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Simplicity of stable principal sheaves

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Abstract

Let M be a compact connected Kähler manifold, and let G be a connected complex reductive linear algebraic group. We prove that a principal G-sheaf on M admits an admissible Einstein–Hermitian connection if and only if the principal G-sheaf is polystable. Using this it is shown that the holomorphic sections of the adjoint vector bundle of a stable principal G-sheaf on M are given by the center of the Lie algebra of G. The Bogomolov inequality is shown to be valid for polystable principal G-sheaves.

1. *Introduction*

Stable vector bundles on curves were introduced by Mumford in the context of geometric invariant theory [6], and stable vector bundles on higher-dimensional varieties were introduced by Takemoto [12]. On the other hand, the notion of an Einstein–Hermitian connection originated in physics. Hitchin and Kobayashi made a very precise conjecture connecting these two notions, which is known as the Hitchin–Kobayashi correspondence; in the special case of degree zero vector bundles over compact Riemann surfaces, their conjecture is an earlier theorem due to Narasimhan and Seshadri [7]. The Hitchin–Kobayashi correspondence was first proved by Donaldson for complex projective varieties [3] and, subsequently, Uhlenbeck and Yau extended it to compact Kähler manifolds [13]. We will now very briefly recall these results and their generalizations.

A holomorphic vector bundle E over a compact connected Kähler manifold M admits an Einstein–Hermitian connection if and only if E is polystable [3, 13]. More generally, a reflexive sheaf on M admits an admissible Einstein–Hermitian connection if and only if it is polystable [2]. For any connected reductive linear algebraic group G defined over $\mathbb{C}$, a holomorphic principal G-bundle E_G over M admits an Einstein–Hermitian connection if and only if E_G is polystable [1, 10].

On the other hand, the principal bundle analog of a torsion-free sheaf, which is called a principal G-sheaf, was introduced in [4].

In Theorem 3.1, we prove that a principal G-sheaf admits an admissible Einstein–Hermitian connection if and only if it is polystable.

A stable vector bundle E over M is simple; that is, any holomorphic global endomorphism of E is multiplication by a constant scalar. We prove the following generalization of it (see Proposition 3.3).

The global holomorphic sections of the adjoint vector bundle of a stable principal G-sheaf coincide with the center of the Lie algebra of G.

This last result is new even for usual principal bundles.

We also prove a Bogomolov-type inequality for polystable principal G-sheaves (see Corollary 3.2).

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2. *Preliminaries*

Let M be a compact connected Kähler manifold equipped with a Kähler form ω. The degree of a torsion-free coherent analytic sheaf F on M is defined to be

$$\mathrm{degree}(F) := \int_M c_1(F)\omega^{d-1} \in \mathbb{R},$$

where d is the complex dimension of M. For any holomorphic vector bundle F defined over a dense open subset

$$U \overset{\iota}{\hookrightarrow} M$$

with complement U^c that is a complex analytic subspace of complex codimension at least two, and such that the direct image $\iota_* F$ is a coherent analytic sheaf, we have

$$\mathrm{degree}(F) := \mathrm{degree}(\iota_* F).$$

The real number $\mathrm{degree}(F)/\mathrm{rank}(F)$ is denoted by $\mu(F)$, and it is called the *slope* of F.

Definition 2.1. By a *big open subset of* M we will mean a dense open subset U of M such that the complement U^c is a complex analytic subspace of M of complex codimension at least two.

We recall that F is called *stable* if $\mu(F') < \mu(F)$ for all $F' \subset F$ with $0 < \mathrm{rank}(F') < \mathrm{rank}(F)$, where $\mu(V) := \mathrm{degree}(V)/\mathrm{rank}(V)$. Under the same conditions, F is called *semistable* if $\mu(F') \leqslant \mu(F)$. A semistable sheaf is called *polystable* if it is a direct sum of stable sheaves. Therefore, a polystable sheaf is a direct sum of stable sheaves of same slope.

Let G be a connected reductive linear algebraic group defined over the field of complex numbers. The Lie algebra of G will be denoted by $\mathfrak{g}$. Let

$$\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \tag{2.1}$$

be the semisimple part of $\mathfrak{g}$. Set

$$Z := G/[G, G] \tag{2.2}$$

to be the quotient group, which is a product of copies of $\mathbb{G}_m = \mathbb{C}^*$.

