A BABYLONIAN TOWER THEOREM FOR PRINCIPAL BUNDLES OVER PROJECTIVE SPACES

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ABSTRACT. We generalise the variant of the Babylonian tower theorem for vector bundles on projective spaces proved by I. Coandă and G. Trautmann (2006) to the case of principal G-bundles over projective spaces, where G is a linear algebraic group defined over an algebraically closed field. In course of the proofs some new insight into the structure of such principal G-bundles is obtained.
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Let G be a linear algebraic group defined over an algebraically closed field k. A principal G-bundle over a projective space $\mathbb{P}_n$ is called *split* if it admits a reduction of structure group to a maximal torus of G. Since a finite dimensional T-module, where T is a torus defined over k, splits into a direct sum of one-dimensional T-modules, the adjoint bundle of a split G-bundle decomposes into a direct sum of line bundles. When G is reductive, also the converse holds:

Proposition 1: *Let G be a reductive linear algebraic group. Let $\mathcal{E}$ be a principal G-bundle over $\mathbb{P}_n$ and $\mathrm{ad}(\mathcal{E})$ its adjoint bundle. If $\mathrm{ad}(\mathcal{E})$ splits as a direct sum of line bundles, then $\mathcal{E}$ is split.*

For $k = \mathbb{C}$ this is proved in [3], Theorem 4.3, using arguments extracted from Grothendieck's paper [11]. A proof in any characteristics is presented in Section 5. Using this result and the method from [7] we will prove the following:

Theorem 1: *Let G be a linear algebraic group, and let $\mathcal{E}$ be a principal G-bundle over $\mathbb{P}_n$ with adjoint bundle $\mathrm{ad}(\mathcal{E})$. Assume that $\mathcal{E}$ can be extended to a principal G-bundle over $\mathbb{P}_{n+m}$ for some $m > \Sigma_{i>0} \dim \mathrm{H}^1(\mathrm{ad}(\mathcal{E})(-i))$.*
If $\mathrm{char}(k) = 0$ or if $\mathrm{char}(k) = p > 0$ and G is reductive, then $\mathcal{E}$ is split as a principal bundle.

Theorem 1 also holds for arbitrary algebraic groups. This follows from the proof of Proposition 1.1 below. When $k = \mathbb{C}$ and G is a (finite dimensional) complex Lie group, one can use arguments analogous to those below to prove that the adjoint bundle of an analytic principal G-bundle on $\mathbb{P}_n(\mathbb{C})$ splits as a direct sum of line bundles, if it satisfies the extension assumption in Theorem 1.

As a byproduct of the proof of Theorem 1 one gets the following theorem.

Theorem 2: *Let $\mathcal{E}$ be a principal G-bundle over $\mathbb{P}_n$. If $\mathrm{H}^1(\mathrm{ad}(\mathcal{E})(-i)) = 0$ for all $i > 0$, then $\mathrm{ad}(\mathcal{E})$ splits as a direct sum of line bundles. If, moreover, G is reductive then $\mathcal{E}$ itself is split.*

Theorem 2 was proved by Mohan Kumar [15] under the assumption that $G = \mathrm{GL}_r(k)$, and it was proved in [2] under the assumption that $k = \mathbb{C}$ with G reductive. Again, the first assertion of Theorem 2 remains valid when $k = \mathbb{C}$, G is a (finite dimensional) complex Lie group, and $\mathcal{E}$ is a complex analytic principal G-bundle.

1. Some non-abelian cohomology

The following Proposition enables us to work with Zariski open subsets of $\mathbb{P}_n$ instead of étale covers. As before, k will denote an algebraically closed field.

1.1. Proposition: *a) Let G be an algebraic group over k. Then any principal G-bundle over $\mathbb{P}_n$ is Zariski locally trivial.*
b) For an abelian variety A over k, any algebraic principal A-bundle over $\mathbb{P}_n$ is trivial.

Proposition 1.1 will be proved in Section 4. One should note, however, that b) is not valid for complex analytic principal bundles with an abelian variety as the structure group.

We use the paper of Frenkel [10] as a reference for basic non-abelian cohomology. Let X be a topological space and $\mathcal{G}$ a sheaf of (not necessarily abelian) groups. For $U \subset X$ open, let e_U denote the unit element of $\mathcal{G}(U)$.

One defines, using Čech 1-cocycles and their equivalence relation, the *first cohomology set* $\mathrm{H}^1(X, \mathcal{G})$. It has a *marked element* corresponding to the 1-cocycle (e_X) on the open cover $\{X\}$ of X.

If $c \in \mathrm{H}^1(X, \mathcal{G})$ is represented by $(g_{ij}) \in \mathrm{Z}^1(\mathcal{U}, \mathcal{G})$ for some open cover $\mathcal{U}$ of X then one gets a *principal $\mathcal{G}$-bundle* by gluing the sheaves $\mathcal{G}|U_i$ with $g_{ij} \cdot - : \mathcal{G}|U_{ji} \xrightarrow{\sim} \mathcal{G}|U_{ij}$. In this way, $\mathrm{H}^1(X, \mathcal{G})$ parametrises the isomorphism classes of principal $\mathcal{G}$-bundles (locally trivial with respect to the topology of X).

A class $c \in \mathrm{H}^1(X, \mathcal{G})$ can also be used to define *twists* of sheaves of groups which are acted on by $\mathcal{G}$. For that let $\mathcal{A}$ be any other sheaf of groups and assume that there is an action $\mathcal{G} \times \mathcal{A} \to \mathcal{A}$. Then a sheaf $\mathcal{A}^c$ of groups is defined by gluing with the isomorphisms $g_{ij} \cdot - : \mathcal{A}|U_{ji} \xrightarrow{\sim} \mathcal{A}|U_{ij}$ of the action.

This twisting is obviously an exact functor on the category of $\mathcal{G}$-sheaves of groups.

