THE SCALAR INVARIANTS OF A GENERAL GRAVITATIONAL METRIC

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1. INTRODUCTION

THE absolute scalar invariants of order two associated with the four-dimensional Riemannian metric,

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu, \tag{1}$$

which is used in the general theory of relativity, have a special significance as the field equations are partial differential equations of second order. The metric tensor $g_{\mu\nu}$ and the Riemann-Christoffel curvature tensor R_{hijk} have ten and twenty independent components respectively. At the same time it can be shown that there are only sixteen independent differential equations for the determination of these scalar invariants. Thus one finds that there are only fourteen[1] (*i.e.*, $10 + 20 - 16$) independent scalar invariants of the second order.

So far as we know, no attempt has been made to obtain a set of fourteen independent scalar invariants. It is the object of this paper to present such a set. We find it convenient in the present investigation to set up two tensors, R_{ij} and C_{hijk}, each representing ten linear combinations of R_{hijk}-components, so that a necessary and sufficient condition for

$$R_{hijk} = 0 \tag{2}$$

is that both

$$\text{(i) } R_{ij} = 0, \text{ (ii) } C_{hijk} = 0. \tag{3}$$

We have

$$R_{ij} = g^{hk} R_{hijk}, \tag{4}$$

and as the only linear combinations of R_{hijk}-components which preserve all the symmetry properties are

$$\text{(i)}\ a\,(g_{hj}\,g_{ik}-g_{hk}\,g_{ij})\,R, \tag{5}$$

$$\text{(ii)}\ b\,(g_{hj}\,R_{ik}+g_{ik}\,R_{hj}-g_{hk}\,R_{ij}-g_{ij}\,R_{hk}), \tag{6}$$

we consider

$$C_{hijk}=R_{hijk}+a\,(g_{hj}\,g_{ik}-g_{hk}\,g_{ij})$$
$$+\,b\,(g_{hj}\,R_{ik}+g_{ik}\,R_{hj}-g_{hk}\,R_{ij}-g_{ij}\,R_{hk}), \tag{7}$$

a and b being unknown scalars. The condition

$$g^{hk}\,C_{hijk}=0 \tag{8}$$

implies, for n dimensions,

$$1+2b-bn=0,\ a\,(n-1)+b=0, \tag{9}$$

which means that the tensor C_{hijk} is Weyl's conformal curvature tensor, having only ten independent components for $n=4$. Weyl's projective tensor is apparently simpler but it has not the same symmetry properties and we do not find it as useful as the conformal curvature tensor in our work.

2. A Set of Fourteen Scalar Invariants

The fourteen invariants and the various tensors associated with them are defined as follows:—

$$\left.\begin{aligned}
A_{hijk}&=C_{hipq}\,C_{rsjk}\,g^{pr}\,g^{qs}\\
B_{hijk}&=C_{hipq}\,A_{rsjk}\,g^{pr}\,g^{qs}\\
D_{hijk}&=B_{hijk}-\tfrac{1}{12}\,J_2\,(g_{hj}\,g_{ik}-g_{hk}\,g_{ij})-\tfrac{1}{4}\,J_1\,C_{hijk}\\
\overline{D}_{hijk}&=(J_3)^{-\frac{1}{2}}\cdot D_{hijk}\\
E_{hijk}&=C_{hipq}\,D_{rsjk}\,g^{pr}\,g^{qs}\\
F_{hijk}&=C_{hipk}\,E_{rsjk}\,g^{pr}\,g^{qs}
\end{aligned}\right\} \tag{10}$$

$$Q_{\gamma}{}^{\mu}=R_{\alpha}{}^{\mu}\,R_{\gamma}{}^{\alpha}; \tag{11}$$

$$I_1=R_{\mu}{}^{\mu},\ I_2=R_{\alpha}{}^{\mu}\,R_{\mu}{}^{\alpha},\ I_3=R_{\alpha}{}^{\mu}\,R_{\beta}{}^{\alpha}\,R_{\mu}{}^{\beta},$$
$$I_4=R_{\alpha}{}^{\mu}\,R_{\beta}{}^{\alpha}\,R_{\delta}{}^{\beta}\,R_{\mu}{}^{\delta}; \tag{12}$$

$$J_1 = A_{hijk} g^{hj} g^{ik}, \; J_2 = B_{hijk} g^{hj} g^{ik},$$
$$J_3 = E_{hijk} g^{hj} g^{ik}, \; J_4 = F_{hijk} g^{hj} g^{ik}; \tag{13}$$

$$K_1 = C_{hijk} R^{hj} R^{ik}, \; K_2 = A_{hijk} R^{hj} R^{ik}, \; K_3 = D_{hijk} R^{hi} R^{jk}$$
$$K_4 = C_{hijk} Q^{hi} Q^{jk}, \; K_5 = A_{hijk} Q^{hj} Q^{ik}, \; K_6 = D_{hijk} Q^{hj} Q^{ik}. \tag{14}$$

