"LĪLĀVATĪ"—A NEW ANALOGUE COMPUTER FOR SOLVING LINEAR SIMULTANEOUS EQUATIONS AND RELATED PROBLEMS

Part I. General Principles and Design of Model I

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Received July 27, 1958

1. Introduction

Linear simultaneous equations in a number of variables occur in various problems in physics and engineering. If the number of variables is larger than 5 or 6, the solution of these equations by the usual numerical methods is very cumbersome and it is necessary to adopt analogue methods for solving them. Closely related to this problem are the problems of the inversion of a matrix of order n and determination of the eigenvalues of such a matrix; the latter, in turn, is related to the solution of secular equations occurring in the theory of vibrations. Here again, an analogue method of solution is highly desirable if n is not small.

A few designs of such analogue machines have been given earlier [Berry *et al.* (1946), Barker (1956), Haupt (1950), Mallock (1933), Mitra (1955), Ryder (1955)] and a review of the subject will be found in the book by Soroka (1954). These involve either mechanical arrangements, which are in general not capable of great accuracy, or electrical systems. The latter are capable of great accuracy, in view of the high precision attainable in the measurement of electrical quantities. However, the circuits employed must be an exact analogy of the equations to be solved, if high precision is to be expected. An essential process which has to be set up in analogy in any such arrangement is a multiplication and addition, *e.g.*, in the equation

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}\, x_n = b_1 \tag{1}$$

one has to multiply each of the quantities x_s by a_{rs} and then add them.

In most of the electrical analogue machines constructed so far, the physical quantities are all voltages and the multiplication is obtained either by using potentiometers or transformers as shown in Figs. 1 (*a*) and (*b*).

While it is true that in an open circuit the ratio of voltages x and ax can be set to any desirable value very accurately, this is not exactly maintained when current is drawn. This inaccuracy can be reduced by making the potentiometer resistance small, compared with that of the measuring instrument. But in principle the accuracy is restricted, *e.g.*, to attain an accuracy of 0·1% it will be necessary to use an extremely sensitive measuring instrument. The essential difficulty is that these designs do not involve null settings in which the current through the circuit to be measured is zero.

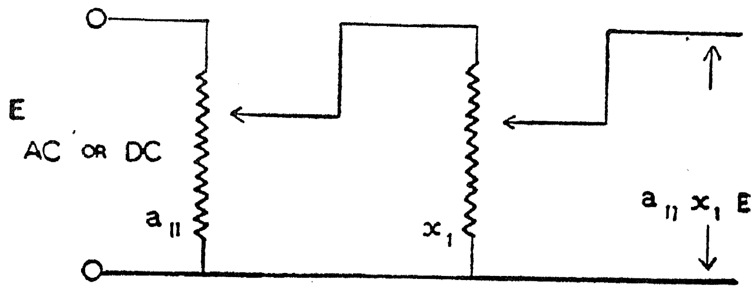

FIG. 1 (*a*)

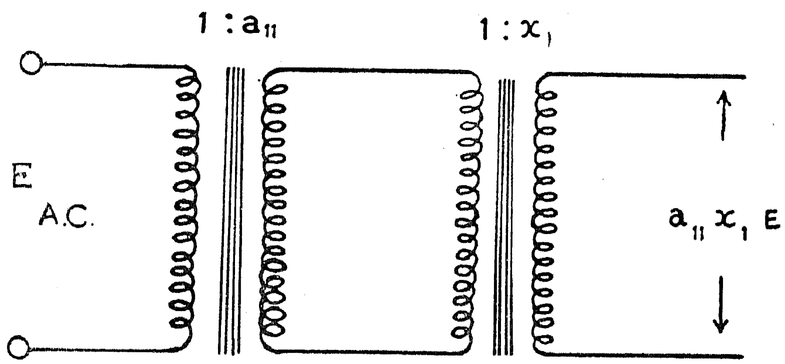

FIG. 1 (*b*)

FIGS. 1 (*a*) and (*b*). Multiplication technique in previous computers.

A second defect in principle, which again can be made negligibly small by increasing the ratio of two resistances in the potentiometric machines [Berry (1946), Mitra (1955)], or by feed back methods in the case of transformer computers [Mallock (1933), Barker (1956)] is the non-linearity of the components involved, due to the loading effects.

2. MULTIPLICATION BY USING OHM'S LAW

It occurred to the authors that both these difficulties could be obviated by using a different method of multiplication. Instead of having both x and $y = ax$ as voltages one can have x as a current, a as a resistance through which this current passes, when the voltage across the ends of the resistance will be exactly $y = ax$ (Fig. 2). By its very principle, the operation here is exactly linear—it is in fact accurate to the same degree that Ohm's law is.