In particular, a new sheaf of groups $\mathcal{G}^c$ is obtained by gluing the sheaves $\mathcal{G}|U_i$ with $g_{ij} \cdot - \cdot g_{ij}^{-1} : \mathcal{G}|U_{ji} \xrightarrow{\sim} \mathcal{G}|U_{ij}$ with respect to the action of inner automorphisms. Let $\phi_i : \mathcal{G}^c|U_i \xrightarrow{\sim} \mathcal{G}|U_i$ be the resulting isomorphisms.

There exists a *bijection* $\mathrm{H}^1(X, \mathcal{G}^c) \xrightarrow{\sim} \mathrm{H}^1(X, \mathcal{G})$ which is constructed by sending (the class of) $(f_{ij}) \in \mathrm{Z}^1(\mathcal{U}, \mathcal{G}^c)$ to (the class of) $(\phi_i(f_{ij}) \cdot g_{ij}) \in \mathrm{Z}^1(\mathcal{U}, \mathcal{G})$. This bijection sends the marked element of $\mathrm{H}^1(X, \mathcal{G}^c)$ to c.

Let, now, $1 \to \mathcal{G}' \xrightarrow{u} \mathcal{G} \xrightarrow{p} \mathcal{G}'' \to 1$ be a short exact sequence of sheaves of groups on X. This means that p is an epimorphism of sheaves and that, for every open subset $U \subseteq X$, $u(U)$ maps $\mathcal{G}'(U)$ isomorphically onto $\operatorname{Ker} p(U)$. In particular, every inner automorphism of $\mathcal{G}(U)$ induces, via $u(U)$, an automorphism of $\mathcal{G}'(U)$. It follows that, if one twists $\mathcal{G}'$ as above, one obtains a new sheaf of groups $\mathcal{G}'^c$ with an exact sequence $1 \to \mathcal{G}'^c \to \mathcal{G}^c \to \mathcal{G}''^c \to 1$.

1.2. Lemma: *Under the above hypothesis, there exists a canonical map*

$$\mathrm{H}^1(X, \mathcal{G}'^c) \to \mathrm{H}^1(X, \mathcal{G})$$

sending the marked element of $\mathrm{H}^1(X, \mathcal{G}'^c)$ to c and whose image is $\mathrm{H}^1(p)^{-1}(\mathrm{H}^1(p)(c))$.

Proof. One uses the H^1 part of the cohomology exact sequence associated to the short exact sequence of sheaves of groups : $1 \to \mathcal{G}'^c \to \mathcal{G}^c \to \mathcal{G}''^c \to 1$. $\qquad\square$

1.3. Lemma: *Let X be an algebraic scheme over k and $Y \subset X$ a closed subscheme defined by an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ with $\mathcal{I}^2 = 0$. Let G be a linear algebraic group, let $\mathcal{O}_X(G)$ denote the sheaf of morphisms from open sets of X to G, and let $L(G)$ denote the Lie algebra of G. Then there is a short exact sequence of sheaves of groups*

$$0 \to L(G) \otimes_k \mathcal{I} \to \mathcal{O}_X(G) \to \mathcal{O}_Y(G) \to 1.$$

Proof. Since G is smooth, $\mathcal{O}_X(G) \to \mathcal{O}_Y(G)$ is an epimorphism. In order to identify its kernel, we may assume that G is a closed subgroup of $\mathrm{GL}_r(k)$ for some r. The group $\mathrm{GL}_r(k)$ is an open subset of the affine space $\mathrm{Mat}_r(k)$ of $r \times r$ matrices. Now, one has an exact sequence

$$0 \to \mathrm{Mat}_r(k) \otimes_k \mathcal{I} \xrightarrow{\varepsilon} \mathcal{O}_X(\mathrm{GL}_r(k)) \to \mathcal{O}_Y(\mathrm{GL}_r(k)) \to 1,$$

in which ε is defined by $A \otimes f \mapsto e + Af$ as truncated exponential, with e denoting the unit $r \times r$ matrix. Let $I_G \subset k[(t_{ij})_{1 \leq i,j \leq r}]$ be the ideal of polynomials vanishing on G. Then, for an element $\gamma \in \mathrm{Mat}_r(k) \otimes_k \mathcal{I}(U)$, where U is open affine in X, $\varepsilon(\gamma)$ belongs to $\mathcal{O}_X(G)(U)$ if and only if $F(\varepsilon(\gamma)) = 0$, for every polynomial $F \in I_G$. One may write $\gamma = A_1 \otimes f_1 + \cdots + A_m \otimes f_m$ with $A_1, \ldots, A_m \in \mathrm{Mat}_r(k)$ and with $f_1, \ldots, f_m \in \mathcal{I}(U)$ linearly independent over k. Now, the Taylor expansion of any $F \in k[t_{ij}]$ at the identity $e \in \mathrm{Mat}_r(k)$, which reads as

$$F(e + A \cdot f) = \sum_{i,j} \frac{\partial F}{\partial t_{ij}}(e) \cdot a_{ij} \cdot f = (d_e F)(A) \cdot f,$$

yields the formula $F(\varepsilon(\gamma)) = (d_e F)(A_1) \cdot f_1 + \cdots + (d_e F)(A_m) \cdot f_m$, since $F(e) = 0$ and $\mathcal{I}(U)^2 = 0$.