A *principal G-sheaf* on M is a triple of the form (E_G, E, ψ), where

- E_G is a rational principal G-bundle over M, which means that E_G is a holomorphic principal G-bundle over some big open subset U of M;
- the holomorphic principal Z-bundle over U

$$E_Z := E_G(Z), \tag{2.3}$$

  obtained by extending the structure group of E_G using the quotient map $G \to Z$ in (2.2), extends to a holomorphic principal Z-bundle over M;
- E is a torsion-free coherent analytic sheaf on M;
- the isomorphism

$$\psi : E_G(\mathfrak{g}') \longrightarrow E|_U \tag{2.4}$$

  is a holomorphic isomorphism of vector bundles over a big open subset U over which E_G is a holomorphic principal G-bundle, where $E_G(\mathfrak{g}')$ is the vector bundle over U associated to E_G for the G-module $\mathfrak{g}'$ defined in (2.1).

Lemma 2.2. *Let (E_G, E, ψ) be a principal G-sheaf as above. The principal G-bundle E_G on U can be extended, as a principal G-bundle, to the open subset $U_E \subset M$ on which E is locally free.*

Proof. The integer $\dim_{\mathbb{C}} \mathfrak{g}'$ will be denoted by n. Therefore $\operatorname{rank}(E) = n$. Let $F_{\mathrm{GL}(\mathfrak{g}')}$ be the principal $\mathrm{GL}(\mathfrak{g}')$-bundle over U_E defined by $E|_{U_E}$. In view of the isomorphism ψ in (2.4), the subset $U \subset M$ is contained in U_E. The restriction of $F_{\mathrm{GL}(\mathfrak{g}')}$ to U will be denoted by $F'_{\mathrm{GL}(\mathfrak{g}')}$.

Let

$$\gamma : G \longrightarrow \mathrm{GL}(\mathfrak{g}') \tag{2.5}$$

be the homomorphism defined by the adjoint action. We note that the isomorphism ψ in (2.4) gives an identification of $F'_{\mathrm{GL}(\mathfrak{g}')}$ with the principal $\mathrm{GL}(\mathfrak{g}')$-bundle obtained by extending the structure group of E_G using γ. In particular, we get a reduction of the structure group of $F'_{\mathrm{GL}(\mathfrak{g}')}$ to the subgroup $\gamma(G) \subset \mathrm{GL}(\mathfrak{g}')$.

Let $\widetilde{Z} \subset G$ be the center. Let $\widetilde{Z}_0 \subset \widetilde{Z}$ be the connected component containing the identity element. Therefore $\widetilde{Z}_0$ is a product of copies of $\mathbb{G}_m = \mathbb{C}^*$. The homomorphism γ in (2.5) factors as

$$G \xrightarrow{\rho_3} G/\widetilde{Z}_0 \xrightarrow{\rho'_2} G/\widetilde{Z} \xhookrightarrow{\rho_2} \mathrm{Aut}(\mathfrak{g}') \xhookrightarrow{\rho_1} \mathrm{GL}(\mathfrak{g}'), \tag{2.6}$$

where $\mathrm{Aut}(\mathfrak{g}')$ is the group of all Lie algebra automorphisms of $\mathfrak{g}'$. We will construct the extension of E_G to U_E, step by step, using these homomorphisms.

Giving a reduction of the structure group of the principal $\mathrm{GL}(\mathfrak{g}')$-bundle $F_{\mathrm{GL}(\mathfrak{g}')}$ to the subgroup $\mathrm{Aut}(\mathfrak{g}')$ in (2.6) is equivalent to constructing a holomorphic homomorphism of vector bundles

$$\beta : E|_{U_E} \otimes E|_{U_E} \longrightarrow E|_{U_E}$$

that satisfies the following two conditions:

- the homomorphism β makes $E|_{U_E}$ a Lie algebra bundle over U_E, and
- for each point $x \in U_E$, the Lie algebra $(E_x, \beta(x))$ is isomorphic to $\mathfrak{g}'$.

The open subset U of U_E is big. Hence the Lie algebra bundle structure

$$E_G(\mathfrak{g}') \otimes E_G(\mathfrak{g}') \longrightarrow E_G(\mathfrak{g}')$$

of $E_G(\mathfrak{g}')$ extends uniquely to a holomorphic homomorphism

$$\beta : E|_{U_E} \otimes E|_{U_E} \longrightarrow E|_{U_E} \tag{2.7}$$

using ψ. To check that for all $x \in U_E$ the fiber $(E_x, \beta(x))$ is a Lie algebra, we note that the homomorphism of sheaves given by β satisfies both the Jacobi identity and the anti-commutativity condition because they are satisfied over the dense open subset U. This immediately implies that $(E_x, \beta(x))$ is a Lie algebra.

