A CLASSICAL LIMIT OF HEAVY HOMOGENEOUS SPHERICAL MASSES

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IN a recent investigation about highly collapsed stars with dense neutron cores having small radius (10^6 cm.) and exceedingly high density (10^{14} g. cm.$^{-3}$) F. Zwicky[1] has considered the classical treatment of proper and effective masses of material spheres. His conclusion is that, gravitation being a 'co-operative phenomenon', the classical theory cannot effectively bring out the distinction between the two kinds of mass even with the help of Einstein's familiar relation,

$$E = Mc^2 \tag{1}$$

Here M is the mass equivalent of the energy E and c is the velocity of light. The object of this note is to show that the Newtonian theory is powerful enough to bring out the distinction between proper and effective masses unambiguously. It is achieved by deriving the formula (7) from first principles. It is not at all suggested that the formula so derived should be used in preference to the corresponding relativistic formula in considerations of heavy masses. The new formula has an interest of its own as it gives an upper limit on the effective mass of a homogeneous sphere which is similar to Schwarzschild's limit. Incidentally, one finds that Volkoff's[2] solutions for a material sphere which give arbitrarily large masses and radii are either mathematical abnormalities, being without Newtonian analogues, or physically significant inasmuch as they show the superiority of general relativity over the other theory.

Consider first a homogeneous sphere of terrestrial dimensions and weight. For in this case one does not go very much astray in taking the conventional definition of homogeneity; also the difference between the proper and effective masses is negligible. If ρ is the density and a the radius the proper mass is supposed to be M:

$$M = \frac{4\pi}{3} \rho a^3 \tag{2}$$

The gravitational potential energy is

$$-\frac{3}{5} \frac{M^2}{a} G, \tag{3}$$

where G is the constant of gravitation and hence on using (1) the effective mass may be stated as M′:

$$M' = M - \frac{3}{5}\frac{M^2}{a}\frac{G}{c^2}. \tag{4}$$

As Zwicky has pointed out the formula (4) is inaccurate and very misleading for large condensed masses such as the supernovæ. The inaccuracy arises through ignoring the distribution of the mass equivalent of the gravitational potential energy the magnitude of which, in its turn, is dependent upon the nature of distribution itself.

The difficulty envisaged by Zwicky can be surmounted by discussing the whole question from a more fundamental standpoint. We will discuss the case of a homogeneous sphere although the same argument can be pursued *mutatis mutandis* for a more complicated distribution. In an idealized case one may consider a flat space-time populated by particles of atomic dimensions each of the same size and mass m. Each particle is supposed to be outside the spheres of influence of all others so that it satisfies Newton's first law. To avoid the complication of mass varying with velocity all the particles may be supposed to be at rest. The mass of a particle is then the pull of the rest of the universe on it and it is defined as its proper mass for that state. Let a homogeneous sphere of radius r be built up in the same space-time frame so that the number of particles per unit volume is constant, being n. Obviously it will not be right to say that the effective density or mass per unit volume of the sphere is mn which is really the proper density. Let m' be the average effective mass of a particle in the sphere of radius r. Similarly let $m' + \delta m'$ be the average effective mass when the spehre is of radius $r + \delta r$. In building up the shell of thickness δr the gain in proper mass is

$$4\,\pi y^2 \delta y m n,$$

while the loss due to the gravitational energy* is

$$\frac{4\,\pi\, r^2\, m' n\, G}{3\, c^2}\, 4\,\pi\, r^2\, \delta\, r\, mn$$

* It is important to recognise that this loss is

$$\frac{4\pi r^2\, m' n\, G}{3c^2} \cdot 4\,\pi\, r^2\, \delta\, r\, mn \tag{A}$$

and not

$$\frac{4\,\pi\, r^2\, mn\, G}{3c^2} \cdot 4\,\pi\, r^2\, \delta\, r\, mn \tag{B}$$

This distinction is in fact the crucial point of the present paper. The particles, $4\pi r^2\,\delta r n$ in number are brought from infinity to the surface of a sphere of radius r. The sphere has the proper mass $4\pi\, r^2 mn/3$ and the effective mass $4\pi r^2 m'n/3$. In calculating the gravitational potential of the

Hence we have

$$\frac{4\pi}{3}(r+\delta r)^3(m'+\delta m')\,n=\frac{4\pi}{3}r^3 m'n+4\pi r^2\,\delta r\,mn-\frac{4\pi}{3}\frac{r^2}{c^2}m'n\,\mathrm{G}\cdot 4\pi r^2\,\delta r\,mn,$$

or

$$4\pi r^2\,\delta r\,m'n+\frac{4\pi r^3}{3}\,\delta m'n=4\pi r^2\,\delta r\,mn-\frac{4\pi}{3}\frac{r^2}{c^2}m'n\,\mathrm{G}\cdot 4\pi r^2\,\delta r\,mn,$$

so that

$$\frac{r}{3}\frac{dm'}{dr}=m-m'-mm'nk\,r^2, \tag{5}$$

where

$$\mathrm{K}=\frac{4\pi\mathrm{G}}{3c^2}\,. \tag{6}$$

Solving (5) we get

$$m'=m-\frac{1}{r^3}e^{-\frac{3k\,mn}{2}r^2}\int_0^r 3k\,m^2 r^4 n\,e^{\frac{3k}{2}mn\,r^2}\,dr \tag{7}$$