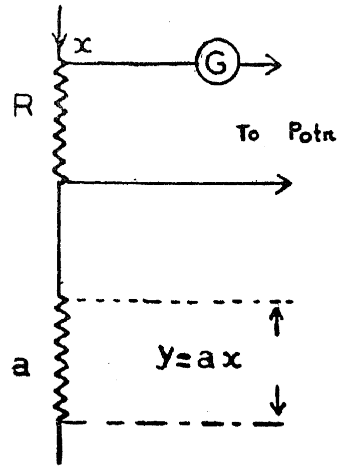

FIG. 2. Technique of multiplication using Ohm's law.

Resistances can be wound to a high degree of accuracy and they are very stable. The current x can be measured by incorporating a standard resistance in the circuit and measuring the voltage developed by means of a potentiometric arrangement. This is a null method and does not disturb the value of x.

The analogue circuit of equation (1) can thus be set up as in Fig. 3. The equality of the two sides of the equations is adjusted by making the current through the galvanometer G to be zero. This again being a null adjustment does not affect the current through any of the circuits.

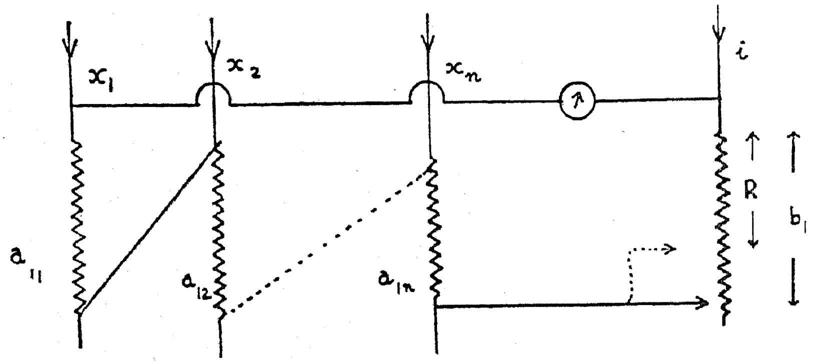

FIG. 3. Analogy of a single equation.

The voltage b_r can be made equal to the required value by either of the following methods:

(*a*) The voltage can be directly measured by a potentiometric arrangement and adjusted to the required value by varying the current i through R (Fig. 3) or (*b*) the current may be kept constant but the resistance R, over which the voltage is taken, may be varied.

Both are useful and either has been used, according to the nature of the problem to be solved.

Thus the essential principle of the new machine is to carry out multiplication by making use of Ohm's law and addition by connecting the individual voltages in series. This has resulted in making the computer inexpensive and at the same time capable of high accuracy.

3. Basic Circuit of the Linear Simultaneous Equation Solver

Suppose the n equations to be solved are

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}\, x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}\, x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n \end{array} \qquad (2)$$

or in general

$$\sum_{s=1}^{n} a_{rs}\, x_s = b_r \qquad (r = 1,\ 2, \ldots\ldots n) \qquad (3)$$

Here the values of x_1 to x_n are the same for all the equations while a_{rs} and b_r are the data which are fed in. An analogy of this can be achieved by employing the circuit shown in Fig. 4. The quantities a_{rs} are represented by resistances and b_r are fed in as voltages, equal to the required values.

Obviously the voltages developed across the points A_1 and B_1 is $\sum_{s=1}^{n} a_{1s}\, x_s$ being the sum of the voltages developed across the resistances a_{1s} ($s =$ 1 to n) by the passage of currents of magnitude x_s. The voltage across C_1 and D_1 is equal to b_1. If these are made equal, then the equation $\sum_s a_{1s}\, x_s = b_1$ is satisfied. If now the cross-connections shown for the first row in Fig. 4 are made successively for each of the rows 1 to n, and currents x_1 to x_n are so adjusted that for every row the galvanometer shows null deflection, then obviously the n equations (2) are all satisfied. Now one has only to measure the n currents, by the method mentioned in the last section, to obtain a solution of the simultaneous equations.