For any $x \in U$, the Lie algebra $E_G(\mathfrak{g}')_x$ is isomorphic to the semisimple Lie algebra $\mathfrak{g}'$, and hence the Killing form of $E_G(\mathfrak{g}')_x$ is non-degenerate. We note that the condition that the Killing form on a Lie algebra $\mathfrak{h}$ is non-degenerate is equivalent to the condition that the element in $(\bigwedge^{\mathrm{top}} \mathfrak{h}) \otimes (\bigwedge^{\mathrm{top}} \mathfrak{h})^*$ given by the Killing form on the Lie algebra $\mathfrak{h}$ is non-zero. Let

$$s \in H^0\left(U_E, \left(\bigwedge^{\mathrm{top}} E\right) \otimes \left(\bigwedge^{\mathrm{top}} E\right)^*\right)$$

be the section given by the Killing forms for the Lie algebra bundle $E|_{U_E}$. We know that $s|_U$ is nowhere vanishing. Since U is a big open subset of U_E, we conclude that the above section s does not vanish anywhere. Hence the fibers of $E|_{U_E}$ are all semisimple. Finally, by the rigidity of semisimple Lie algebras, the fibers of $E|_{U_E}$ are all isomorphic to $\mathfrak{g}'$.

Therefore, we get a reduction of the structure group of $F_{\mathrm{GL}(\mathfrak{g}')}$ to the subgroup $\mathrm{Aut}(\mathfrak{g}')$ in (2.6). The principal $\mathrm{Aut}(\mathfrak{g}')$-bundle giving the reduction of the structure group will be denoted by $F_{\mathrm{Aut}(\mathfrak{g}')}$.

The restriction of the homomorphism β in (2.7) to $U \subset U_E$ coincides with the Lie algebra bundle $E_G(\mathfrak{g}')$ using the isomorphism ψ. Hence the principal $\mathrm{Aut}(\mathfrak{g}')$-bundle obtained by extending the structure group of E_G using the homomorphism $\rho_2 \circ \rho_2' \circ \rho_3$ in (2.6) is identified with $F_{\mathrm{Aut}(\mathfrak{g}')}|_U$.

The subgroup $G/\widetilde{Z} \subset \mathrm{Aut}(\mathfrak{g}')$ is the connected component containing the identity element, and U is a dense open subset of U_E. Hence $F_{\mathrm{Aut}(\mathfrak{g}')}$ has a natural reduction of the structure group to $G/\widetilde{Z}$. This reduction of the structure group is uniquely determined by the condition that its restriction to U coincides with the principal $G/\widetilde{Z}$-bundle obtained by extending the structure group of E_G using the homomorphism $\rho_2' \circ \rho_3$ in (2.6). The principal $G/\widetilde{Z}$-bundle over U_E giving the reduction of the structure group of $F_{\mathrm{Aut}(\mathfrak{g}')}$ will be denoted by $F_{G/\widetilde{Z}}$.

The kernel of the homomorphism ρ_2' is a finite group. Since U is a big open subset of U_E, the homomorphism of fundamental groups induced by the inclusion map of U in U_E is surjective. Therefore, the principal $G/\widetilde{Z}$-bundle $F_{G/\widetilde{Z}}$ has a natural lift of the structure group to $G/\widetilde{Z}_0$. This lift is uniquely determined by the condition that its restriction to U coincides with the principal $G/\widetilde{Z}_0$-bundle obtained by extending the structure group of E_G using the homomorphism ρ_3 in (2.6). Let $F_{G/\widetilde{Z}_0}$ be the principal $G/\widetilde{Z}_0$-bundle over U_E giving the above lift of $F_{G/\widetilde{Z}}$.