If $\varepsilon(\gamma) \in \mathcal{O}_X(G)(U)$, then it follows that $(\mathrm{d}_e F)(A_\mu) = 0$, $\mu = 1, \ldots, m$, for any $F \in I_G$. But the intersection of the kernels of the differentials $\mathrm{d}_e F : \mathrm{Mat}_r(k) \to k$ for all the $F \in I_G$ is exactly the tangent space $\mathrm{T}_e G = L(G)$. Consequently, the kernel of $\mathcal{O}_X(G)(U) \to \mathcal{O}_Y(G)(U)$ is $L(G) \otimes_k \mathcal{I}(U)$. We have thus established the exact diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L(G) \otimes_k \mathcal{I} & \xrightarrow{\varepsilon_G} & \mathcal{O}_X(G) & \longrightarrow & \mathcal{O}_Y(G) & \longrightarrow & 1 \\
 & & \uparrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L(\mathrm{GL}_r) \otimes_k \mathcal{I} & \xrightarrow{\varepsilon} & \mathcal{O}_X(\mathrm{GL}_r) & \longrightarrow & \mathcal{O}_Y(\mathrm{GL}_r) & \longrightarrow & 1
\end{array}
$$

with ε_G induced by ε.   □

1.4. Remark: The action of $\mathcal{O}_X(G)$ on itself deduced from the action of G on itself by inner automorphisms induces, via the exact sequence from Lemma 1.3, an action of $\mathcal{O}_X(G)$ on $L(G) \otimes_k \mathcal{I}$. On the other hand, the action of $\mathcal{O}_X(G)$ on $L(G) \otimes_k \mathcal{O}_X$ (identified with the sheaf of morphisms from open sets of X to the vector space $L(G)$) deduced from the adjoint action of G on $L(G)$ induces, via the exact sequence :

$$0 \to L(G) \otimes_k \mathcal{I} \to L(G) \otimes_k \mathcal{O}_X \to L(G) \otimes_k \mathcal{O}_Y \to 0,$$

an action of $\mathcal{O}_X(G)$ on $L(G) \otimes_k \mathcal{I}$. These two actions of $\mathcal{O}_X(G)$ on $L(G) \otimes_k \mathcal{I}$ coincide since they obviously coincide in the case $G = GL_r$.

1.5. Lemma: *Under the assumptions of Lemma 1.3, let $\mathcal{F}$ be a principal G-bundle over X and let $\mathcal{E} = \mathcal{F}|Y$. Then there exists a canonical map*

$$\mathrm{H}^1(Y, \mathrm{ad}(\mathcal{E}) \otimes_{\mathcal{O}_Y} \mathcal{I}) \xrightarrow{\alpha} \mathrm{H}^1(X, \mathcal{O}_X(G))$$

sending 0 to the isomorphism class of $\mathcal{F}$, and whose image is the set of isomorphism classes of principal G-bundles $\mathcal{F}'$ over X such that $\mathcal{F}'|Y \simeq \mathcal{E}$.

Proof. $\mathcal{F}$ corresponds to an element $c \in \mathrm{H}^1(X, \mathcal{O}_X(G))$. If one uses the adjoint action of $\mathcal{O}_X(G)$ on $L(G) \otimes_k \mathcal{O}_X$, then the corresponding twisted sheaf $(L(G) \otimes_k \mathcal{O}_X)^c$ is exactly $\mathrm{ad}(\mathcal{F})$. The conclusion follows now from Lemma 1.3 and Lemma 1.2, taking into account that, according to the above Remark 1.4, one has an exact sequence :

$$0 \to (L(G) \otimes_k \mathcal{I})^c \to (L(G) \otimes_k \mathcal{O}_X)^c \to (L(G) \otimes_k \mathcal{O}_Y)^c \to 0,$$

hence : $(L(G) \otimes_k \mathcal{I})^c \simeq \mathrm{Ker}(\mathrm{ad}(\mathcal{F}) \to \mathrm{ad}(\mathcal{F})|Y) \simeq \mathrm{ad}(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{I} \simeq \mathrm{ad}(\mathcal{E}) \otimes_{\mathcal{O}_Y} \mathcal{I}$.   □

Notice, for further use, that, by construction, *the map α in the statement of the previous lemma is functorial in $(X, Y, \mathcal{F})$.*

2. Proof of Theorem 1

First, let us recall a result which is implicit in Kempf's paper [14]. For an explicit proof see [7].

2.1. Lemma: *Let E be an algebraic vector bundle on $\mathbb{P}_n$, $n \geq 2$, $H \subset \mathbb{P}_n$ a hyperplane, $x \in \mathbb{P}_n \setminus H$ and $p : \mathbb{P}_n \setminus \{x\} \to H$ the central projection. If E and $p^*(E|H)$ are isomorphic, as vector bundles, over each infinitesimal neighborhood of H in $\mathbb{P}_n$, then E splits into a direct sum of line bundles.*

In characteristic 0 one can generalise the above lemma to principal bundles :

2.2. Lemma: *Assume that $\mathrm{char}(k) = 0$ and let G be a linear algebraic group over k. Let $\mathcal{E}$ be a principal G-bundle on $\mathbb{P} = \mathbb{P}_n, n \geq 2$, and let H and p be as in the previous lemma. If $\mathcal{E}$ and $p^*(\mathcal{E}|H)$ are isomorphic as principal G-bundles over each infinitesimal neighborhood of H in $\mathbb{P}_n$, then $\mathcal{E}$ is split.*

Proof. Let $c \in \mathrm{H}^1(\mathbb{P}, \mathcal{O}_\mathbb{P}(G))$ be the class of $\mathcal{E}$. Let $R_u G$ be the unipotent radical of G, $Q = G/R_u G$ the reductive quotient and $\rho : G \to Q$ the canonical surjection. We will show that $\mathrm{H}^1(\mathbb{P}, \mathcal{O}_\mathbb{P}(R_u G)^c) = \{e\}$ (see (II) below) and that $(L(Q) \otimes_k \mathcal{O}_\mathbb{P})^c$ is a direct sum of line bundles (see (I) below).