One of us has verified by direct calculation that the fourteen invariants defined by (12), (13) and (14) are independent in the algebraic sense. The actual computation was simplified by neglecting the cross terms of $g_{\mu\gamma}$ and $R_{\mu\gamma}$ and it was found that the reduced expressions of the fourteen invariants were functions of fourteen independent variables, the Jacobian being non-zero.

3. Gravitational Interpretations

The vanishing of the fourteen invariants does not imply the vanishing of R_{hijk}. We give later an example of a non-flat metric for which all the invariants vanish.

We start our discussion with a class of metrics for which the number of non-zero R_{hijk}-components is essentially six, the number of independent scalar invariants also being six. This enables us to obtain a number of interesting results bringing out the importance of the invariants.

We proceed to consider a line-element

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + g_{44}(dx^4)^2, \tag{15}$$

which has the additional property that

$$R_{1212}, \; R_{1313}, \; R_{1414}, \; R_{2323}, \; R_{2424}, \; R_{3434} \tag{16}$$

constitute a complete set of independent and surviving components of R_{hijk}. The line-element for example,

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + (dx^4)^2, \tag{17}$$

where g_{11}, g_{22}, g_{33} are functions of x^4 only, is found to satisfy (15) and (16). A general spherically symmetrical line-element, which is of considerable importance in the general theory of relativity, is given by

$$ds^2 = -Adr^2 - B(d\theta^2 + \sin^2\theta d\phi^2) + 2Cdrdt + Ddt^2, \tag{18}$$

A, B, C, D being functions of r and t. The surviving components of R_{hijk} are given by

$$R_{1313} = R_{1212} \sin^2 \theta, \; R_{1414}, \; R_{2323},$$

$$R_{3434} = R_{2424} \sin^2 \theta, \; R_{3143} = R_{2142} \sin^2 \theta. \tag{19}$$

By a suitable transformation of co-ordinates

$$\bar{r} = \bar{r}(r, t), \; \bar{t} = \bar{t}(r, t), \; \bar{\theta} = \theta, \; \bar{\phi} = \phi, \tag{20}$$

(18) can be reduced to the form,

$$ds^2 = -\bar{A}(d\bar{r})^2 - \bar{B}(d\bar{\theta}^2 + \sin^2 \bar{\theta}\, d\bar{\phi}^2) + D d\bar{t}^2, \tag{21}$$

so that the only surviving components of $\bar{R}_{hijk}$ are

$$\bar{R}_{1313} = \bar{R}_{1212} \sin^2 \bar{\theta}, \; \bar{R}_{3434} = \bar{R}_{2424} \sin^2 \bar{\theta}, \; \bar{R}_{1414}, \; \bar{R}_{2323}. \tag{22}$$

Thus a general spherically symmetrical line-element is also reducible to the form (15) complying with (16). We, therefore, discuss the line-element (15) in detail. Only two of the independent components of C_{hijk} are non-zero while only four of the R_{ij}-components survive. It is obvious that the six invariants

$$I_1, \; I_2, \; I_3, \; I_4, \; J_1, \; J_2 \tag{23}$$

of which the first four are exclusively functions of R_{ij} and the last two are exclusively functions of the two independent components of C_{hijk}, are algebraically independent. It follows that if R_{hijk} vanishes all the six vanish and that the converse of this also holds. Again if C_{hijk} is zero, J_1 and J_2 also vanish and conversely the vanishing of J_1 and J_2 implies that of C_{hijk}. Let us consider what happens if (15) represents a space of constant curvature K_0, the necessary and sufficient conditions for this being

$$R_{hijk} = K_0 (g_{hj} g_{ik} - g_{hk} g_{ij}). \tag{24}$$

On using (24) we get

$$I_1 = 4(-3K_0), \; I_2 = 4(-3K_0)^2, \; J_1 = 0,$$

$$I_3 = 4(-3K_0)^3, \; I_4 = 4(-3K_0)^4, \; J_2 = 0, \tag{25}$$

Conversely, if (25) is satisfied, we can show that

$$R_1^1 = R_2^2 = R_3^3 = R_4^4 = -3K_0, \quad R_\nu^\mu = 0\ (\mu \neq \nu), \quad C_{hijk} = 0 \tag{26}$$

On expressing these in terms of R_{hijk}, we find that these imply (24). Thus for the special line-element under consideration the necessary and sufficient conditions that the line-element may represent (1) a flat space, (ii) a Riemannian space of constant curvature, can be expressed in terms of these invariants. All these results are true for a general spherically symmetrical line-element which has been shown to be a particular case of the line-element under consideration. In addition, we find that due to the relations (22) there are only four independent scalar invariants associated with a general spherically symmetrical line-element and may be supposed to be given by I_1, I_2, I_3 and J_1. Incidentally it may be noted that the tensor D_{hijk} vanishes for the line-element (15) satisfying (16).