The adjustment of the n currents so as to simultaneously satisfy all the n equations may be done by the well-known Gauss-Seidel iterative process [*cf.* Berry (1945) and Young (1956)]. In using this process, one starts with an arbitrary set of values $x_1^{(0)}$ to $x_n^{(0)}$ and the first row is connected up. Then the current in the first circuit (*viz.*, $x_1^{(0)}$) alone is varied to $x_1^{(1)}$ such that balance is obtained. Using now the new value of the current $x_1^{(1)}$ and the old values $x_2^{(0)}$ ...... $x_n^{(0)}$, the second row is connected up and $x_2^{(0)}$ is then varied to $x_2^{(1)}$ to get the balance. The process is continued until a new set of variables $x_1^{(1)}$ to $x_n^{(1)}$ is obtained. The whole set of operations is then repeated. It can be shown that if the original equations are properly set up, then the process converges in general. The conditions for maximum convergence, etc., are dealt with in articles by Berry (1945), Hotelling (1943) and Young (1956).

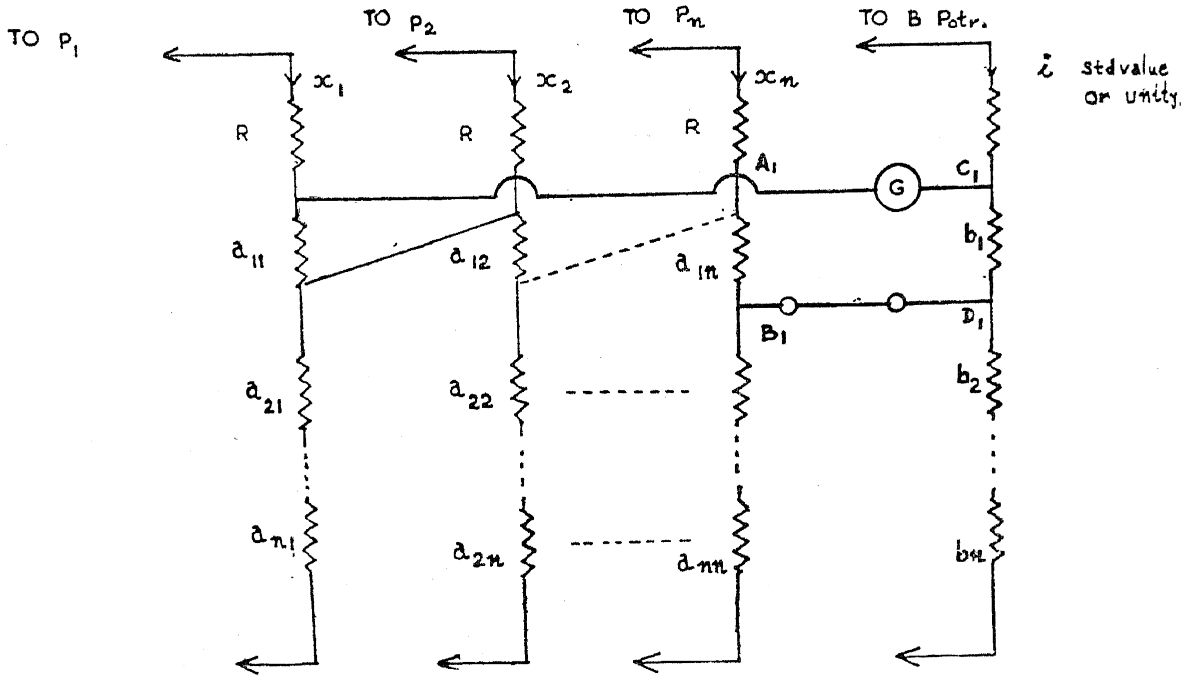

FIG. 4. Analogy of a set of simultaneous equations.

It is found that the Gauss-Seidel process does not always converge. However, it is possible to work out a different iterative process which always converges and is generally applicable to the solution of any set of n simultaneous equations with real coefficients. The details of this method together with its application to our analogue computer will be discussed in Part III.

4. AUXILIARY CIRCUITS AND ARRANGEMENTS

(i) *Plus-Minus Switch.*—In actual practice, several other considerations have to be taken into account before the above circuit is utilised for building a convenient machine. In the above discussion, we did not consider the problem of having a_{rs}, x_s and b_r to be either positive or negative. A reversal of sign can readily be made with electrical voltages by interchanging the two terminals across which the voltage is measured. Current can also be similarly reversed. The basic component which can do this is the well-known 6-key commutator. Commercially available double pole double throw switches (DPDT) can be made to serve this purpose admirably.

Fig. 5 (*a*) shows how voltages $\pm b_r$ can be obtained by throwing the switch either to the left or to the right.

