THE GRAVITATIONAL EQUATIONS OF MOTION IN RELATIVITY

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§1. IN a recent paper, which will here be referred to as E. I. H., Einstein, Infeld and Hoffmann[1] have given a new relativity theory of gravitation. It marks an important departure from the well-known general theory which the authors describe as classical relativity. The essence of the new theory lies in its possession of the three ideal features which were the desiderata of the earlier theory. Einstein[2,3,4] has stated them in a number of publications during the last six years or so. They may be expressed briefly as follows : (A) Field equations should give both matter and motion, (B) the geodesic postulate[5] being redundant should have no place in the field theory, (C) the duality of the field energy and matter[4,6] being unsatisfactory the problem of gravitating matter should be solved without the use of an extraneous energy-momentum tensor such as $T_{\mu\nu}$. The three ideas are interconnected. It is essential to understand them if the significance of the results arrived at in this paper is to be grasped.

A logical working out of these ideas which covers the entire domain of phenomena of general relativity is a definite advance over the latter theory inasmuch as a greater economy and generality are achieved in the choice of the number and nature of the basic postulates and hypotheses. The solution of the problem of n bodies given in E. I. H. is such a logical working out of the new ideas. Thus the gravitational equations of motion are obtained purely from the field equations,

$$G_{\mu\nu} = 0, \tag{1}$$

the solution being subject to the condition that the space-time frame is flat at infinity. This method must be distinguished from the earlier and unsatisfactory attempts of de Sitter, Levi Civita and Eddington and Clark[7] who had to use not only the more complicated equations,

$$G_{\mu\nu} - \tfrac{1}{2} G g_{\mu\nu} = - k T_{\mu\nu}, \tag{2}$$

but the equations of geodesics also, *viz.*,

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0. \tag{3}$$

(The equations so far written appear in the usual notation and therefore no attempt will be made to define the various symbols.) The latter procedure is in contravention of Einstein's ideas and it introduces the difficulties of infinite self-energy which are quite common in the quantum theories of fundamental particles.

§2. The problem[5] that we take up is this. Following the procedure of E. I. H. one gets as a solution of (1),

$$g_{\mu\nu} = g_{\mu\nu}\,(m_1, m_2, \ldots m_n) \tag{4}$$

and the $3n$ equations of motion of the particles are obtained, those of m_1 being, say,

$$f^m\,(m_1, m_2, \ldots m_n) = 0 \qquad m = 1, 2, 3. \tag{5}$$

For the line-element given by (4) we have the equations of geodesics which with the time co-ordinate τ as the independent variable can be cast into the form

$$g^m\,(m_1, m_2, \ldots m_n) = 0 \qquad m = 1, 2, 3. \tag{6}$$

If we put $m_1 = 0$ in (5)

$$f^m\,(0, m_2, \ldots m_n) = 0 \tag{7}$$

are the equations of a particle of zero mass in the field of $n-1$ particles. If, however, the geodesic postulate were valid the required equations would be

$$g^m\,(0, m_2, \ldots m_n) = 0. \tag{8}$$

In general the two sets of equations, (7) and (8), would not be identical since the geodesic postulate has been weeded out as extraneous and unnecessary in the treatment of E.I.H. It is interesting to find out therefore at what stage the new equations of motion deviate from the geodesic postulate. When the necessary calculations are carried out we discover the surprising result that the equations (7) and (8) are identical in form and content to the second order of masses, that is, as far as the right-hand side of (5) is computed in E.I.H., in the two-body problem. Thus the geodesic postulate is found to be consistent with the equations of motion of the new theory at least over the first two phases of approximation. What will happen at higher approximations cannot be foretold without doing very lengthy calculations. In what follows the consistency will be established and an explanation will also be provided how it arises.

§3. Consider* the flat Galilean space given by the matrices $\eta_{\mu\nu}$ and $\eta_{\mu\nu}$ whose principal diagonals alone have non-zero members, being 1, –1,

* A full exposition of E.I.H. will be found in the author's paper referred to as (6).

