THE DOPPLER EFFECT IN THE FIELD OF A THICK SPHERICAL SHELL

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1. Introduction

It is well known that the gravitational potential is constant in the cavity of a thick heterogeneous shell of matter, having a distribution of spherical symmetry, and that the gravitational force is consequently nil at each point. The relativistic field of such a shell has been obtained by J. T. Combridge (1926). A more general discussion may be found in a later treatment by A. J. Carr (1933). The relativistic analogue of the classical result of a constant potential in the cavity is found to be in the Euclidean metric representing its space-time. There is however another result which is purely relativistic, having no classical analogue for this gravitational situation. It is the Doppler effect in the cavity. That the Riemannian metric of a gravitational field gives rise to a Doppler effect is a fact that has been known since the early days of general relativity. In the present note we consider an observer situated within the cavity of a spherical shell and a light source situated outside the shell. The observer, moving as he is in a flat space-time, experiences the shift on account of the conditions of continuity to be satisfied at the boundaries of the shell. Apart from any possible cosmological application of this, we find that the result which has not been explicitly stated before is novel enough to be placed on record here. Moreover the cavity in this case presents the only known example of a Euclidean pocket in a Riemannian space-time. In the relativistic analogue of a thick homœoid of matter, in the cavity of which there is no gravitational force, a similar Euclidean pocket is expected to exist. But this problem, as far as we know, has not yet been worked out. We give here just the few essential details showing how the pocket arises and how the constant of Doppler effect is fixed by the boundary conditions. This constant depends upon the constant L appearing in the following discussion. We find that Combridge's treatment is defective on account of his assumption $L = 1$.

2. The Field

For a shell of radii a and b ($b > a > 0$) we assume the distribution of matter to be of spherical symmetry and obtain the metric

$$ds^2 = -e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + e^{\gamma} dt^2 \qquad (1$$

where, in the usual notation,

$$e^{-\lambda} = 1 - \frac{2}{r} m(r), \tag{2}$$

$$m(r) = \frac{4\pi G}{c^2} \int_a^r T_4^4 r^2 dr, \tag{3}$$

$$e^{\gamma} = L\left(1 - \frac{2}{r} m(r)\right) e^{\int_a^r F(r)\, dr}, \tag{4}$$

$$F(r) = \frac{8\pi G}{c^2} \cdot r\left(1 - \frac{2}{r} m(r)\right)^{-1} (T_4^4 - T_1^1), \tag{5}$$

for $a \leqslant r \leqslant b$, in the usual notation. L is an arbitrary positive constant which is fixed by the boundary conditions. The only non-zero components of the energy-momentum tensor are T_1^1, T_2^2, T_3^3, T_4^4 and consistency demands that $T_2^2 = T_3^3$. Combridge has taken as the mass equivalent of the distribution,

$$m_1 = \frac{4\pi G}{c^2} \int_a^b T_4^4 r^2 dr. \tag{6}$$

The total energy of the distribution in the shell is

$$\left[e^{-\lambda/2} \cdot \frac{\partial}{\partial r} (e^{\nu/2})\, 4\pi r^2)\right]_a^b = 4\pi m_0 \tag{7}$$

according to a formula given by G. K. Patwardhan and P. C. Vaidya (1943). Since T_1^1 vanishes at $r = a$ and $r = b$, we find that m_1 and m_0 are connected by

$$L^{\frac{1}{2}} \cdot e^{\frac{1}{2}\int_a^b F(r)\, dr} \cdot 4\pi m_1 = 4\pi m_0. \tag{8}$$

We can make $m_1 = m_0$ only by taking

$$L = e^{-\int_a^b F(r)\, dr} \tag{9}$$

and thus the continuity of $g_{\mu\nu}$ is ensured at $r = b$, provided we take Schwarzchild's line-element for $r \geqslant b$,

$$ds^2 = -\left(1 - \frac{2m_0}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) + \left(1 - \frac{2m_0}{r}\right) dt^2. \tag{10}$$

Both Carr and Combridge envisage the case where the integral in (9) vanishes. But it is clear that this takes away the physical interest from the problem. It may be noticed that because Combridge did not introduce L in (4) he did not get the Schwarzchild line-element in the usual form for $r \geqslant b$. We consider it vital not to restrict L as Combridge has done by putting $L=1$. The continuity of $g_{\mu\nu}$ at $r = a$ gives

$$ds^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{-\int_a^b F(r)\,dr} dt^2 \tag{11}$$

for $0 \leqslant r \leqslant a$.

3. The Doppler Effect

We now see that if a pulse of light starts from a stationary source distant $r >> b$ in the region outside the shell and is received at any stationary point on the radial line* within the cavity, there is a violet-shift as given by

$$\left(1 - \frac{2m_0}{r}\right)^{-1} \frac{\lambda^2}{c^2} = e^{\int_a^b F(r)\,dr} \frac{(\lambda - \delta\lambda)^2}{c^2},$$

or

$$\frac{\delta\lambda}{\lambda} = \frac{1}{2} \int_a^b F(r)\,dr - \frac{m_0}{r}, \tag{12}$$

to the first approximation. For a shell of the mass of the sun and of the thickness of the radius of the sun, the constant positive term on the right in (12) is of the order of 10^{-6}. If the second term is neglected, the Doppler effect may be treated as constant for distant sources of light. It is also clear that if the positions of the source and the observer are interchanged, a red-shift is experienced.

Summary

The relativistic field of a thick spherical shell of matter is worthy of notice inasmuch as it provides the only known example of a Euclidean pocket in a Riemannian space-time. The boundary conditions are used to obtain correct expressions for the metric in different parts of the field and an inaccuracy of Combridge's treatment is removed. The constant of the Doppler effect which is a violet-shift, follows from the boundary conditions in an unforced manner.

References

1. Carr, A. J. .. *Proc. Lon. Math. Soc.*, 1933, **35**, Part 7, 523.
2. Combridge, J. T. .. *Phil. Mag.*, 1926, **1**, 276.
3. Patwardhan, G. K., and P. C. Vaidya .. *Bom. U.J.*, 1943, **12**, Part III, 35.

[**Note added in proof*.—It may be noted that Tolman's formula, 116·8 (*Relativity, Thermodynamics and Cosmology*, 1934, p. 289) gives the same result, so far as the pure gravitational effect is concerned, even if the observer is not on the radial line.—V. V. N. and A. P.]