If we put

$$\frac{3k\,mn}{2}=\lambda,$$

$$m'=m-\frac{2\lambda m}{r^3}\left[\frac{r^5}{5}-\frac{2}{35}\lambda r^7+\frac{4}{315}\lambda^2 r^9\dots\right]. \tag{8}$$

With

$$\mathrm{G}=6\cdot 66\times 10^{-8}\ \mathrm{g.}^{-1}\ \mathrm{cm.}^3\ \mathrm{sec.}^{-2}$$

and

$$c=3\times 10^{10}\ \mathrm{cm.\ sec.}^{-1}$$

$$\mathrm{K}=3\cdot 1\times 10^{-28}\ \mathrm{g.}^{-1}\ \mathrm{cm.} \tag{9}$$

Hence even if a very high value is chosen for the density mn say,

$$mn=10^{14}\ \mathrm{g.cm.}^{-3} \tag{10}$$

we find

$$\lambda=4\cdot 6\times 10^{-14}\ \mathrm{cm.}^{-2} \tag{11}$$

The successive terms of the series in (8) thus fall off very rapidly. Denoting the effective density by ρ' and the proper density by ρ, we have

$$m'n\equiv\rho'=\rho-\frac{3\mathrm{K}}{r^3}\rho^2\left[\frac{r^5}{5}-\frac{2}{35}(3\,\mathrm{K}\rho)\,r^7+\frac{4}{315}(3\mathrm{K}\rho)^2\,r^9+\cdots\right] \tag{12}$$

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sphere at its surface it is the latter that must be used. The proper mass has no meaning for dynamical or kinematical purposes: it merely represents the sum of the masses of the constituent particles when their mutual interactions are neglected, the gravitational mass of a sphere is what is called the effective mass and hence we have to choose (A) in place of (B). The distinction between effective and proper masses, which arises through interaction, could not be logically maintained if (B) were the correct expression.

The relation between the effective mass M′ and the proper mass M of a homogeneous sphere of radius r follows from (12):

$$M' = M - \frac{3M^2G}{r^6c^2}\left[\frac{r^5}{5} - \frac{2}{35}\left(\frac{3\,M}{r^3}\frac{G}{c^2}\right)r^7 + \frac{4}{315}\left(\frac{3\,M}{r^3}\frac{G}{c^2}\right)^2 r^9 \ldots\right]$$

or

$$M' = M - \frac{3\,M^2G}{5\,c^2r} + \frac{18}{35}\frac{M^3G^2}{c^4r^2} - \frac{12}{35}\frac{M^4G^3}{c^6r^3}\ldots \tag{13}$$

For the sake of clarity it may be stated that

$$M' = \frac{4\pi}{3}\rho' r^3,\ M = \frac{4\,\pi}{3}\rho r^3. \tag{14}$$

Although ρ' is a function of r its definition allows us to connect it with M′ in the same manner as ρ with M.

For a given proper density ρ we can determine from (7) the maximum value of the effective mass M′. When $\frac{dM'}{dr} = 0$,

$$r^3\frac{d\rho'}{dr} + 3\,r^2\rho' = 0$$

or

$$\frac{r}{3}\frac{dm'}{dr} + m' = 0 \tag{15}$$

Therefore, from (5), we get

$$\frac{4\,\pi G r^2\, m' n}{3\,c^2} = 1 \tag{16}$$

or

$$\frac{M'G}{c^2} = r \tag{17}$$

The Schwarzschild limit is given by

$$\frac{M'G}{c^2} = 2\,r, \tag{18}$$

which may be compared to (17). The classical limit has numerical consequences of the same order as the other.

When Liapounoff[3] obtained his celebrated result that a gravitating homogeneous liquid assumes a spherical form in free space, the gravitational energy being a minimum for the sphere, this distinction between proper and effective masses was unknown. There is reason therefore to expect that his result is not valid for large masses. Bodies of the massiveness of the nebulæ (10^{10} ⊙ or 10^{43} g.) are not at all spherical but largely flat. One is tempted to link this empirical fact with the classical result that a homogeneous sphere which is being built up with proper density ρ has an effective mass increasing with radius till (17) or its equivalent,

$$\rho' = \frac{3}{4\pi}\left(\frac{c^2}{G}\right)^3 \frac{1}{M'^2} \tag{19}$$

is reached after which the effective mass diminishes with radius. If the spherical form can be preserved only at the cost of effective mass when the critical stage in the building-up process is reached it is not so surprising that the heaviest objects of the universe, the nebulæ, appear as flat bodies with central cores. It is interesting to note from (19) that for M′ of the order of an average nebular mass, ρ' is of the order of unity.

Summary

It is shown that the classical theory effectively brings out the distinction between the effective and proper masses of a homogeneous sphere. The formula derived to clarify this point leads to an upper limit on the effective mass of a sphere which is analogous to Schwarzschild's. The bearing of the limit on the flatness of nebulæ is pointed out.

REFERENCES

1. Zwicky, F. .. *Phys. Rev.*, 1939, **55**, 729.
2. Volkoff, G. M. .. *Ibid.*, 1939, **55**, 413.
3. Appell .. *Mechanique Rationelle, tome quatrieme*, 1921, 37.