$-1, -1$ in order. x° is the time co-ordinate and x^1, x^2, x^3 are the space co-ordinates. Throughout the rest of the paper the Greek indices such as μ and ν refer to both space and time and run over 0, 1, 2, 3 while the Latin indices such as m and n take on only the spatial values 1, 2, 3. The dummy-suffix convention is resorted to wherever possible to indicate summation. Thus for the Galilean space

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu = (dx^\circ)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{9}$$

Since the velocity of light is taken as unity in this it is convenient to use an auxiliary time co-ordinate τ so that

$$\frac{\partial\phi}{\partial x^\circ} = \lambda \frac{\partial\phi}{\partial\tau}, \tag{10}$$

where ϕ is a differentiable function of x^μ and λ is a small parameter. This makes $\partial\phi/\partial\tau$ of the same order as $\partial\phi/\partial x^m$.

The line-element of Einstein's[1] solution of the problem of two bodies of masses m_1 and m_2 runs as follows :

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu\, dx^\nu \tag{11}$$

where,

$$h_{oo} = \lambda^2 \left[-\frac{2m_1}{r_1} - \frac{2m_2}{r_2} \right] + \lambda^4 \left[-m_1 \frac{\partial^2 r_1}{\partial\tau^2} - m_2 \frac{\partial^2 r_2}{\partial\tau^2} + 2\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right)^2 + \left(\tfrac{3}{2}\,\dot\eta^s\dot\eta^s - \frac{m^2}{r_{12}}\right)\left(-\frac{2m_1}{r_1}\right) + \left(\tfrac{3}{2}\,\dot\xi^s\dot\xi^s - \frac{m_1}{r_{12}}\right)\left(-\frac{2m_2}{r_2}\right)\right] + 0\,(\lambda^6), \tag{12}$$

$$h_{on} = \lambda^3 \left(\frac{4m_1}{r_1}\,\dot\eta^n + \frac{4m_2}{r_2}\,\dot\xi^n\right) + 0\,(\lambda^5), \tag{13}$$

$$h_{mn} = -\lambda^2\, \delta_{mn} \left(\frac{2m_1}{r_1} + \frac{2m_2}{r_2}\right) + 0\,(\lambda^4). \tag{14}$$

In the above expressions δ_{mn} stands for the Kronecker delta so that

$$\left.\begin{aligned} \delta_{mn} &= 0,\ m \neq n \\ &= 1,\ m = n. \end{aligned}\right\} \tag{15}$$

Also

$$\left.\begin{aligned} r_{12}^2 &= (\eta^m - \xi^m)(\eta^m - \xi^m), \\ r_1^2 &= (x^m - \eta^m)(x^m - \eta^m), \\ r^2_2 &= (x^m - \xi^m)(x^m - \xi^m), \end{aligned}\right\} \tag{16}$$

η^m, ξ^m being the spatial co-ordinates of m_1 and m_2 respectively at time τ. An overhead dot denotes a differentiation with regard to τ. The metric (11) satisfies the field equations (1) only if the six equations of motion for m_1 and m_2 are satisfied. Those for m_1 run as follows :†

$$\ddot{\eta}^m - m_2 \frac{\partial (1/r_{12})}{\partial \eta^m} = m_2 \lambda^2 \Big\{ \Big[\dot{\eta}^s \dot{\eta}^s + \tfrac{3}{2} \dot{\xi}^s \dot{\xi}^s - 4 \dot{\eta}^s \dot{\xi}^s - \frac{4m_2}{r_{12}} - \frac{5m_1}{r_{12}} \Big] \frac{\partial}{\partial \eta^m} (1/r_{12})$$

$$+ [4 \dot{\eta}^s (\dot{\xi}^m - \dot{\eta}^m) + 3 \dot{\eta}^m \dot{\xi}^s - 4 \dot{\xi}^m \dot{\xi}^s] \frac{\partial}{\partial \eta^s} (1/r_{12})$$

$$+ \tfrac{1}{2} \frac{\partial^3 r_{12}}{\partial \eta^m \partial \eta^s \partial \eta^t} \dot{\xi}^s \dot{\xi}^t \Big\} \qquad m = 1, 2, 3. \tag{17}$$

The equations of m_2 can be similarly written down by interchanging the mass-constants as also the co-ordinates. The constant λ^2 on the right-hand side may be absorbed either in m_1 and m_2 which are just constants of integration or by reverting to x° as the independent variable in place of τ.