Let

$$q : G \longrightarrow Z \times (G/\widetilde{Z}_0) \tag{2.8}$$

be the natural projection, where Z is the quotient group in (2.2). The homomorphism q is surjective with a finite kernel. We know that the principal Z-bundle E_Z in (2.3) extends to M. The extension of E_Z to U_E will be denoted by E_Z'. Therefore, the fiber product

$$E_Z' \times_{U_E} F_{G/\widetilde{Z}_0} \longrightarrow U_E$$

is a principal $Z \times (G/\widetilde{Z}_0)$-bundle over U_E. The principal $Z \times (G/\widetilde{Z}_0)$-bundle over U obtained by extending the structure group of E_G using the homomorphism q in (2.8) is clearly identified with $E_Z' \times_{U_E} F_{G/\widetilde{Z}_0}$. We have already noted that the homomorphism of fundamental groups corresponding to the inclusion map $U \hookrightarrow U_E$ is surjective. Hence arguing as before (see the construction of $F_{G/\widetilde{Z}_0}$ from $F_{G/\widetilde{Z}}$) it follows that the principal G-bundle E_G extends to U_E. This completes the proof of the lemma. □

REMARK 2.3. We put down a few observations regarding the above definition of a principal G-sheaf.

(1) Consider the special case where $G = \mathrm{GL}(n, \mathbb{C})$. Then we have $Z = \mathbb{C}^*$. Let E be a torsion-free coherent analytic sheaf on M, and let U be the big open subset of M over which E is locally free. The principal Z-bundle E_Z corresponds to the holomorphic line bundle $\bigwedge^n E$ over U. This line bundle over U extends to M as the holomorphic determinant line bundle $\det E$. (See [**5**, Chapter V, § 6] for the construction of the determinant line bundle of a torsion-free coherent analytic sheaf on M.) Therefore, in the special case of $G = \mathrm{GL}(n, \mathbb{C})$, the principal G-sheaves are not restrictive compared to the torsion-free coherent analytic sheaves on M.

(2) In view of the homomorphism ψ in (2.4) it follows that the vector bundle $E_G(\mathfrak{g}')$ over U extends to M as a coherent analytic sheaf. Since the adjoint vector bundle $\mathrm{ad}(E_G)$ over U is a direct sum of $E_G(\mathfrak{g}')$ with a trivial vector bundle of rank $\dim_{\mathbb{C}} Z$, we conclude that $\mathrm{ad}(E_G)$ extends to M as a coherent analytic sheaf. Therefore, the direct image $\iota_*\mathrm{ad}(E_G)$ is a coherent analytic sheaf on M, where $\iota : U \to M$ is the inclusion map; see [**11**, p. 364, Théorème 1].

(3) If M is a complex projective manifold, and the principal Z-bundle E_Z is algebraic, then from the given condition that the open subset $U \subset M$ is big it follows that E_Z extends to M as a holomorphic principal Z-bundle.

(4) The above-mentioned big open subset U is not a part of the definition of a principal G-sheaf. In other words, we do not distinguish between the two principal G-sheaves

given by (E, U, E_G, ψ) and (E, U', E'_G, ψ'), respectively, where $E_G|_{U\cap U'} = E'_G|_{U\cap U'}$ and $\psi|_{U\cap U'} = \psi'|_{U\cap U'}$. However, in view of Lemma 2.2, we may take U to be the unique largest open subset of M over which the torsion-free coherent analytic sheaf E is a vector bundle. In this sense, there is a natural choice of the big open subset U.

A principal G-sheaf (E_G, E, ψ) is called *stable* if for every triple of the form (U', Q, σ), where

- $U' \overset{\iota}{\hookrightarrow} M$ is a big open subset contained in the open subset of M over which E_G is a holomorphic principal G-bundle,
- $Q \subset G$ is a maximal proper parabolic subgroup, and
- the reduction
$$\sigma : U' \longrightarrow (E_G|_{U'})/Q \tag{2.9}$$
is a holomorphic reduction of the structure group of $E_G|_{U'}$ to the subgroup Q such that the direct image $\iota_*\sigma^*T_{\mathrm{rel}}$ is a coherent analytic sheaf on M, where T_{rel} is the relative tangent bundle for the natural projection $(E_G|_{U'})/Q \to U'$, and ι is the above inclusion map,

the inequality
$$\mathrm{degree}(\sigma^*T_{\mathrm{rel}}) > 0 \tag{2.10}$$
holds. Under the same conditions, a principle G-sheaf (E_G, E_m, ψ) is called *semistable* if $\mathrm{degree}(\sigma^*T_{\mathrm{rel}}) \geqslant 0$. (See [**4**, Corollary 5.7; **9**].)