Now, according to a result of G.D. Mostow [17] (which is valid only in characteristic 0, see [5] or [6]) there exists a *Levi subgroup* Λ of G, i.e., a closed subgroup such that, denoting by u the inclusion $\Lambda \hookrightarrow G$, the composition $\rho \circ u : \Lambda \to Q$ is an isomorphism. Since $\mathrm{H}^1(\mathbb{P}, \mathcal{O}_\mathbb{P}(R_u G)^c) = \{e\}$, Lemma 1.2 implies that $\mathrm{H}^1(\rho)^{-1}(\mathrm{H}^1(\rho)(c)) = \{c\}$. In particular, if $c_\Lambda \in \mathrm{H}^1(\mathbb{P}, \mathcal{O}_\mathbb{P}(\Lambda))$ is defined by $\mathrm{H}^1(\rho \circ u)(c_\Lambda) = \mathrm{H}^1(\rho)(c)$ then $c = \mathrm{H}^1(u)(c_\Lambda)$, i.e., $\mathcal{E}$ admits a reduction of structure group to Λ. Let $\mathcal{E}_\Lambda$ be the principal Λ-bundle defined by c_Λ. Then

$$\mathrm{ad}(\mathcal{E}_\Lambda) := (L(\Lambda) \otimes_k \mathcal{O}_\mathbb{P})^{c_\Lambda} \simeq (L(Q) \otimes_k \mathcal{O}_\mathbb{P})^c$$

is a direct sum of line bundles. Since $\Lambda \simeq Q$ is reductive and $\mathrm{char}(k) = 0$, [3], Theorem 4.3., implies that $\mathcal{E}_\Lambda$ admits a reduction of structure group to a maximal torus T of Λ.

This proves the lemma, modulo the two technical facts (I) and (II) quoted above.

(I) For any closed normal connected subgroup N of G there is the induced exact sequence of Lie algebras :

$$0 \to L(N) \to L(G) \to L(G/N) \to 0$$

and the associated exact sequence of locally free sheaves

$$0 \to (L(N) \otimes_k \mathcal{O}_\mathbb{P})^c \to (L(G) \otimes_k \mathcal{O}_\mathbb{P})^c \to (L(G/N) \otimes_k \mathcal{O}_\mathbb{P})^c \to 0.$$

One sees easily that each of the three bundles occurring in the last exact sequence satisfies the hypothesis of Lemma 2.1, hence is a direct sum of line bundles. Moreover, since $n \geq 2$, this exact sequence splits, so that $(L(N) \otimes_k \mathcal{O}_\mathbb{P})^c$ is a direct summand of $(L(G) \otimes \mathcal{O}_\mathbb{P})^c$.

(II) We show now that $\mathrm{H}^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(R_u G)^c) = \{e\}$.

To prove that we consider the central series

$$R_u G = C^0 \supset C^1 \supset \ldots \supset C^n = \{e\}$$

of $R_u G$. Each of the groups C^i is a closed connected normal subgroup of G and the quotients C^i/C^{i+1} are abelian and unipotent. This implies that the exponential map

$$L(C^i/C^{i+1}) \to C^i/C^{i+1}$$

is an isomorphism of algebraic groups. Using again the twisting by c, which is induced by the inner automorphisms of G, we obtain the exact sequences

$$0 \to (L(C^{i+1}) \otimes_k \mathcal{O}_{\mathbb{P}})^c \to (L(C^i) \otimes_k \mathcal{O}_{\mathbb{P}})^c \to (L(C^i/C^{i+1}) \otimes_k \mathcal{O}_{\mathbb{P}})^c \to 0$$

and, according to (I), $(L(C^{i+1}) \otimes_k \mathcal{O}_{\mathbb{P}})^c$ is a direct summand to $(L(C^i) \otimes_k \mathcal{O}_{\mathbb{P}})^c$ for $i \geq 0$.

It follows that also $(L(C^i/C^{i+1}) \otimes_k \mathcal{O}_{\mathbb{P}})^c$ is a direct sum of line bundles. Since $n \geq 2$, $\mathrm{H}^1(\mathbb{P}, (L(C^i/C^{i+1}) \otimes_k \mathcal{O}_{\mathbb{P}})^c) = 0$ for $i \geq 0$, and then also $\mathrm{H}^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(C^i/C^{i+1})^c) = \{e\}$. This proves that $\mathrm{H}^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(R_u G)^c) = \{e\}$. $\qquad\square$

We are able, now, to prove Theorem 1. Using the notation from the preparations preceding the proof of the Theorem in [7], suppose that there exists a principal G-bundle $\mathcal{F}$ over $\mathbb{P}_{n+m}$ such that $\mathcal{F}|L \simeq \mathcal{E}$. We shall construct a homogeneous ideal $J \subset R$, generated by $\Sigma_{i>0} \dim \mathrm{H}^1(\mathrm{ad}(\mathcal{E})(-i))$ homogeneous elements such that, for any $i \geq 0$,

$$\mathcal{F}|L_i \cap X \simeq \pi^* \mathcal{E}|L_i \cap X,$$

where X is the closed subscheme of $\mathbb{P}_{n+m}$ defined by the ideal JS and $\pi : \mathbb{P}_{n+m} \backslash L' \to L$ the central projection. The inequality imposed on m implies that there exists $p \in L' \simeq \mathbb{P}_{m-1}$ such that the polynomials from J vanish in p. The linear span P of p and L is contained in X, hence $\mathcal{F}|L_i \cap P \simeq \pi^* \mathcal{E}|L_i \cap P$, for any $i \geq 0$.

Recall that $P \simeq \mathbb{P}_{n+1}$ and that the schemes $L_i \cap P$ are the infinitesimal neighborhoods in P of the hyperplane L of P. Therefore, if $\mathrm{char}(k) = 0$, Lemma 2.2 implies that $\mathcal{F}|P$ is split and so is $\mathcal{E} \simeq \mathcal{F}|L$.

In the case of a reductive linear algebraic group in arbitrary characteristic, we know that also $\mathrm{ad}(\mathcal{F})|L_i \cap P \simeq \pi^* \mathrm{ad}(\mathcal{E})|L_i \cap P$ (as vector bundles on $L_i \cap P$), and then Lemma 2.1 implies that $\mathrm{ad}(\mathcal{E})$ splits as a direct sum of line bundles. From Proposition 1 one deduces that $\mathcal{E}$ is split in this case, too.