The equations of a geodesic are given by

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0. \tag{18}$$

Taking τ as the independent variable we restate (18) as

$$\lambda^2 \ddot{x}^m + \lambda^2 \dot{x}^m (d^2\tau/ds^2)/(d\tau/ds)^2 + \Gamma^m_{00} + 2\lambda \dot{x}^n \Gamma^m_{n0} + \lambda^2 \dot{x}^p \dot{x}^q \Gamma^m_{pq} = 0. \tag{19}$$

The three equations (19) together with the equation supplied by the metric, *viz.*,

$$\lambda^2 (ds/d\tau)^2 = g_{oo} + 2 g_{no} \lambda \dot{x}^n + \lambda^2 g_{pq} \dot{x}^p \dot{x}^q \tag{20}$$

are equivalent to the four equations (18). From (19) and (20) one concludes that

$$0 = \lambda^2 \ddot{x}^m + \Gamma^m_{oo} + 2\lambda \dot{x}^n \Gamma^m_{no} + \lambda^2 \dot{x}^p \dot{x}^q \Gamma^m_{pq}$$

$$- \lambda^2 \dot{x}^m \cdot \tfrac{1}{2} \frac{d}{d\tau} \log (g_{oo} + 2 g_{no} \lambda \dot{x}^n + \lambda^2 g_{pq} \dot{x}^p \dot{x}^q). \tag{21}$$

For the evaluation of the various terms in the last equation it is enough to note that

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \tag{22}$$

where

$$h^{oo} = \lambda^2 \left(\frac{2m_1}{r_1} + \frac{2m_2}{r_2} \right) + 0\,(\lambda^4), \tag{23}$$

$$h^{on} = \lambda^3 \left(\frac{4m_1}{r_1} \dot{\eta}^n + \frac{4m_2}{r_2} \dot{\xi}^n \right) + 0\,(\lambda^5), \tag{24}$$

$$h^{mn} = \delta_{mn} \lambda^2 \left(\frac{2m_1}{r_1} + \frac{2m_2}{r_2} \right) + 0\,(\lambda^4). \tag{25}$$

† The notation is the same as that in Reference (6), the overhead indices being dropped for suffixes. (17) is the same as (11·67) of Reference (6).

Thus

$$\Gamma^m_{00} = \tfrac{1}{2} g^{m\mu} \left[\frac{\partial h_{\mu 0}}{\partial x^0} + \frac{\partial h_{\mu 0}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^\mu}\right]$$

$$= -\lambda^2 \frac{\partial}{\partial x^m}\left[-\frac{m_1}{r_1} - \frac{m_2}{r_2}\right]$$

$$+ \lambda^4 \Big\{ -\frac{\partial}{\partial \tau}\left[\frac{4m_1}{r_1}\dot{\eta}^m + \frac{4m_2}{r_2}\dot{\xi}^m\right] + 2\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)\frac{\partial}{\partial x^m}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)$$

$$+ \tfrac{1}{2}\frac{\partial}{\partial x^m}\Big[-m_1\frac{\partial^2 r_1}{\partial \tau^2} - m_2\frac{\partial^2 r_2}{\partial \tau^2} + 2\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)^2$$

$$- \frac{2m_1}{r_1}\left(\tfrac{3}{2}\dot{\eta}^s\dot{\eta}^s - \frac{m_2}{r_{12}}\right) - \frac{2m_2}{r_2}\left(\tfrac{3}{2}\dot{\xi}^s\dot{\xi}^s - \frac{m_1}{r_{12}}\right)\Big]\Big\}$$

