. \qquad (9)$$

G. K. Patwardhan and P. C. Vaidya have shown in a paper, which is awaiting publication, how (7) arises from the principles of conservation. It is implicit in Tolman's[2] exposition of these principles that the continuity of

$$-g^{\alpha 4}\sqrt{-g}\,A^{\gamma}_{\mu\alpha}$$
$$+\tfrac{1}{2}\,g^{4}_{\mu}\,g^{\alpha\beta}\,\sqrt{-g}\,A^{\gamma}_{\alpha\beta}\quad \mu=1,2,3,4;\ \gamma=1,2,3 \quad (10)$$

"On the boundary is necessary. Here

$$A^{\gamma}_{\alpha\beta}=-\Gamma^{\gamma}_{\alpha\beta}+\tfrac{1}{2}g^{\gamma}_{\alpha}\,\Gamma^{\nu}_{\beta\nu}+\tfrac{1}{2}g^{\gamma}_{\beta}\,\Gamma^{\nu}_{\alpha\nu} \qquad (11)$$

This continuity ensures the conservation of energy and momentum provided

$$\frac{\partial}{\partial x^4}\iiint\Big(-g^{\alpha 4}\sqrt{-g}\,A^{4}_{\mu\alpha}$$
$$+\tfrac{1}{2}\,g^{4}_{\mu}\,g^{\alpha\beta}\sqrt{-g}\,A^{4}_{\alpha\beta}\Big)dx^1\,dx^2\,dx^3=0, \quad (12)$$

the integral being taken over the whole space for $\mu=1,\ 2,\ 3,\ 4$.

It is believed that the results (7), (8) and (9) are new. We are publishing elsewhere the full implications of Tolman's tacit assumption regarding the continuity of (12) which leads to (7). He seems to be unaware of (7).[3]

Lastly, another new point that has struck us in this connection may be noted. For the geodesies on the boundary surface we have

$$g_{\mu\nu}\frac{dx^{\mu}}{ds}\,\frac{dx^{\gamma}}{ds}=\pm 1, 0, \qquad (13)$$

$$f_{\mu}\,\frac{dx^{\mu}}{ds}=0, \qquad (14)$$

$$\left(-\frac{\partial f}{\partial x^{\mu}}\,\Gamma^{\mu}_{\alpha\beta}+\frac{\partial^2 f}{\partial x^{\alpha}\partial x^{\beta}}\right)\frac{dx^{\alpha}}{ds}\,\frac{dx^{\beta}}{ds}=0 \quad (15)$$

along with (1). If there is to be no ambiguity about these geodesies (15) must be the same whether it is couched in terms of $g_{\mu\nu}$ or $g'_{\mu\nu}$. That this is true has been verified for spherical distributions.

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[1] Eddington, Sir A. S., *The Mathematical Theory of Relativity*, 1924, 93.

[2] Tolman, R. C., *Relativity, Thermodynamics and Cosmology*, 1934, 232.

[3] —, *Phys. Rev.*, 1939, **55**, 364.