Finally, J is constructed, as in the proof of the Theorem in [7], by a standard technique borrowed from infinitesimal deformation theory, using Lemma 1.5 above. Explicitly:

Suppose that $J \subset R$ has already been constructed such that $\mathcal{F}|L_i \cap X \simeq \pi^* \mathcal{E}|L_i \cap X$. We enlarge J in degree $\geq i+1$ to an ideal J' as to obtain also $\mathcal{F}|L_{i+1} \cap X' \simeq \pi^* \mathcal{E}|L_{i+1} \cap X'$, where X' is the closed subscheme of $\mathbb{P}_{n+m}$ defined by the ideal $J'S$.

To do so we put $X_j = L_j \cap X$. Using the notation of [7], the ideal sheaf $\mathcal{I}_{X_i}$ of X_i in X_{i+1} is isomorphic to $\mathcal{O}_L(-i-1) \otimes R_{i+1}/J_{i+1}$ and satisfies $\mathcal{I}_{X_i}^2 = 0$. By Lemma 1.5 there is a canonical map

$$\mathrm{H}^1(L, \mathrm{ad}(\mathcal{E})(-i-1) \otimes R_{i+1}/J_{i+1}) \overset{\alpha}{\longrightarrow} \mathrm{H}^1(X_{i+1}, \mathcal{O}_{X_{i+1}}(G))$$

such that $\alpha(0) = [\pi^*\mathcal{E}|X_{i+1}]$ and the image of α is the set of all classes $[\mathcal{F}']$ of principal bundles $\mathcal{F}'$ on X_{i+1} such that $\mathcal{F}'|X_i \simeq \pi^*\mathcal{E}|X_i$. By assumption the class of $\mathcal{F}|X_{i+1}$ belongs to this set, hence $[\mathcal{F}|X_{i+1}] = \alpha(\xi)$ for some $\xi \in \mathrm{H}^1(L, \mathrm{ad}(\mathcal{E})(-i-1)) \otimes R_{i+1}/J_{i+1}$. Let $\xi_1, \ldots, \xi_s$ be a basis of $\mathrm{H}^1(L, \mathrm{ad}(\mathcal{E})(-i-1))$. Then

$$\xi = \xi_1 \otimes \bar{f}_1 + \cdots + \xi_s \otimes \bar{f}_s$$

with unique residue classes $\bar{f}_\nu \in R_{i+1}/J_{i+1}$, $f_\nu \in R_{i+1}$. Let

$$J' := J + Rf_1 + \cdots + Rf_s$$

and let $X' \subset X$ be the variety of $J'S \supset JS$.

Then $X'_i = L_i \cap X' = L_i \cap X = X_i$ and $X_i \subset X'_{i+1} \subset X_{i+1}$.

According to the functoriality of the maps α in Lemma 1.5, there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}^1(L, \mathrm{ad}(\mathcal{E})(-i-1)) \otimes R_{i+1}/J_{i+1} & \overset{\alpha}{\longrightarrow} & \mathrm{H}^1(X_{i+1}, \mathcal{O}_{X_{i+1}}(G)) \\
\downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} \\
\mathrm{H}^1(L, \mathrm{ad}(\mathcal{E})(-i-1)) \otimes R_{i+1}/J'_{i+1} & \overset{\alpha'}{\longrightarrow} & \mathrm{H}^1(X'_{i+1}, \mathcal{O}_{X'_{i+1}}(G))
\end{array}
$$

where ρ' and ρ denote the natural quotient maps. By definition of α' in Lemma 1.5, $[\pi^*\mathcal{E}|X'_{i+1}] = \alpha'(0)$. Since $\rho'(\xi) = 0$, it follows that

$$[\mathcal{F}|X'_{i+1}] = [\pi^*\mathcal{E}|X'_{i+1}].$$

This completes the inductive construction of J and the proof of Theorem 1.

3. Proof of Theorem 2

We use a trick of Mohan Kumar [15], to show that, under the hypothesis of the theorem, $\mathrm{ad}(\mathcal{E})$ can be extended to a vector bundle on $\mathbb{P}_{n+1}$.

Embed $\mathbb{P}_n$ as the hyperplane H of $\mathbb{P}_{n+1} =: P$ of equation $X_{n+1} = 0$ and let H_i denote its ith infinitesimal neighbourhood, of equation $X_{n+1}^{i+1} = 0$. Let $x \in P \setminus H$ and let $\pi_x : P \setminus \{x\} \to H$ be the projection. Using Lemma 1.5 and the vanishing conditions in the hypothesis one shows, by induction on $i \geq 0$, that if $\mathcal{F}$ is a principal G-bundle over H_i such that $\mathcal{F}|H \simeq \mathcal{E}$ then $\mathcal{F} \simeq \pi_x^*\mathcal{E}|H_i$. In particular, if $y \in P \setminus H$ is another point and $\pi_y : P \setminus \{y\} \to H$ the corresponding projection, then $\pi_y^*\mathcal{E}|H_i \simeq \pi_x^*\mathcal{E}|H_i$, $\forall i \geq 0$. This implies that $\pi_y^*\mathrm{ad}(\mathcal{E})|H_i \simeq \pi_x^*\mathrm{ad}(\mathcal{E})|H_i$, $\forall i \geq 0$.

Both $\pi_x^*\mathrm{ad}(\mathcal{E})$ and $\pi_y^*\mathrm{ad}(\mathcal{E})$ can be extended to *reflexive sheaves* $\mathcal{A}_x$ and $\mathcal{A}_y$ on P. The sheaf $\mathcal{H}om_{\mathcal{O}_P}(\mathcal{A}_x, \mathcal{A}_y)$ is reflexive, hence, for $j = 0,\ 1$, $\mathrm{H}^j(\mathcal{H}om_{\mathcal{O}_P}(\mathcal{A}_x, \mathcal{A}_y)(-i-1)) = 0$ for $i >> 0$. It follows that

$$\mathrm{Hom}_{\mathcal{O}_P}(\mathcal{A}_x, \mathcal{A}_y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{H_i}}(\mathcal{A}_x|H_i, \mathcal{A}_y|H_i)$$