$$+ O(\lambda^6); \qquad (26)$$

$$2\lambda\dot{x}^n\,\Gamma^m_{n0} = 2\lambda\dot{x}^n\cdot\tfrac{1}{2}g^{m\mu}\left(\frac{\partial h_{n\mu}}{\partial x^0} + \frac{\partial h_{\mu 0}}{\partial x^n} - \frac{\partial h_{n0}}{\partial x^\mu}\right)$$

$$= -\lambda^2\dot{x}^n\frac{\partial}{\partial \tau}h_{mn} - \lambda\dot{x}^n\frac{\partial h_{m0}}{\partial x^n} + \lambda\dot{x}^n\frac{\partial h_{n0}}{\partial x^m}$$

$$= -2\lambda^4\dot{x}^m\frac{\partial}{\partial \tau}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right) - 4\lambda^4\dot{x}^s\frac{\partial}{\partial x^s}\left(\frac{m_1}{r_1}\dot{\eta}^m + \frac{m_2}{r_2}\dot{\xi}^m\right)$$

$$+ 4\lambda^4\dot{x}^s\frac{\partial}{\partial x^m}\left(\frac{m_1}{r_1}\dot{\eta}^s + \frac{m_2}{r_2}\dot{\xi}^s\right) + O(\lambda^6); \qquad (27)$$

$$\lambda^2\dot{x}^p\dot{x}^q\,\Gamma^m_{pq} = \lambda^2\dot{x}^p\dot{x}^q\cdot\tfrac{1}{2}g^{m\mu}\left[\frac{\partial h_{p\mu}}{\partial x^q} + \frac{\partial h_{q\mu}}{\partial x^p} - \frac{\partial h_{pq}}{\partial x^\mu}\right]$$

$$= -2\lambda^4\dot{x}^m\dot{x}^q\frac{\partial}{\partial x^q}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right) - \lambda^4\dot{x}^s\dot{x}^s\frac{\partial}{\partial x^m}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right) + O(\lambda^6). \qquad (28)$$

$$\tfrac{1}{2}\lambda^2\dot{x}^m\frac{d}{d\tau}\log\left[g_{00} + 2g_{n0}\lambda\dot{x}^n + \lambda^2\dot{x}^p\dot{x}^q g_{pq}\right]$$

$$= \tfrac{1}{2}\lambda^2\dot{x}^m\frac{d}{d\tau}\log\left(1 + h_{00} - \lambda^2\dot{x}^p\dot{x}^p\right) + O(\lambda^6)$$

$$= \tfrac{1}{2}\lambda^4\dot{x}^m\frac{d}{d\tau}\left[-\frac{2m_1}{r_1} - \frac{2m_2}{r_2} - \dot{x}^p\dot{x}^p\right] + O(\lambda^6). \qquad (29)$$

Making the necessary substitutions for the various terms in (19) we get

$$\lambda^2\left[\ddot{x}^m - \frac{\partial}{\partial x^m}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)\right]$$

$$+ \lambda^4\Big\{\tfrac{1}{2}\frac{\partial}{\partial x^m}\left(m_1\frac{\partial^2 r_1}{\partial \tau^2} + m_2\frac{\partial^2 r_2}{\partial \tau^2}\right) - 2\dot{x}^m\frac{\partial}{\partial \tau}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right) - \dot{x}^m\frac{d}{d\tau}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)$$

$$\dot{x}^m\dot{x}^s\dot{x}^s + \dot{x}^s\left[4\dot{\eta}^m\frac{\partial}{\partial x^s}\left(\frac{m_1}{r_1}\right) + 4\dot{\xi}^m\frac{\partial}{\partial x^s}\left(\frac{m_2}{r_2}\right) - 2\dot{x}^m\frac{\partial}{\partial x^s}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)\right]$$