Fig. 5 (*b*) shows how the current through the resistors a_{rs} can be either made to flow in one direction or in the opposite direction by throwing the switch to the left or to the right thus getting a current $\pm x_s$. Fig. 5 (*c*)

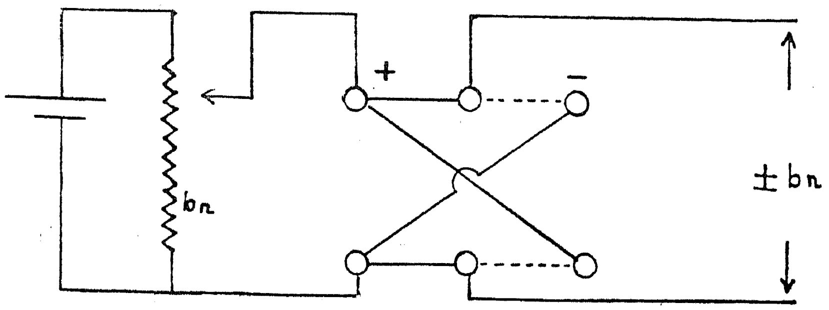

FIG. 5 (*a*)

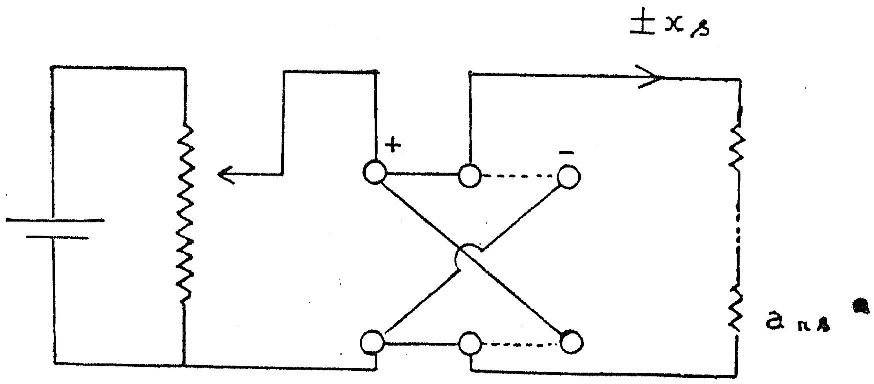

FIG. 5 (*b*)

shows the method of affixing a *plus* or *minus* sign to the quantity $| a_{rs} |$ which is represented by a resistance. Essentially, it is necessary to have a quantity $\pm | a_{rs} | x_s$ in which x_s is given (as a current) and a_{rs} is represented by the magnitude of a resistance. The product $| a_{rs} | x_s$ is the voltage developed across the resistance. Now if this voltage is measured in the same sense as the current x_s the resulting value would obviously be $+ | a_{rs} | x_s$, while if it is measured in the opposite sense it would be $- | a_{rs} | x_s$.

Actually the term $a_{rs} x_s = \pm \mid a_{rs} \mid x_s$ is one in a series of similar quantities which are all to be added together, to give a total voltage. Consequently, the way in which the particular voltage $\mid a_{rs} \mid x_s$ can be added either with a *plus* or *minus* sign to the series may be understood from Fig. 5 (*c*). If the switch is thrown to the left, the top terminal of resistance a_{rs} is connected to the incoming line and the bottom is connected to the outgoing line and consequently a voltage $+ \mid a_{rs} \mid x_s$ is added. If the switch is thrown to the right, then the sign of the voltage added is reversed owing to the commutator action of the switch.

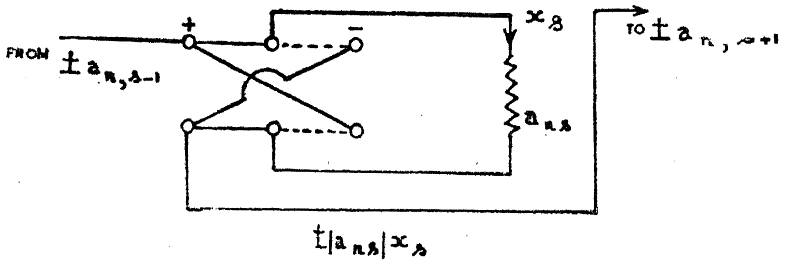

FIG. 5 (*c*)

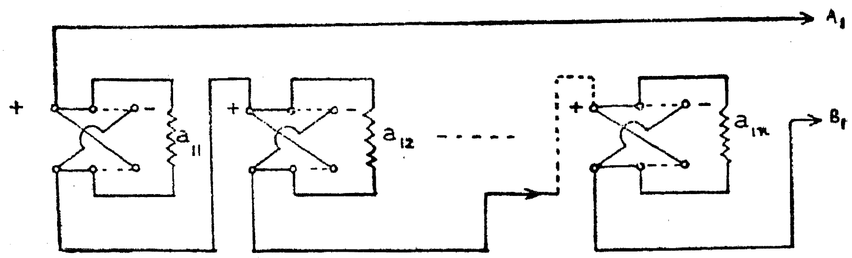

FIG. 5 (*d*)

FIG. 5. *Plus-Minus* switches.