for $i >> 0$. For $i >> 0$, any isomorphism $\mathcal{A}_x|H_i \xrightarrow{\sim} \mathcal{A}_y|H_i$ can be lifted to a morphism $\mathcal{A}_x \to \mathcal{A}_y$ which must be an isomorphism on a (Zariski) open neighbourhood U of H in P, that is, $\pi_x^*\mathrm{ad}(\mathcal{E})$ and $\pi_y^*\mathrm{ad}(\mathcal{E})$ are isomorphic over U. But $P\setminus U$ must be 0-dimensional, hence it has codimension ≥ 2. It follows that $\pi_x^*\mathrm{ad}(\mathcal{E})$ and $\pi_y^*\mathrm{ad}(\mathcal{E})$ are isomorphic over $P\setminus\{x,y\}$, hence they can be glued and one gets a vector bundle $\tilde{A}$ on P extending $\mathrm{ad}(\mathcal{E})$. Since $\tilde{A}|H_i \simeq \pi_x^*\mathrm{ad}(\mathcal{E})|H_i$, $\forall i \geq 0$, Lemma 2.1 implies that $\tilde{A}$ splits, hence $\mathrm{ad}(\mathcal{E}) \simeq \tilde{A}|H$ splits.

4. Proof of Proposition 1.1

A theorem of Chevalley says that there is a short exact sequence of groups

$$1 \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow 1, \tag{1}$$

where A is an abelian variety over k and H is an affine algebraic group over k; for a modern proof see [9]. We will show that any algebraic principal A-bundle over $\mathbb{P}_n$ is trivial.

Let E_A be a principal A-bundle over $\mathbb{P}_n$. To prove that E_A is trivial, it suffices to show that E_A admits a section over the generic point. Indeed, if s is a section of E_A over a Zariski open subset of $\mathbb{P}_n$, then s extends to a section of the pullback of E_A over some blow-up of $\mathbb{P}_n$. Since an abelian variety does not have any rational curves, the section over the blow-up of $\mathbb{P}_n$ descends to a section of E_A over $\mathbb{P}_n$.

There is a separable extension K' of the function field K of $\mathbb{P}_n$ over which E_A has a rational point. Hence E_A over K' is trivial. There is an inflation homomorphism $\mathrm{H}^1(K', A) \longrightarrow \mathrm{H}^1(K, A)$ whose composition with the natural homomorphism $\mathrm{H}^1(K, A) \longrightarrow \mathrm{H}^1(K', A)$ is multiplication by d, where d is the degree of the field-extension. So the class in $\mathrm{H}^1(K, A)$ given by E_A is torsion. We noted earlier that a principal A-bundle over $\mathbb{P}_n$ is trivial if its restriction to K is trivial. Therefore, the class in $\mathrm{H}^1(K, A)$ given by E_A being torsion it follows that the class in $\mathrm{H}^1(\mathbb{P}_n, A)$ given by E_A is torsion. Consequently, the principal A-bundle E_A over $\mathbb{P}_n$ admits a reduction of structure group to a finite group-scheme. Since the fundamental group-scheme of $\mathbb{P}_n$ is trivial [18, p. 93, Corollary], it follows that any principal bundle over $\mathbb{P}_n$ with a finite group-scheme as the structure group is trivial. Hence E_A is trivial.

Since any principal A-bundle over $\mathbb{P}_n$ is trivial, using (1) it follows that any principal G-bundle over $\mathbb{P}_n$ admits a reduction of structure group to the subgroup H. Therefore, to prove the Proposition it suffices to show that any principal H-bundle over $\mathbb{P}_n$ is Zariski locally trivial.

Let E_H be a principal H-bundle over $\mathbb{P}_n$. Let $H_0 \subset H$ be the connected component of H containing the identity element. Let $E_{H/H_0} = E_H \times_H (H/H_0)$ be the principal (H/H_0)-bundle over $\mathbb{P}_n$ obtained by extending the structure group of E_H using the quotient map $H \longrightarrow H/H_0$. Since $\mathbb{P}_n$ is simply connected, it follows immediately that E_{H/H_0} is a trivial principal (H/H_0)-bundle. Therefore, E_H admits a reduction of structure group to H_0.

Let E_{H_0} be a principal H_0-bundle over $\mathbb{P}_n$. To prove the Proposition it is enough to show that E_{H_0} is Zariski locally trivial.

We will prove that H_0 is acceptable in the sense of [19, p. 188, Definition]. But before that we will show that E_{H_0} is Zariski locally trivial assuming that H_0 is acceptable.

So assume that H_0 is acceptable. Since k is algebraically closed, any principal H_0-bundle over $\operatorname{Spec} k$ is trivial. Hence using [19, p. 189, Theorem A] it follows that the restriction of E_{H_0} to some open subscheme of any affine chart of $\mathbb{P}_n$ is trivial.

It follows from [8, hypothesis (1), p. 97] or [8, p. 110, Theorem 3.2], and the assumption that k is algebraically closed, that the principal H_0-bundle E_{H_0} is Zariski locally trivial under the assumption that H_0 is acceptable.

To prove that H_0 is acceptable, let $R_u H_0$ be the unipotent radical of H_0. So we have a short exact sequence of groups

$$1 \longrightarrow R_u H_0 \longrightarrow H_0 \longrightarrow Q_0 \longrightarrow 1\,,$$

where Q_0 is reductive. Note that Q_0 is connected as H_0 is so. From [20, p. 137, Theorem 1.1] we know that Q_0 is acceptable. Hence it suffices to show that $R_u H_0$ is acceptable.

The unipotent group $R_u H_0$ has a filtration of normal subgroups

$$e = U_0 \subset U_1 \subset \cdots \subset U_i \subset \cdots \subset U_{d-1} \subset U_d = R_u H_0\,,$$

where $d = \dim R_u H_0$, and U_i/U_{i-1} is the additive group $\mathbb{G}_a$ for each $i \in [1, d]$ (see [13, p. 123, Theorem 19.3]). Therefore, the group $R_u H_0$ is acceptable if $\mathbb{G}_a$ is acceptable. But

$$\mathrm{H}^1_{et}(\mathbb{A}^1, \mathbb{G}_a) = \mathrm{H}^1_{et}(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) = 0\,.$$

Hence $\mathbb{G}_a$ is acceptable. This completes the proof of Proposition 1.1.