$$+ 4\,\dot\eta^m \frac{\partial}{\partial\tau}\left(\frac{m_1}{r_1}\right) + \frac{4\,m_1}{r_1}\,\ddot\eta^m + 4\,\dot\xi^m \frac{\partial}{\partial\tau}\left(\frac{m_2}{r_2}\right) + 4\,\frac{m_2}{r_2}\,\ddot\xi^m$$

$$+ \frac{\partial}{\partial x^m}\left[-2\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)^2 + \frac{m_1}{r_1}\left(\tfrac{3}{2}\,\dot\eta^s\dot\eta^s - \frac{m_2}{r_{12}}\right) + \frac{m_2}{r_2}\left(\tfrac{3}{2}\,\dot\xi^s\dot\xi^s - \frac{m_1}{r_{12}}\right)\right]$$

$$- 4\,\dot x^s \frac{\partial}{\partial x^m}\left(\frac{m_1}{r_1}\,\dot\eta^s + \frac{m_2}{r_2}\,\dot\xi^s\right) + \dot x^s \dot x^s \frac{\partial}{\partial x^m}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)\Bigg\} + 0\,(\lambda^6). \tag{30}$$

In simplifying (30) we note that

$$\frac{\partial r_1}{\partial\tau} = -\frac{\partial r_1}{\partial x^s}\,\dot\eta^s \tag{31}$$

$$\frac{\partial^2 r_1}{\partial\tau^2} = -\frac{\partial r_1}{\partial x^s}\,\ddot\eta^s + \frac{\partial^2 r_1}{\partial x^p\,\partial x^q}\,\dot\eta^p\dot\eta^q \tag{32}$$

$$\frac{\partial^2 r_2}{\partial\tau^2} = -\frac{\partial r_2}{\partial x^s}\,\ddot\xi^s + \frac{\partial^2 r_2}{\partial x^p\,\partial x^q}\,\dot\xi^p\dot\xi^q \tag{33}$$

$$\frac{\partial}{\partial\tau}\left(\frac{m_1}{r_1}\right) = \dot\eta^s \frac{\partial}{\partial\eta^s}\left(\frac{m_1}{r_1}\right) = -\,\dot\eta^s \frac{\partial}{\partial x^s}\left(\frac{m_1}{r_1}\right) \tag{34}$$

$$\frac{\partial}{\partial\tau}\left(\frac{m_2}{r_2}\right) = -\dot\xi^s \frac{\partial}{\partial x^s}\left(\frac{m_2}{r_2}\right) \tag{35}$$

$$\frac{d}{d\tau}\left(\frac{m_1}{r_1}\right) = (\dot x^s - \dot\eta^s)\frac{\partial}{\partial x^s}\left(\frac{m_1}{r_1}\right) \tag{36}$$

$$\frac{d}{d\tau}\left(\frac{m_2}{r_2}\right) = (\dot x^s - \dot\xi^s)\frac{\partial}{\partial x^s}\left(\frac{m_2}{r_2}\right) \tag{37}$$

$$\ddot x^s = \frac{\partial}{\partial x^s}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right) + 0\,(\lambda^2) \tag{38}$$

$$\ddot\eta^s = \frac{\partial}{\partial x^s}\left(\frac{m_2}{r_{12}}\right) + 0\,(\lambda^2) \tag{39}$$

$$\ddot\xi^s = \frac{\partial}{\partial x^s}\left(\frac{m_1}{r_{12}}\right) + 0\,(\lambda^2). \tag{40}$$