REMARK 2.4. We put down two remarks on the above definition.

(1) If F is a coherent analytic subsheaf of a torsion-free coherent analytic sheaf E over M, then there is a big open subset $U' \overset{\iota}{\hookrightarrow} M$ such that the restriction $E' := E|_{U'}$ is locally free, and furthermore, the restriction $F' := F|_{U'}$ is a sub-bundle of E'. The double dual $(F^* \otimes (E/F))^{**}$ is a coherent analytic sheaf on M extending the vector bundle $(F')^* \otimes (E'/F')$ on U'. Hence the direct image $\iota_*((F')^* \otimes (E'/F'))$ is a coherent analytic sheaf on M [**11**, p. 364, Théorème 1]. Consequently, in the special case of $G = \mathrm{GL}(n, \mathbb{C})$, the above definitions of stability and semistability coincide with the usual definitions of stability and semistability of torsion-free coherent analytic sheaves on M.

(2) If M is a complex projective manifold, and σ^*T_{rel} is algebraic, then the condition in the above definition that $\iota_*\sigma^*T_{\mathrm{rel}}$ is a coherent analytic sheaf on M is automatically satisfied.

By a *Levi subgroup* of a parabolic subgroup $P \subset G$ we will mean a connected reductive subgroup of P with projection to the quotient $P/R_u(P)$ that is an isomorphism, where $R_u(P)$ is the unipotent radical of P.

A principal G-sheaf (E_G, E, ψ) is called *polystable* if either (E_G, E, ψ) is stable, or there is a pair $(L(P), E_{L(P)})$ satisfying the following three conditions.

- $L(P) \subset P \subset G$ is a Levi subgroup of some parabolic subgroup P of G.
- $E_{L(P)} \subset E_G|_U$ is a holomorphic reduction of the structure group to $L(P) \subset G$ over the big open subset U over which E_G is a holomorphic principal G-bundle, such that the adjoint vector bundle $\mathrm{ad}(E_{L(P)})$ extends to M as a coherent analytic sheaf.
- The principal $L(P)$-bundle $E_{L(P)}$ is stable, and furthermore, for each character χ of $L(P)$ which is trivial on the center of G, the line bundle $E_{L(P)}(\chi)$ over U associated to $E_{L(P)}$ for the character χ is of degree zero.

From the second condition in the above definition of polystability it follows that the direct image $\iota_*\mathrm{ad}(E_{L(P)})$ is a coherent analytic sheaf on M, where $\iota : U \hookrightarrow M$ is the inclusion map [**11**, p. 364, Théorème 1]. The principal $L(P)$-bundle $E_{L(P)}$ over U may be considered as a principal $L(P)$-sheaf using the torsion-free sheaf $\iota_*\mathrm{ad}(E_{L(P)})$ on M. We note that the coherent analytic sheaf $\iota_* \mathrm{ad}(E_{L(P)})$ is independent of the choice of U. Also, the condition that the principal $L(P)$-bundle $E_{L(P)}$ is stable is independent of the choice of the coherent analytic sheaf extending $\mathrm{ad}(E_{L(P)})$.

More details on principal G-sheaves can be found in [**4**], where they were introduced.

3. *Einstein–Hermitian connection*

Let E be a torsion-free coherent analytic sheaf on the Kähler manifold (M, ω). Let $U \subset M$ be the big open subset over which E is locally free. A smooth Hermitian metric h on $E|_U$ is called an *admissible Einstein–Hermitian metric* if the curvature tensor $\Omega(h)$ of the Chern connection corresponding to h is locally square integrable on M, and also

$$\Lambda_\omega \Omega(h) = c \cdot \mathrm{Id}_E \tag{3.1}$$

on U, where c is some complex number and Λ_ω is the adjoint of multiplication of differential forms by the Kähler form ω. The main theorem of [**2**] says that a reflexive sheaf E on M admits an admissible Einstein–Hermitian metric if and only if E is polystable [**2**, p. 40, Theorem 3].

We will define admissible Einstein–Hermitian connections on principal G-sheaves.

Fix a maximal compact subgroup

$$K(G) \subset G. \tag{3.2}$$

If E'_G is a holomorphic principal G-bundle over a complex manifold, and $E'_{K(G)} \subset E'_G$ is a C^∞ reduction of the structure group of E'_G to the subgroup $K(G)$, then the G-bundle E'_G has a unique complex connection which is induced by a connection on $E'_{K(G)}$ (see [**1**, p. 220; **10**, p. 24]). This unique connection will be called the *Chern connection*.