5. Proof of Proposition 1

The aim in this section is to prove Proposition 1 for algebraically closed fields k of arbitrary characteristics. (for $k = \mathbb{C}$ this follows from [3])

Proposition: *Let G be reductive linear algebraic group defined over k. Let $\mathcal{E}$ be a principal G-bundle over $\mathbb{P}_n$ and $\mathrm{ad}(\mathcal{E})$ its adjoint bundle. If $\mathrm{ad}(\mathcal{E})$ splits as a direct sum of line bundles, then $\mathcal{E}$ is split.*

Proof. We recall that if the characteristic of the base field k is positive, then a principal bundle $\mathcal{F}$ over a smooth variety X defined over k is called *strongly semistable* if the pull back $(F_X^n)^* \mathcal{F}$ over X is semistable for all $n \geq 1$, where F_X^n is the n-fold composition of

the Frobenius morphism $F_X : X \to X$. For our convenience, when the characteristic of k is zero, by a strongly semistable principal bundle we will mean a semistable bundle.

Since the tangent bundle $T\mathbb{P}_n$ is semistable of positive degree, any semistable vector bundle over $\mathbb{P}_n$ is strongly semistable [16, p. 316, Theorem 2.1(1)].

Let $\mathcal{E}$ be a principal G-bundle over $T\mathbb{P}_n$, where G is a reductive linear algebraic group defined over k. Then $\mathcal{E}$ admits a unique Harder-Narasimhan reduction; see [4]. In general some conditions are needed for the uniqueness part of the Harder-Narasimhan reduction. (See [4, p. 208, Proposition 3.1] for the existence of Harder-Narasimhan reduction, and [4, p. 221, Corollary 6.11] for the uniqueness; the fact that any any semistable vector bundle over $\mathbb{P}_n$ is strongly semistable ensures Proposition 6.9 in [4, p. 219] remains valid without the assumption on the height.)

Now we assume that the adjoint vector bundle $\mathrm{ad}(\mathcal{E})$ is a direct sum of line bundles. This immediately implies that the Harder-Narasimhan filtration of $\mathrm{ad}(\mathcal{E})$ is a filtration of subbundles of $\mathrm{ad}(\mathcal{E})$ (in general it is only a filtration of subsheaves with each successive quotient being torsionfree). Therefore, the Harder-Narasimhan reduction of $\mathcal{E}$ is defined over entire $\mathbb{P}_n$.

Let

$$\mathcal{E}_P \subset \mathcal{E} \tag{2}$$

be a principal P-bundle giving the Harder-Narasimhan reduction of $\mathcal{E}$ over $\mathbb{P}_n$; here $P \subset G$ is a parabolic subgroup.

Any principal G-bundle over $\mathbb{P}_1$ is split (see [12]). Therefore, the proposition is proved for $n = 1$. Henceforth, we will assume that $n \geq 2$.

Consider the short exact sequence of groups

$$1 \to R_u(P) \to P \to Q(P) \to 1 \,, \tag{3}$$

where $R_u(P)$ is the unipotent radical of P, and $Q(P)$ is the Levi quotient of of P. This short exact sequence is right split. Fix a subgroup of P that projects isomorphically to $Q(P)$. This subgroup will be denoted by $\Lambda(P)$. We will show that $\mathcal{E}_P$ admits a reduction of structure group to the subgroup $\Lambda(P)$ of P.

To prove this first note that giving a reduction of structure group of $\mathcal{E}_P$ to $\Lambda(P)$ is equivalent to giving a section of the fibre bundle $\mathcal{E}_P/\Lambda(P)$ over $\mathbb{P}_n$. Let $\mathcal{E}_P(R_u(P))$ be the group-scheme over $\mathbb{P}_n$ associated to $\mathcal{E}_P$ for the adjoint action of P on the normal subgroup $R_u(P)$ in (3). The fibre bundle $\mathcal{E}_P/\Lambda(P)$ is a torsor for $\mathcal{E}_P(R_u(P))$. In other words, the fibres of $\mathcal{E}_P(R_u(P))$ have a natural free transitive action on the fibres of $\mathcal{E}_P/\Lambda(P)$. Torsors for $\mathcal{E}_P(R_u(P))$ are parametrised by $\mathrm{H}^1(\mathbb{P}_n, \mathcal{E}_P(R_u(P)))$. Therefore, to prove that $\mathcal{E}_P$ admits a reduction of structure group to the subgroup $\Lambda(P)$ it suffices to show that

$$\mathrm{H}^1(\mathbb{P}_n, \mathcal{E}_P(R_u(P))) = 0 \,. \tag{4}$$

Consider the upper central series $\{\mathcal{G}_i\}_{i \geq 0}$ for $R_u(P)$. So $\mathcal{G}_0 = R_u(P)$ and $\mathcal{G}_{i+1} = [R_u(P), \mathcal{G}_i]$ for all $i \geq 0$. This central series is preserved by the adjoint action of P,

and each successive quotient is an abelian unipotent group. For an abelian unipotent group, the exponential map from its Lie algebra is well-defined, and it is an isomorphism. Also, for any line bundle ξ on $\mathbb{P}_n$ we have $\mathrm{H}^1(\mathbb{P}_n, \xi) = 0$ (recall that $n > 1$). Therefore it follows that (4) holds.

Let

$$\mathcal{E}_{\Lambda(P)} \subset \mathcal{E}_P \tag{5}$$

be a reduction of structure group of $\mathcal{E}_P$ to $\Lambda(P)$. Let $\mathcal{E}'_{Q(P)}$ be the principal $Q(P)$-bundle obtained by extending the structure group of $\mathcal{E}_P$ using the projection in (3). We note that the principal $Q(P)$-bundle $\mathcal{E}'_{Q(P)}$ is canonically identified with $\mathcal{E}_{\Lambda(P)}$.