(*a*) for voltage (b_r).
(*b*) for current (x_s).
(*c*) for a single resistance (a_{rs}).
(*d*) for a series of resistances in a row.

It may be mentioned that the sign *plus* or *minus* corresponding to the left or right position of the switch is maintained irrespective of the sign of the current x_s. The actual voltage added may be negative for the *plus* setting of the switch if x_s itself is negative, but the algebraic magnitude which is added is only $+ \mid a_{rs} \mid x_s$.

Obviously, a series of such voltages can be added together each with the appropriate sign, and the full circuit diagram for a single row giving $\pm \mid a_{11} \mid x_1 \pm \mid a_{12} \mid x_2 \pm \ldots\ldots \pm \mid a_{1n} \mid x_n$ is shown in Fig. 5 (*d*).

(ii) *Circuit for Setting Resistances.*—The setting up of the problem on the machine is done by adjusting the resistances a_{rs} (including the signs by the *plus-minus* switches) and the voltages b_r to those demanded by the particular set of equations to be solved. If these have to be made accurate to 1% (say) then the resistance value must be adjusted correct to one unit if the maximum is 100 units. One way of doing this is to have a two-decade dial resistance box (accuracy $0 \cdot 1\%$) for each of the components a_{rs} and b_r. We would thus require $n\,(n + 1)$ resistance boxes. This would obviously make the whole unit very costly.

The incorporation of a Wheatstone's network would enable one to avoid this. One makes use of the most sensitive form of an one-to-one ratio Wheatstone's network as an auxiliary circuit and employs a proper switching arrangement to bring each of the resistances a_{rs} which are normally in open circuit, to the fourth arm of the network. Then it is enough to have wirewound potentiometers, such as those used for radio volume controls, for the resistances a_{rs} and these could be adjusted to be equal to the required value by means of the Wheatstone's bridge. The same galvanometer which is used for balancing the main circuits can be used for this adjustment also, if suitable switches are provided.

(iii) *Measurement of Current and Voltage.*—This is best done by means of an auxiliary potentiometer which is standardised by means of a standard cell. However, in the very first design of the machine, this was not done and the measurements were made by means of an ammeter and a high resistance voltmeter.

The use of n ammeters for measuring the currents x_1 to x_n and n voltmeters for measuring the voltages b_1 to b_n was avoided by the following arrangements.

It is obviously unnecessary to know the value of the current while doing the iterative process. Consequently, the adjustments were made with the circuit shown in Fig. 4. After the adjustments were completed, an ammeter, with a series resistance adjusted so as to make its total resistance exactly equal to R (*i.e.* 100Ω) was introduced in place of the resistance R in each of the circuits in turn, and the currents x_1 to x_n were read off. Since the total resistance in the circuits were unchanged, the current could be accurately read in this way. The switching in of the ammeter was done by means of n DPDT switches, the circuit for one of which is indicated in Fig. 6. Normally the resistances R will be included in the circuit and for measuring the current in a particular circuit, the appropriate switch has to be pressed.

The voltmeter could be connected in turn to the ends of the resistances b_1 to b_n by means of a multipole switch and the voltages could be read off.

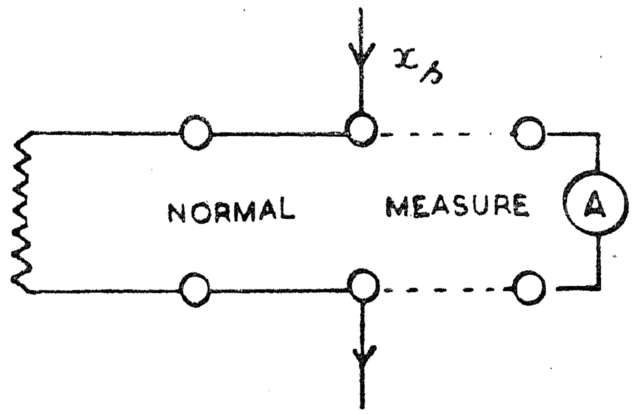

FIG. 6. Switch for reading current.

This multipole switch is a ganged one and is also used to connect up the appropriate rows (Fig. 7).