Consider the quotient group Z of G defined in (2.2). We note that Z, which is a product of copies of $\mathbb{C}^*$, is a finite quotient of $\widetilde{Z}_0$, the connected component, containing the identity element, of the center of G. Let $E_Z = E_G(Z)$ be the holomorphic principal Z-bundle over U constructed in (2.3) from E_G. By the definition of a principal G-sheaf, the principal Z-bundle E_Z in (2.3) extends to a holomorphic principal Z-bundle over M. The holomorphic extension of E_Z to M is clearly unique. Since any holomorphic line bundle over M has a unique Einstein–Hermitian connection, any holomorphic principal Z-bundle over M also has a unique Einstein–Hermitian connection.

Let (E_G, E, ψ) be a principal G-sheaf on M. Let $U \subset M$ be the big open subset over which E_G is a holomorphic principal G-bundle (see Remark 2.3(4)). An *Einstein–Hermitian connection* on (E_G, E, ψ) is a Chern connection ∇ on the principal G-bundle E_G over U satisfying the following two conditions.

(1) The connection on the principal Z-bundle $E_Z := E_G(Z)$ (defined in (2.3)) induced by ∇ coincides with the unique Einstein–Hermitian connection on the extension of E_Z to M (recall that E_Z extends holomorphically to M, and the extension has a unique Einstein–Hermitian connection).

(2) The connection on $E|_U$ induced by ∇ and ψ is an admissible Einstein–Hermitian connection on the reflexive sheaf $E^{\vee\vee}$ (the connection ∇ induces a connection on the associated vector bundle $E_G(\mathfrak{g}')$ in (2.4), and using the isomorphism ψ in (2.4), this induced connection gives a connection on $E|_U$).

Theorem 3.1. *A principal G-sheaf (E_G, E, ψ) over a compact connected Kähler manifold (M, ω) admits an admissible Einstein–Hermitian connection if and only if (E_G, E, ψ) is polystable.*

Proof. First assume that (E_G, E, ψ) is polystable. Let $\mathrm{ad}(E_G) := E_G(\mathfrak{g})$ be the adjoint vector bundle defined over the big open subset $U \subset M$ over which E_G is a holomorphic principal G-bundle. We will prove that $\mathrm{ad}(E_G)$ is polystable.

If (E_G, E, ψ) is stable, then imitating the first part of the proof of [**1**, Theorem 2.6] we derive that $\mathrm{ad}(E_G)$ is polystable. If (E_G, E, ψ) is polystable but not stable, then we recall from the definition of polystability that there is a Levi subgroup $L(P) \subset G$, of some parabolic subgroup P of G, and a holomorphic reduction of the structure group $E_{L(P)} \subset E_G$ over U such that $E_{L(P)}$ is stable, and furthermore, for any character χ of $L(P)$ trivial on the center of G, the line bundle over U associated to $E_{L(P)}$ for χ is of degree zero. Therefore, by the previous reasoning, the adjoint vector bundle $\mathrm{ad}(E_{L(P)})$ is polystable. Using [**2**, Theorem 3], the reflexive sheaf $\iota_*\mathrm{ad}(E_{L(P)})$ on M admits an admissible Einstein–Hermitian connection, where ι is the inclusion map of U in M. (We recall that $\iota_*\mathrm{ad}(E_{L(P)})$ is a coherent analytic sheaf on M.) Following the argument in the first part of the proof of [**1**, Theorem 2.6] and using [**2**, p. 49, Proposition 3] it follows that an admissible Einstein–Hermitian connection on $\iota_*\mathrm{ad}(E_{L(P)})$ is induced by a Chern connection on the holomorphic principal $L(P)$-bundle $E_{L(P)}$. Let ∇ be a Chern connection on $E_{L(P)}$ inducing an admissible Einstein–Hermitian connection on $\iota_*\mathrm{ad}(E_{L(P)})$.

We will show that

- ∇ induces a connection on E_G, and
- the induced connection on E_G is Einstein–Hermitian.

The proof of the fact that the induced connection on E_G is Einstein–Hermitian crucially uses the given condition that for any character χ of $L(P)$, which is trivial on the center of G, the degree of the line bundle over U associated to $E_{L(P)}$ for χ is zero.