So, on neglecting terms of the order of λ^6 we get from (30)

$$\lambda^2\left[\ddot x^m - \frac{\partial}{\partial x^m}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)\right]$$

$$= \lambda^4\left\{\left[\dot x^s\dot x^s - 4\,\dot x^s\dot\eta^s + \tfrac{3}{2}\,\dot\eta^s\dot\eta^s - \frac{4\,m_1}{r_1} - \frac{4\,m_2}{r_2} - \frac{2\,m_2}{r_{12}}\right]\frac{\partial}{\partial x^m}\left(\frac{m_1}{r_1}\right)\right.$$

$$+ (\dot x^s\dot x^s - 4\,\dot x^s\dot\xi^s + \tfrac{3}{2}\,\dot\xi^s\dot\xi^s - \frac{4\,m_1}{r_1} - \frac{4\,m_2}{r_2} - \frac{2\,m_1}{r_{12}}\Big]\frac{\partial}{\partial x^m}\left(\frac{m_2}{r_2}\right)$$

$$+ [4\,\dot x^s(\dot\eta^m - \dot x^m) + 3\,\dot x^m\dot\eta^s - 4\,\dot\eta^m\dot\eta^s]\frac{\partial}{\partial x^s}\left(\frac{m_1}{r_1}\right)$$

$$+ [4\, \dot{x}^s (\dot{\xi}^m - \dot{x}^m) + 3\, \dot{x}^m \dot{\xi}^s - 4\, \dot{\xi}^m \dot{\xi}^s] \frac{\partial}{\partial x^s}\left(\frac{m_2}{r_2}\right)$$

$$+ \frac{4\, m_1}{r_1} \frac{\partial}{\partial x^m}\left(\frac{m_2}{r_{12}}\right) + \frac{4\, m_2}{r_2} \frac{\partial}{\partial x^m}\left(\frac{m_1}{r_{12}}\right) - \tfrac{1}{2}\, m_1 \left(\frac{\partial r_1}{\partial x^s}\right) \frac{\partial}{\partial x^s}\left(\frac{m_2}{r_{12}}\right)$$

$$- \tfrac{1}{2}\, m_2 \left(\frac{\partial r_2}{\partial x^s}\right) \frac{\partial}{\partial x^s}\left(\frac{m_1}{r_{12}}\right)$$

$$+ \tfrac{1}{2}\, m_1 \frac{\partial^3 r_1}{\partial x^m\, \partial x^p\, \partial x^q}\, \dot{\eta}^p \dot{\eta}^q + \tfrac{1}{2}\, m_2 \frac{\partial^3 r_2}{\partial x^m\, \partial x^p\, \partial x^q}\, \dot{\xi}^p \dot{\xi}^q \Big\}. \tag{41}$$

If by changing the density of the first body we can make approximately,

$$m_1 = 0, \; m_1/r_1 = 0, \;\; \partial/\partial x^m\, (m_1/r_1) = 0 \tag{42}$$

we have

$$r_2 = r_{12} \tag{43}$$

and (41) reduces to

$$\ddot{x}^m - \frac{\partial}{\partial x^m}\left(\frac{m_2}{r_{12}}\right) = \lambda^4\, m_2 \left\{\left[\dot{x}^s \dot{x}^s + \tfrac{3}{2}\, \dot{\xi}^s \dot{\xi}^s - 4\, \dot{x}^s \dot{\xi}^s - \frac{4\, m_2}{r_2}\right] \frac{\partial}{\partial x^m}\left(\frac{1}{r_{12}}\right)\right.$$

$$+ [4\, \dot{x}^s (\dot{\xi}^m - \dot{x}^m) + 3\, \dot{x}^m \dot{\xi}^s - 4\, \dot{\xi}^m \dot{\xi}^s] \frac{\partial}{\partial x^s}\left(\frac{1}{r_{12}}\right)$$

$$\left. + \tfrac{1}{2} \frac{\partial^3 r_{12}}{\partial x^m\, \partial x^p\, \partial x^q}\, \dot{\xi}^p \dot{\xi}^q \right\}. \tag{44}$$

This is precisely what (17) reduces to when $m_1 = 0$. Hence the consistency of the new method with the geodesic postulate is established as far as this test goes. We could have tested the consistency further by comparing (41) itself with the form of equations for m_3 in the field of m_1 and m_2, as given by the new method, when m_3 is negligible. E.I.H. does not given the three-body equations and we have therefore to be content with the comparison afforded by the two-body problem. As far as our calculations of the three-body problem go (41) is found to be consistent with the corresponding form of Einstein's equations of motion.