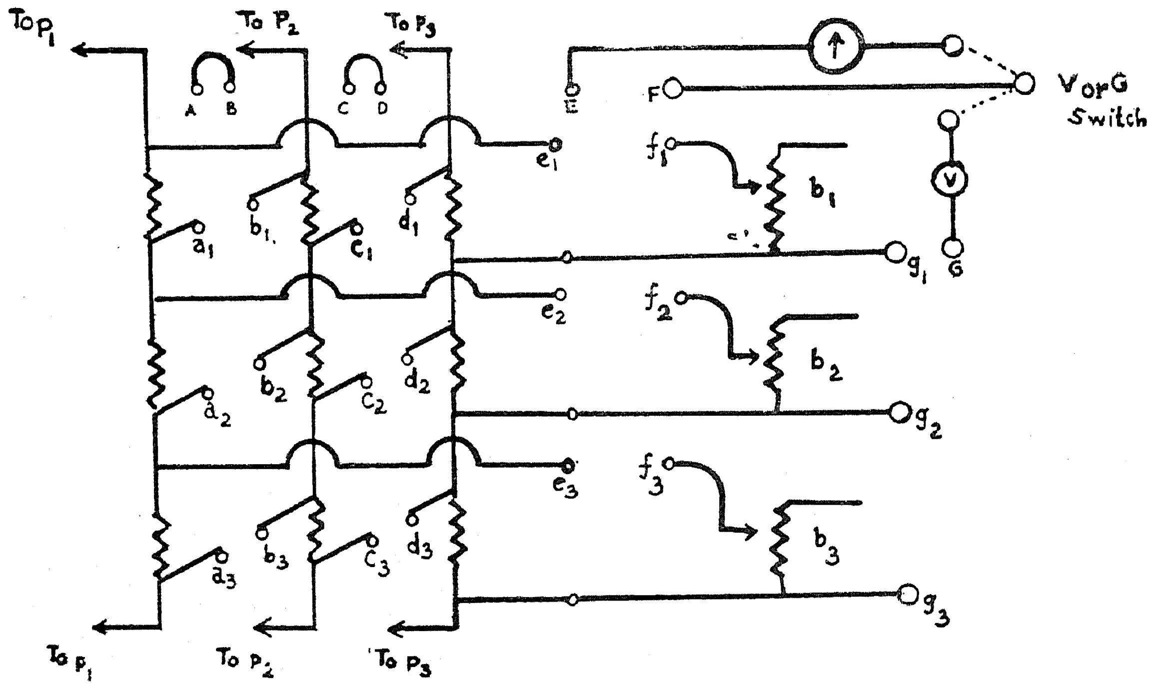

FIG. 7. Circuit diagram of selector switch.

(iv) *Selector Switch*.—The purpose of the selector switch is to set up each of the n equations, in turn, in the analogy. It has to make $(n + 2)$ contacts each time. The usual commercially available multipole ganged switches can only connect a common terminal (pole) to one of the several points (ways). Here, since each one of the points in a particular row has either to be connected to the other or to be isolated from the other, the ganged switch should have twice the number of poles, *i.e.*, $2(n+1)$ poles and n ways.

5. Details of Model I Employing Meters for Measurement

(i) *Description of the Model.*—The first prototype unit constructed was a 3-equation solver which made use of a microammeter for measuring both current and voltage. A photograph of it is shown in Fig. 8 and a short account of its design is given below. It is proposed to name this machine as "Līlāvatī" (Model I) after the title of the celebrated book on algebra by Bhāskarāchāryā.

In this model a set of three wirewound volume controls (0–1000 Ω, 0–100 Ω, 0–10 Ω mounted at the top of the instrument—marked a_{rs} in Fig. 8) is used for the accurate adjustment of each coefficient and a 2-pole 9-way selector switch W_S (mounted on the left side) brings each set to the fourth arm of the Wheatstone's network whose other arms are constituted by two standard 100 Ω coils and a three decade resistance box R. Thus the coefficients can be set to three significant figures.

The potentiometers P_1, P_2 and P_3 and b_r-potentiometers (Fig. 7) are also wirewound volume controls having coarse and fine adjustments (0–100 Ω and 0–10 Ω). The voltage sources used are 4·5 V. dry batteries, and they are found to be quite steady as only currents of the order of a few milliamperes are drawn. The on-off switches for the current circuits are located at the top of the left side panel, and for the voltage circuits, at the top of the right side panel (not seen in the photograph). The *plus-minus* switches for currents (x_S) and for voltages (b_r) are respectively located just above the P-potentiometers and b-potentiometers. The 9 switches (all shown in *plus*-position) for setting $\pm$ are at the bottom-left of the front panel.