The associated vector bundle $E_G(\mathfrak{g}')$ in (2.4) is identified with the vector bundle associated to $E_{L(P)}$ for the $L(P)$-module $\mathfrak{g}'$. Therefore, ∇ induces a connection on $E_G(\mathfrak{g}')$. Consider the connection on $E|_U$ induced by this connection on $E_G(\mathfrak{g}')$ using the isomorphism ψ in (2.4). It can be shown that this connection gives an admissible Einstein–Hermitian connection on $\iota_*(E|_U)$. Indeed, this follows from the facts that ∇ is Einstein–Hermitian, and for any character χ of $L(P)$ trivial on the center of G, the line bundle over U associated to $E_{L(P)}$ for χ is of degree zero. Consequently, $\iota_*(E|_U)$ is polystable. Therefore, the vector bundle $\mathrm{ad}(E_G)$, which is a direct sum of $E|_U$ with a trivial vector bundle, is polystable (note that the degree of $E|_U$ is zero).

Since $\mathrm{ad}(E_G)$ is polystable, the reflexive sheaf $\iota_*\mathrm{ad}(E_G)$ admits an admissible Einstein–Hermitian connection [**2**, p. 40, Theorem 3]. Now using the argument in the first part of the proof of [**1**, Theorem 3.7] together with [**2**, p. 49, Proposition 3] we conclude that (E_G, E, ψ) admits an admissible Einstein–Hermitian connection.

To prove the converse, assume that (E_G, E, ψ) admits an admissible Einstein–Hermitian connection. Let ∇ be an admissible Einstein–Hermitian connection on (E_G, E, ψ). The connection on $\mathrm{ad}(E_G)$ over U induced by ∇ gives an admissible Einstein–Hermitian connection on the reflexive sheaf $\iota_*\mathrm{ad}(E_G)$. (In Remark 2.3(2) we noted that $\iota_*\mathrm{ad}(E_G)$ is a coherent analytic sheaf on M.) Therefore, $\mathrm{ad}(E_G)$ over U is polystable. From this it is straight forward to deduce that (E_G, E, ψ) is polystable; see [**10**, pp. 28–29]. This completes the proof of the theorem. □

We have the following corollary.

Corollary 3.2. *Let (E_G, E, ψ) be a principal G-sheaf over a compact connected Kähler manifold (M, ω) of (complex) dimension d. Let $U \subset M$ be a big open subset over which E_G is a principal G-bundle. If (E_G, E, ψ) is polystable, then*

$$(2\dim_{\mathbb{C}}\mathfrak{g}\cdot c_2(\iota_*\mathrm{ad}(E_G)) - (\dim_{\mathbb{C}}\mathfrak{g} - 1)c_1(\iota_*\mathrm{ad}(E_G))^2)\omega^{d-2} \geqslant 0,$$

where $\mathfrak{g}$ is the Lie algebra of G, and $\iota : U \hookrightarrow M$ is the inclusion map.

Proof. It was shown in the proof of Theorem 3.1 that the reflexive sheaf $\iota_* \mathrm{ad}(E_G)$ is polystable. Hence the corollary follows from [**2**, p. 40, Corollary 3]. □

PROPOSITION 3.3. *Let (E_G, E, ψ) be a stable principal G-sheaf on M. Let $U \subset M$ be the open dense set over which E_G is a holomorphic principal G-bundle. Then*

$$H^0(U, \mathrm{ad}(E_G)) = \mathfrak{z}(\mathfrak{g}),$$

where $\mathfrak{z}(\mathfrak{g})$ is the center of the Lie algebra $\mathfrak{g}$ of G.

Proof. Let ∇ be an admissible Einstein–Hermitian connection on the stable principal G-sheaf (E_G, E, ψ), which exists by Theorem 3.1. We recall from the proof of Theorem 3.1 that the connection on $\mathrm{ad}(E_G)$ induced by ∇, which we will henceforth denote by ∇', is in fact an admissible Einstein–Hermitian connection on $\iota_* \mathrm{ad}(E_G)$, where ι is the inclusion map of U in M.

