

A PRACTICAL FINANCIAL TRANSACTION.

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A MESSENGER prize of the Institute of Actuaries, London, was awarded in 1933, for the first time, to an Indian, Mr. D. P. Misra,¹ for demonstrating that multiplicity of the rate of interest is possible for a financial transaction and for propounding the need of a criterion by which a unique rate of interest may be determined in such cases of ambiguity. It had been taken for granted before, presumably without any mathematical proof to support it, that a financial transaction admits only one rate of interest; and hence the examples given by Misra in which two or more rates of interest become possible evoked much interest among the leading actuaries of the world.

A precise definition of a practical financial transaction must first be stated as it is intended to treat it here mathematically. The definition proposed by us is this: throughout a practical financial transaction, on the basis of a uniform rate of interest, the original creditor always remains the creditor, and the original debtor always the debtor and that the two are quits only when the transaction comes to an end. This definition is not satisfied by the examples of Misra wherein multiplicity arises.

The definition of a practical financial transaction may now be expressed as a set of mathematical conditions. Let a financial transaction start with A lending a sum a_0 to B on a certain date, say, January 1st, 1935. If A is the creditor $a_0 > 0$. Exactly a year from that date A lends another sum a_1 to B or B returns a sum $-a_1$ to A according as a_1 is +ve or -ve. If $y\%$ is the rate of interest we may denote $1+y/100$ by x . According to the definition of a practical transaction $a_0x + a_1 \geq 0$ on January 1st, 1936. If $a_0x + a_1 = 0$ the transaction becomes complete and the rate of interest is $10^2(-a_1/a_0 - 1)\%$. If the transaction is not complete on January 1st, 1936, let it be completed exactly n years after the initial payment. If at the end of the r th year from the beginning A lends a_r to B or B returns $-a_r$ to A according as a_r is +ve or -ve, then the following algebraic conditions hold good :

¹ *Journal of the Institute of Actuaries*, Vol. 64, pp. 71-97. *Vide* also Dr. Steffensen's paper in the same volume.

$$\begin{aligned} f_n(x) &\equiv a_0x^n + a_1x^{n-1} + \dots + a_n = 0 \\ f_r(x) &\equiv a_0x^r + a_1x^{r-1} + \dots + a_r > 0, r = 0, 1, \dots, n-1 \end{aligned} \quad \dots \quad (I)$$

If $a_r = 0$ it means of course that no payment occurs between the parties at the end of the r th year. The uniqueness will be established when we prove that if there exists a root of $f_n(x) = 0$, say, $x = a > 1$ satisfying all the conditions (I) then there cannot exist $\beta > 1$ satisfying $f_n(\beta) = 0$. This is an algebraic problem and we proceed to prove now a slightly more general problem :

If $a \geq 0$ satisfies all the conditions (I) then $f_n(x) = 0$ has no other positive root.

We have

$$f'_n(x) = [f_{n-1} + xf_{n-2} + \dots + x^rf_{n-r-1} + \dots + x^{n-1}f_0] \quad \dots \quad (II)$$

It is clear from (II) that at $x = a$ $f'_n(a)$ is positive. As in (II) $f'_{n-1}(x)$, $f'_{n-2}(x)$, etc., may also be expressed by similar finite series. Hence $f'_{n-1}(a)$, $f'_{n-2}(a)$, etc., are all positive. It follows therefore that $f''_n(x)$ is +ve at $x = a$. Proceeding in this manner it becomes clear that $f'''_n(a)$, $f^{IV}_n(a)$, \dots $f^n_n(a)$ are all positive and that the higher derivatives vanish. Using the Taylor polynomial expansion we find that $f_n(x) > 0$ for $x > a$ and that $f_n(x) = 0$ cannot therefore have another root greater than a .

We can also show that there cannot exist another root β of $f_n(x) = 0$ such that $0 < \beta < a$. For obviously $f_0(\beta) = f_0(a) > 0$. Hence $af_0(a) + a_1 > \beta f_0(\beta) + a_1$ or $f_1(a) > f_1(\beta)$. Since $f_1(a) > 0$ we obtain similarly $f_2(a) > f_2(\beta)$. Applying this process continuously the conclusion is reached that $f_n(a) > f_n(\beta)$. But $f_n(a) = 0$. Hence β cannot be a root of $f_n(x) = 0$.

It is this algebraic proposition that supplies a proof of the uniqueness of the rate of interest in a practical financial transaction. We have considered the simple case where payments and returns are made at yearly intervals, but a more sophisticated case where payments and returns are made at any intervals which however can be expressed by commensurable integers reduces to a similar algebraic problem to which the result of the general proposition established becomes applicable.

Incidentally the general algebraic proposition is also valid if some, but not all, $f_r(x)$ vanish at $x = a$. The case is certainly trivial where all $f_r(x)$ vanish at $x = a$. It is also evident that the root a is not repeated except possibly when $a = 0$.

When this investigation was being carried out a new theorem in the theory of equations occurred to us. In view of the close connection between the mathematical arguments of this paper and this theorem it may not be

out of place to state and prove it here briefly. Following the notation of the paper we state the theorem as follows :

If $f_r(x) = 0$ has p roots each greater than a , then there are at least p changes of sign in the series $f_0(a), f_1(a), \dots, f_r(a)$ where $a \geq 0$.

Consider $(x - \beta) f_r(x) = \psi_{r+1}(x)$ for $\beta > a$. If $\psi_{r+1}(x) \equiv b_0 x^{r+1} + b_1 x^r + \dots + b_{r+1}$ we may define $\psi_t(x) \equiv b_0 x^t + b_1 x^{t-1} + \dots + b_t$ for $t = 0, 1, \dots, r$; so that $\psi_t(a) = (a - \beta)f_{t-1}(a) + a_t$, $\psi_0(a) = f_0(a)$ and $\psi_{r+1}(a) = (a - \beta)f_r(a)$.

Now consider the signs in the two series :

$$\begin{array}{ccccccc} f_0(a) & f_1(a) & \dots & \dots & f_r(a) \\ \psi_0(a) & \psi_1(a) & \dots & \dots & \psi_{r+1}(a). \end{array}$$

The two series begin with the same sign and end with opposite signs. Corresponding to a change of sign from $f_{t-1}(a)$ to $f_t(a)$ the sign of $\psi_t(a)$ agrees with the latter; and corresponding to a continuation of sign in the first from $f_{t-1}(a)$ to $f_t(a)$ the sign of $\psi_t(a)$ remains ambiguous. When a zero value comes in either series we may take the left hand sign.

Hence it follows that the ψ -series has at least one more change of sign than the f -series. If the f -series is perfectly general so also is the ψ series, the increase in changes of sign in the latter series being due to the fact that $\psi_{r+1}(x) = 0$ has one more root (*viz.*, β) exceeding a than $f_r(x) = 0$. This completes the proof of the theorem, which may be regarded as a generalisation of Descartes' rule of signs.

Summary.

A definition of a practical financial transaction is given and it is proved straight from the definition that a practical transaction admits only one rate of interest. The transactions considered by Misra in his paper are thus shown to be only mathematically possible. Two new algebraic theorems are stated and proved.