LETTERS TO THE EDITOR

On Poincaré's Theorem in Stellar Dynamics

THE major problems of stellar dynamics are concerned with systems of discrete gravitating particles which are virtually free from an external field of force. Poincaré's theorem[1] for such systems is usually expressed in the form

$$\frac{d^2}{dt^2}\Sigma\, m\rho^2 = 4\mathrm{T} + 2\mathrm{V} = 4h - 2\mathrm{V}, \qquad (1)$$

where m is a typical mass of the system, ρ its distance from the origin, and T and V are the total kinetic and potential energies h being their sum. Without any loss of generality the centre of mass of the system may be supposed to be permanently at rest at the origin. From (1) it follows that when the steady state is reached

$$2\mathrm{T} + \mathrm{V} = 0 \qquad (2)$$

and $\mathrm{T} = -h, \quad \mathrm{V} = 2h. \qquad (3)$

The simplifying hypothesis of a steady state does not always fit in with facts and it is therefore worthwhile looking into possible rates of condensation and disintegration for such systems. These considerations also help us to determine the age of a system.

Let us consider the parameters S, R defined by

$$\mathrm{S} = \Sigma m\rho^2 - 2ht^2 = \mathrm{MR}^2 - 2ht^2, \qquad (4)$$

where M is the total mass of the system. (1) is now equivalent to

$$\ddot{\mathrm{S}} = -2\mathrm{V} \qquad (5)$$

Hence $\ddot{\mathrm{S}} \geqslant 0 \qquad (6)$

Also $\dot{\mathrm{S}} = 2\,(\Sigma\, m\rho\dot{\rho} - 2ht) \qquad (7)$

We have

$$\frac{\dot{\mathrm{S}}^2}{4} = \mathrm{S}\,[\Sigma\, m\dot{\rho}^2 - 2h] + 2h\,\Sigma\, m\,(\rho - \dot{\rho}t)^2 - \sum_{m}\sum_{m'} mm'\,(\rho\dot{\rho}' - \rho'\dot{\rho}) \qquad (8)$$

Now $\Sigma\, m\,\dot{\rho}^2 - 2h = -2\mathrm{V} \geqslant 2\mathrm{T}' \qquad (9)$

where T′ is the kinetic energy of rotation. If the resultant angular momentum is a it can be easily shown[2] that

$$\mathrm{T}' \geqslant a^2/(\mathrm{S} + 2ht^2). \qquad (10)$$

We may therefore write

$$\mathrm{T}' = a^2 z/(\mathrm{S} + 2ht)^2 \qquad (11)$$

where $\mathrm{Z} \geqslant 1 \qquad (12)$

Thus

$$\frac{\dot{S}^2}{4} = S\left[-2V - \frac{2\alpha^2 z}{S + 2ht^2}\right]$$

$$+ 2h\Sigma\, m\,(\rho - \dot{\rho}t)^2 - \Sigma\Sigma\, mm'\,(\dot{\rho}\rho' - \rho'\dot{\rho})^2. \quad (13)$$

Using (5), the last equation can be restated as

$$S\ddot{S} - \frac{\dot{S}^2}{4} - 2\alpha^2 z\,\frac{S}{S + 2ht^2} =$$

$$\Sigma\Sigma\, mm'\,(\dot{\rho}\rho' - \rho'\dot{\rho})^2 - 2h\Sigma\, m\,(\rho - \dot{\rho}t)^2. \quad (14)$$

It is obvious that for particles satisfying the equation of Milne's law of recession,[3] *viz.*,

$$\dot{\rho} = \rho/t, \quad \dot{\rho}' = \rho'/t, \text{etc.} \quad (15)$$

the right-hand side of (14) vanishes. But (15) cannot be rigorously true for a classical system of gravitating particles as (14) clearly shows.

It is possible to discuss to some extent the evolution of a stellar cluster with the use of the parameter S. For if $\dot{S}$ is non-negative when $t = 0$, S is always positive and increasing by virtue of (6). In case $\dot{S}$ is negative initially S may attain a minimum or decrease indefinitely. The system is hyperbolic, elliptic or parabolic in character according as h is positive, negative or zero respectively. For an elliptic or parabolic system S will always be positive and if $\dot{S}$ is initially negative so that the system is shrinking ($\dot{R} < 0$) in its early stages S attains a minimum before proceeding to increase indefinitely. For a hyperbolic system S always increases or decreases or the first phase of decrease is followed by the second of uninterrupted increase.

These simple characteristics of the behaviour of S can be utilized in inferring the past and predicting the future of stellar clusters. R may very conveniently be taken as a measure of dispersion since the variation in the masses of stars is generally small. For an expanding hyperbolic cluster let $R = R_0$ at $t = 0$ and $R = R_1$ at $t = t_1$. Then

$$R_1^2 - \frac{h}{M}\, t_1^2 > R_0^2.$$

Hence $$\left\{\frac{(R_1^2 - R_0^2)M}{h}\right\}^{\frac{1}{2}} > t_1 \quad (16)$$

Similarly a lower limit on time can be determined for condensing systems for which h is negative.

Stellar considerations apart, the statement of Poincaré's theorem in the form (5) with the immediate conclusion (6) and the equation (14) are believed to be two new results sufficiently important in their own rights.

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1 Smart, W. M., *Stellar Dynamics*, 1938, 310.
2 Birkhoff, G. D., *Dynamical System*, 1927, 266.
3 Narlikar, V. V., *Nature*, 1935, **149**, 135.