A (0–100 μA) microammeter M was wired up as a multimeter with ranges 0–1 mA, 0–10 mA, 0–1 V. and 0–10 V., the voltmeter sensitivity being 10,000 Ω per volt. The resistance of the ammeter was made to be accurately equal to 100 Ω for both the ranges for reasons already mentioned in § 4 (iii). This was used for the measurement of currents (x_S) and voltages (b_r). The range selector R_S is just below the meter and the sign reversal switch for the meter is located by its side.

The equation selector E_S is an 8-pole 4-way switch. Since separate potentiometers were used for obtaining each one of the voltages b_r in the right-hand side, only seven of the poles A, B, C, D, E, F and G with their ways a_i, b_i, c_i, d_i, e_i, f_i and g_i ($i = 1$ to 3) were used for equation selection (Fig. 7). The eighth pole (not shown) was used to light up the pilot lamps mounted in between each of the coarse and fine potentiometers. These indicate the appropriate P-potentiometer which is to be adjusted each time during the iterative process.

A separate 2-way switch (G–V) is provided to either select the voltmeter for reading the voltage or to select the galvanometer for balancing.

A (0–500 μA) meter G of resistance 100 Ω was used as the galvanometer with a shunt of 100 Ω, which could be removed when needed by means of a switch S. This galvanometer was also used for setting of resistances in the Wheatstone's network. For this purpose, the Equation Selector E_S should be kept in the fourth position.

The three DPDT switches C_1, C_2, C_3 which have to be pressed to read off the currents in a particular circuit are seen just above E_S.

(ii) *Operation.*—This machine has been tested out during the last few months and has been found to be highly satisfactory from the operational point of view. Even an inexperienced person can handle the computer quite easily.

The feeding in of the coefficients a_{rs} and voltages b_r are readily done and in order to carry out the iterative process one has only to move the selector E_S to the three different positions, namely 1, 2 and 3, each time adjusting the proper P-potentiometer for null reading in the galvanometer. Since the pilot lamps indicate the appropriate P-potentiometer each time, it is entirely unnecessary to remember which control has to be operated each time.

A larger version of this computer could readily be constructed. However, this was not taken up, as it was found that by making use of potentiometric principles considerable economy in material and also in the number of operations could be obtained. These are discussed in the next paper.

6. APPLICATIONS TO RELATED PROBLEMS

The computer has several other applications such as the computation of the inverses, products and eigenvalues of matrices.

(i) *Inversion of Matrices.*—For the computation of the inverse matrix one can use the method given by Hartree (1955) which requires the solution of n sets of simultaneous equations, if the matrix is of order n. Consequently the computer can be directly used for this purpose. To give a simple example if one wants to get the inverse of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ one has to solve the two sets of simultaneous equations,

$$\begin{aligned} ax_1 + bx_2 &= 1 \\ cx_1 + dx_2 &= 0 \end{aligned} \quad (4) \text{ and } \quad \begin{aligned} ay_1 + by_2 &= 0 \\ cy_1 + dy_2 &= 1 \end{aligned} \quad (5)$$

If (x_1, x_2) and (y_1, y_2) are the solutions obtained for two equations (4) and (5) respectively, then the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$.

(ii) *Multiplication of Matrices.*—If two matrices have to be multiplied, one has to simply feed, each time, the columns of the second matrix as currents and set the rows of the first matrix as resistances and read off the sums as voltages. For example, if two matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ have to be multiplied, one sets up the values a, b, c, d as resistances, then feeds in first the currents x_1, x_2 and reads off $ax_1 + bx_2$ and $cx_1 + dx_2$, which form the first column of the product on the voltmeter. Similarly by feeding in the currents y_1, y_2 the second column can be obtained.

(iii) *Eigenvalues of Matrices and the Solution of Secular Equations.*—A slight modification of the circuit discussed earlier can be used to solve the secular equations that occur in the theory of vibrations. There are various possible modifications of the circuit depending upon the method adopted for solution. A discussion of some of them is given by Wilson, Decius and Cross (1955). A particular modification for solving equations of the type $Ax = \lambda x$ will be described in Part II of this series, while more general circuits for solving $Ax = \lambda Bx$ will be discussed later.

7. Accuracy

As already pointed out, the unit employing meters for measurement of voltage and current will not be very accurate. In order to improve the accuracy one has to use the null method of measurement. Part II of this series will give a detailed account of the modification made in order to obtain an accuracy of 1% and at the same time operate the computer both with A.C. and D.C. If one desires to increase the accuracy further, say to 0·1% or even less, some other new design features have to be incorporated. It was found that by a slight modification of the Kelvin-Varley slide one can achieve the desired amount of accuracy. Further, using multipole switches, it is possible to reduce the number of coefficient resistance boxes required from $n(n+1)$ to $(n+1)$. Fuller details of this circuit and its applications to various problems will be discussed in Part III.