§ 4. We will now proceed to consider how it is possible to reconcile the equations of motion obtained from

$$G_{\mu\nu} = 0$$

with the geodesic postulate. The clue is provided by the identities,

$$(G^{\mu\nu} = \tfrac{1}{2}\, G\, g^{\mu\nu})_{,\nu} = 0. \tag{45}$$

Certain functions $\overset{k}{C}_m (\tau)$ which enter the method of E.I.H. have the property that when all $\overset{k}{C}_m \to 0$, $G_{\mu\nu}$ also tends to zero. If $\overset{k}{C}_m \neq 0$ for one value of m or k, $G_{\mu\nu} \neq 0$ also. Now

$$\overset{k}{C}_m = 0 \qquad m = 1, 2, 3 \tag{46}$$

are the equations of motion of the kth particle of the system in Einstein's method. Let us see how $G_{\mu\nu}$ can be made to tend to zero so that when the mass of the kth particle is zero (46) is consistent with the equations of geodesics. When $G_{\mu\nu} \neq 0$ its value can be expressed in the normal form:

$$G_{\mu\nu} = \rho_0 u_\mu u_\nu - \rho_1 v_\mu v_\nu - \rho_2 w_\mu w_\nu - \rho_3 x_\mu x_\nu, \quad (47)$$

where u_μ is a timelike unit vector and v_μ, w_μ and x_μ are spacelike unit vectors. Also

$$G = \rho_0 + \rho_1 + \rho_2 + \rho_3 \quad (48)$$

and

$$g_{\mu\nu} = u_\mu u_\nu - (v_\mu v_\nu + w_\mu w_\nu + x_\mu x_\nu). \quad (49)$$

The identities (45) now become

$$\{(\rho_0 - \rho_1 - \rho_2 - \rho_3)\, u^\mu u^\nu + (\rho_0 - \rho_1 + \rho_2 + \rho_3)\, v^\mu v^\nu + (\rho_0 + \rho_1 - \rho_2 + \rho_3)\, w^\mu w^\nu + (\rho_0 + \rho_1 + \rho_2 - \rho_3)\, x^\mu x^\nu)\}_\nu = 0. \quad (50)$$

$$G_{\mu\nu} \to 0 \qquad \text{as } \rho_\mu \to 0. \quad (51)$$

Hence (50) will reduce to the equations of a geodesic,

$$(u^\nu)_\nu\, u^\nu = 0$$

when along with $\rho_\mu \to 0$ we have also

$$\frac{\{(\rho_0 - \rho_1 + \rho_2 + \rho_3) v^\mu v^\nu + (\rho_0 + \rho_1 - \rho_2 + \rho_3) w^\mu w^\nu + (\rho_0 + \rho_1 + \rho_2 - \rho_3) x^\mu x^\nu\}_\nu}{\{(\rho_0 - \rho_1 - \rho_2 - \rho_3) u^\nu\}_\nu} \to 0. \quad (52)$$

It is not necessary that the conditions (52) will always be satisfied. It is the special virtue of Einstein's method that (52) is satisfied at least upto $0\,(\lambda^6)$. Herein lies an explanation of the consistency with the geodesic postulate as discovered by us. Incidentally it is of some astronomical interest to observe that if the bodies of masses m_1 and m_2 are looked upon as the earth and the sun the equation (41) obtained by us gives the motion of the moon in the field of the others.

Summary

We derive an equation (41) for the geodesics in the field of two bodies as given by Einstein, Infeld and Hoffmann. It is shown that when $m_1 = 0$ the equation reduces to that of the two-body problem (17) obtained without any reference to the geodesic postulate. This is the main result of the paper. Although Einstein's new method of deriving gravitational equations of motion is based on the field equations only we show how it can be consistent with the geodesic postulate if the field tensor tends to zero in a particular manner.

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