Let

$$F_1 \subset \cdots \subset F_{m-1} \subset F_m = \mathrm{ad}(\mathcal{E})$$

be the Harder-Narasimhan filtration of $\mathrm{ad}(\mathcal{E})$. From the construction of the Harder-Narasimhan reduction of $\mathcal{E}$ we know that $m = 2m_0 + 1$, and $\mathrm{degree}(F_{(m+1)/2}/F_{(m-1)/2}) = 0$. Furthermore, the subbundle $\mathrm{ad}(\mathcal{E}_P) \subset \mathrm{ad}(\mathcal{E})$ coincides with $F_{(m+1)/2}$, and $\mathrm{ad}(\mathcal{E}'_{Q(P)})$ coincides with the quotient $F_{(m+1)/2}/F_{(m-1)/2}$. In particular, $\mathrm{ad}(\mathcal{E}'_{Q(P)})$ is a semistable vector bundle. We noted earlier that any semistable vector bundle on $\mathbb{P}_n$ is strongly semistable. Therefore, $\mathrm{ad}(\mathcal{E}'_{Q(P)})$ is strongly semistable.

Since $\mathcal{E}'_{Q(P)}$ is identified with $\mathcal{E}_{\Lambda(P)}$, we now conclude that the adjoint vector bundle $\mathrm{ad}(\mathcal{E}_{\Lambda(P)})$ is strongly semistable.

We recall that $\mathrm{ad}(\mathcal{E})$ is a direct sum of line bundles. From the above remark that the subbundle $\mathrm{ad}(\mathcal{E}_P)$ coincides with $F_{(m+1)/2}$ it follows immediately that $\mathrm{ad}(\mathcal{E}_P)$ is also a direct sum of line bundles. Since the adjoint bundle $\mathrm{ad}(\mathcal{E}_{\Lambda(P)})$ is a direct summand of $\mathrm{ad}(\mathcal{E}_P)$, using the Atiyah-Krull-Schmidt theorem (see [1, p. 315, Theorem 3]) we conclude that $\mathrm{ad}(\mathcal{E}_{\Lambda(P)})$ is also a direct sum of line bundle.

Since $\Lambda(P)$ is reductive, the adjoint group

$$H := \Lambda(P)/Z(\Lambda(P)) \tag{6}$$

$\Lambda(P)$ does not admit any nontrivial character; here $Z(\Lambda(P))$ is the center of $\Lambda(P)$. Hence $\det \mathrm{ad}(\mathcal{E}_{\Lambda(P)}) = \bigwedge^{\mathrm{top}} \mathrm{ad}(\mathcal{E}_{\Lambda(P)})$ is a trivial line bundle. On the other hand, we already proved that $\mathrm{ad}(\mathcal{E}_{\Lambda(P)})$ is semistable and it splits into a direct sum of line bundles. Combining these it follows that $\mathrm{ad}(\mathcal{E}_{\Lambda(P)})$ is a trivial vector bundle.

Let $\mathfrak{l}(\mathfrak{p})$ denote the Lie algebra of the group $\Lambda(P)$. Consider the adjoint action of $\Lambda(P)$ on $\mathfrak{l}(\mathfrak{p})$. It gives a homomorphism to the linear group

$$\rho : \Lambda(P) \rightarrow \mathrm{GL}(\mathfrak{l}(\mathfrak{p})) . \tag{7}$$

Let $\mathcal{E}_{\mathrm{GL}(\mathfrak{l}(\mathfrak{p}))}$ denote the principal $\mathrm{GL}(\mathfrak{l}(\mathfrak{p}))$-bundle over $\mathbb{P}_n$ obtained by extending the structure group of $\mathcal{E}_{\Lambda(P)}$ using the homomorphism ρ in (7). We noted earlier that $\mathrm{ad}(\mathcal{E}_{\Lambda(P)})$ is a trivial vector bundle. Therefore, $\mathcal{E}_{\mathrm{GL}(\mathfrak{l}(\mathfrak{p}))}$ is a trivial principal bundle.

Consider the quotient H of $\Lambda(P)$ defined in (6). Let $\mathcal{E}_H$ be the principal H-bundle over $\mathbb{P}_n$ obtained by extending the structure group of $\mathcal{E}_{\Lambda(P)}$ using the quotient map. The

homomorphism ρ in (7) factors through H. Therefore, $\mathcal{E}_{\mathrm{GL}(\mathfrak{l}(\mathfrak{p}))}$ is an extension of structure group of $\mathcal{E}_H$.

Since $\Lambda(P)$ is reductive, the homomorphism ρ gives an embedding of H into $\mathrm{GL}(\mathfrak{l}(\mathfrak{p}))$. We already noted that $\mathcal{E}_{\mathrm{GL}(\mathfrak{l}(\mathfrak{p}))}$ is trivial. Therefore, the reduction

$$\mathcal{E}_H \subset \mathcal{E}_{\mathrm{GL}(\mathfrak{l}(\mathfrak{p}))}$$

is given by a morphism

$$f : \mathbb{P}_n \to \mathrm{GL}(\mathfrak{l}(\mathfrak{p}))/H \, . \tag{8}$$

Since H is a reductive subgroup of $\mathrm{GL}(\mathfrak{l}(\mathfrak{p}))$, the quotient space $\mathrm{GL}(\mathfrak{l}(\mathfrak{p}))/H$ is an affine variety. Therefore, the morphism f in (8) is a constant one. This immediately implies that the principal H-bundle $\mathcal{E}_H$ is trivial. From this it follows that the principal $\Lambda(P)$-bundle $\mathcal{E}_{\Lambda(P)}$ admits a reduction of structure group to the center $Z(\Lambda(P))$.

Since $\mathcal{E}$ is an extension of structure group of $\mathcal{E}_{\Lambda(P)}$, we now conclude that $\mathcal{E}$ admits a reduction of structure group to a maximal torus of G. $\qquad\square$

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