8. SUMMARY

A new electrical analogue computer, 'Līlāvatī', for solving linear simultaneous equations and related problems has been designed. The essential principle of the new machine is to carry out multiplication making use of Ohm's law and addition by connecting the individual voltages in series. Then the well-known Gauss-Seidel iterative process is carried out for obtaining the solution of the simultaneous linear equations. The relative merits of this computer, as compared with other models, the design considerations and the applications to inversion of matrices, products of matrices and the solution of secular equations are also discussed.

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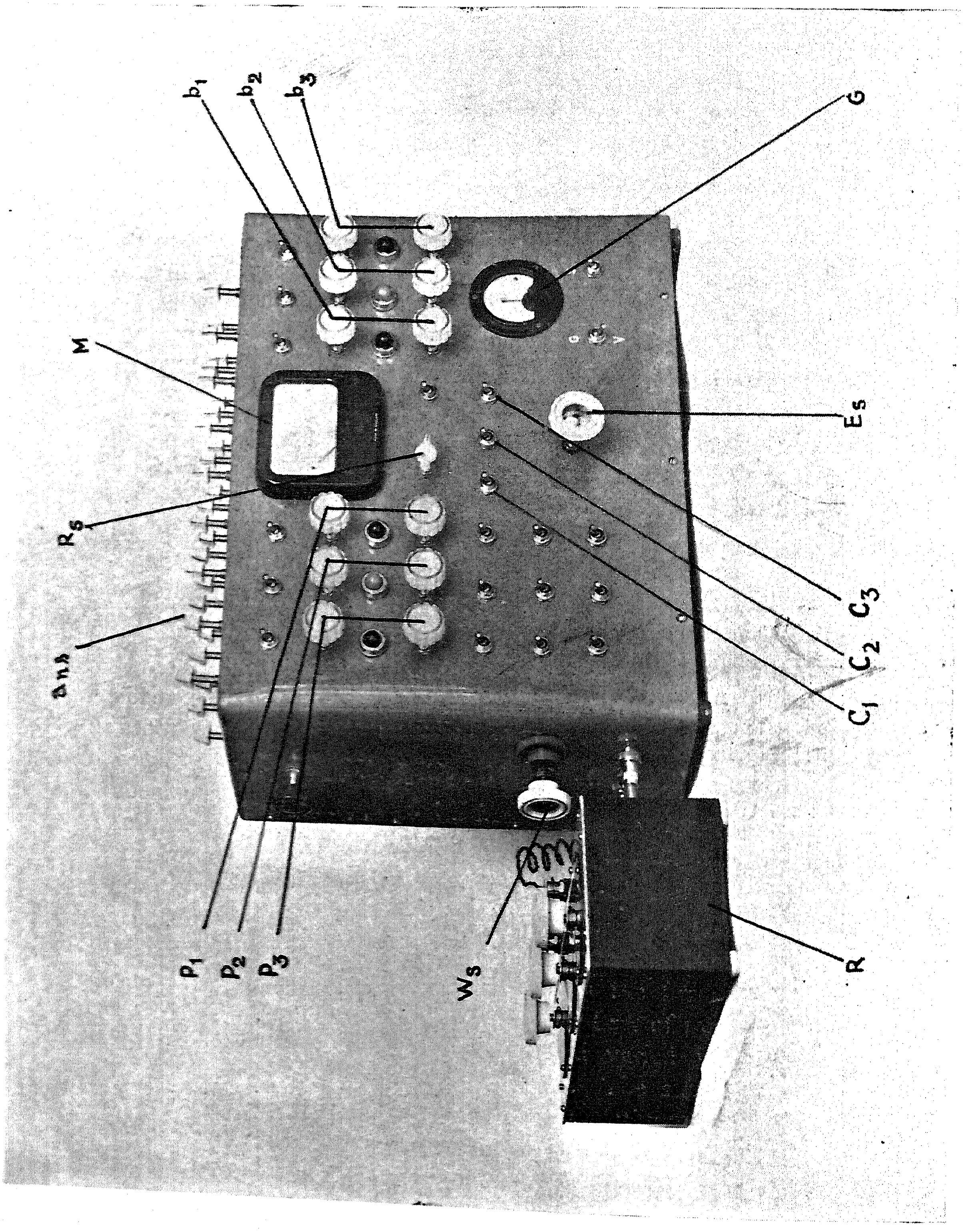

FIG. 8. Photograph of 'Līlāvati'—Model I.