Proposition 3 of [**2**, p. 49] says that any section

$$s \in H^0(U, \mathrm{ad}(E_G)) \tag{3.3}$$

is flat with respect to ∇'. Therefore, any section s as in (3.3) corresponds to an invariant of $\mathfrak{g}$ for the adjoint action on $\mathfrak{g}$ of the closure of the monodromy group for the connection ∇. (Here monodromy corresponds to parallel translations along all piecewise smooth paths in U.) Using [**8**, Proposition 2.1], this implies that if there is a non-zero section s as in (3.3), then the closure of the monodromy group for ∇ is contained in some proper parabolic subgroup of G.

Let P be a maximal proper parabolic subgroup of G that contains the monodromy group for the connection ∇. Let

$$E_P \subset E_G|_U$$

be the holomorphic reduction of the structure group given by the connection ∇ (as P contains the monodromy group for ∇, the connection gives a reduction of the structure group of $E_G|_U$ to P). Therefore, ∇ induces a connection on E_P. The connection on $\mathrm{ad}(E_P)$ induced by this connection on E_P given by ∇ will be denoted by ∇''.

As the connection ∇' on $\mathrm{ad}(E_G)|_U$ is Einstein–Hermitian, and

$$\mathrm{ad}(E_G)|_U = (\mathrm{ad}(E_G)|_U)^*,$$

in particular, $c_1(\mathrm{ad}(E_G)|_U) = 0$, and we conclude that the constant in the Einstein–Hermitian equation (the constant 'c' in (3.1)) vanishes; see [**5**, p. 103, Proposition 2.1]. Since the connection ∇'' is Einstein–Hermitian (as ∇' is so), and the constant in the Einstein–Hermitian equation vanishes, we have

$$\mathrm{degree}(\mathrm{ad}(E_P)) = 0 \tag{3.4}$$

(see [**5**, p. 103, Proposition 2.1]). From (3.4) we have

$$\mathrm{degree}(\sigma^* T_{\mathrm{rel}}) = \mathrm{degree}(\mathrm{ad}(E_G)/\mathrm{ad}(E_P)) = 0,$$

where σ is the section as in (2.9) for the reduction $E_P \subset E_G|_U$, and $\sigma^* T_{\mathrm{rel}}$ is the vector bundle as in (2.10). However, this contradicts the stability condition in (2.10). This completes the proof of the proposition. □

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References

1. B. Anchouche and I. Biswas, 'Einstein–Hermitian connections on polystable principal bundles over a compact Kähler manifold', *Amer. J. Math.* 123 (2001) 207–228.
2. S. Bando and Y.-T. Siu, 'Stable sheaves and Einstein–Hermitian metrics', *Geometry and analysis on complex manifolds* (World Scientific Publishing, River Edge, NJ, 1994) 39–50.
3. S. K. Donaldson, 'Infinite determinants, stable bundles, and curvature', *Duke Math. J.* 54 (1987) 231–247.
4. T. L. Gómez and I. Sols, 'Moduli space of principal sheaves over projective varieties', *Ann. of Math.* 161 (2005) 1037–1092.
5. S. Kobayashi, *Differential geometry of complex vector bundles* (Princeton University Press, Princeton, NJ/Iwanami Shoten, Tokyo, 1987).
6. D. Mumford, 'Projective invariants of projective structures and applications', *1963 Proc. Internat. Congr. Mathematicians*, (Inst. Mittag–Leffler, Ojursholm, 1962) 526–530.
7. M. S. Narasimhan and C. S. Seshadri, 'Stable and unitary vector bundles on a compact Riemann surface', *Ann. of Math.* 82 (1965) 540–567.
8. A. Ramanathan, 'Stable principal bundles on a compact Riemann surface', *Math. Ann.* 213 (1975) 129–152.
9. A. Ramanathan, 'Moduli for principal bundles', *Algebraic geometry*, Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978, Lecture Notes in Mathematics 732 (Springer, Berlin, 1979) 527–533.
10. A. Ramanathan and S. Subramanian, 'Einstein-Hermitian connections on principal bundles and stability', *J. Reine Angew. Math.* 390 (1988) 21–31.
11. J.-P. Serre, 'Prolongement de faisceaux analytiques cohérents', *Ann. Inst. Fourier* 16 (1966) 363–374.
12. F. Takemoto, 'Stable vector bundles on algebraic surfaces', *Nagoya Math. J.* 47 (1973) 29–48.
13. K. Uhlenbeck and S.-T. Yau, 'On the existence of Hermitian–Yang–Mills connections in stable vector bundles', *Comm. Pure Appl. Math.* 39 (1986) 257–293.

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