We return to the discussion of the general line-element (1). It is obvious that if R_{hijk} vanishes all the invariants must vanish. Since there are twenty independent components of R_{hijk} while there are only fourteen independent invariants R_{hijk} may not vanish even if all the invariants are zero. As an example, let us consider the line-element,

$$ds^2 = -A(\xi)\,[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2] \tag{27}$$

$$\xi \equiv (x^1 - x^4).$$

It is conformal to a Euclidian metric and C_{hijk} is zero. Therefore the only possible non-vanishing invariants are I_1, I_2, I_3, I_4. The surviving components of R_j^i are given by

$$R_1^1 = R_1^4 = -R_4^1 = -R_4^4 = \frac{1}{A^2}\frac{d^2A}{d\xi^2} - \frac{3}{2}\cdot\frac{1}{A^3}\left(\frac{dA}{d\xi}\right)^2. \tag{28}$$

It easily follows that I_1, I_2, I_3, I_4 are all zero. Thus (27) is an example of a non-flat line-element for which *all the scalar invariants of the second order vanish.* It represents the gravitational field caused by radiation flowing in the x^1-direction. Tolman[2] has given a first order solution of the problem. As far as we know no such line-element possessing this striking geometrical property has been discovered before.

From the set we have obtained it follows that if R_{ij} is not zero but all the fourteen invariants vanish there are ten restrictions on the ten independent components of C_{hijk} and hence C_{hijk} must vanish. Thus the metric is necessarily conformal to a flat space metric. Alternatively, if R_{ij} vanishes and the invariants also vanish it virtually amounts to only four restrictions on the ten C_{hijk}-components and hence C_{hijk} need not be zero. A result similar to this can be established for a general n-dimensional Riemannian space.

It is well known that a necessary condition that a Riemannian space be of class one[3] is

$$R_{hijk} = \pm (b_{hj}\, b_{ik} - b_{hk}\, b_{ij}), \tag{29}$$

b_{ij} being a symmetrical tensor. For the four-dimensional metric (1), R_{hijk} and b_{ij} have respectively twenty and ten independent components and by the elimination of b_{ij} we can get ten conditions to be satisfied by the R_{hijk}-components. It can be shown that there are only four independent invariants associated with b_{ij}, by the argument used to esbtalish the existence of fourteen independent scalars associated with R_{hijk}. If a line-element is of class one the fourteen invariants would be functions of only four invariants associated with b_{ij}. Hence, by elimination, the necessary condition (29) can be expressed as a set of ten relations between the fourteen invariants. In the case of a general spherically symmetrical line-element the conditions (29) reduce to one[4] which is also found to be sufficient that it may be of class one. Thus for a general spherically symmetrical line-element a necessary and sufficient condition that it may be of class one can be expressed in terms of scalar invariants. A result similar to the one obtained above can be shown to hold good for a general n-dimensional Riemannian space of class one.

Summary

A set of fourteen independent scalar invariants of the second order, associated with a general four-dimensional Riemannian metric, is obtained. The set is found to reduce only to four independent invariants in the case of a general spherically symmetrical line-element. Moreover it is shown that the necessary and sufficient conditions that a general spherically symmetrical line-element may represent (i) a flat space, (ii) a Riemannian space of constant curvature, (iii) a Riemannian space conformal to a flat space,

(iv) a Riemannian space of class one, can be expressed in terms of these invariants. A new line-element which is the first of its kind known to us is also given, representing the gravitational field due to radiation flowing in one direction and for which all the scalar invariants of the second order vanish.

References

1. Thomas, T. Y. .. *Differential Invariants of Generalized Spaces*, 1934, 183.
2. Tolman, R. C. .. *Relativity, Thermodynamics and Cosmology*, 1934, 273.
3. Eisenhart, L. P. .. *Riemannian Geometry*, 1926, 197.
4. Karmarkar, K. R. .. *Proc. Ind. Acad. Sci.*, 1948, **27**, 56-60.