THE THEORY OF CORONÆ AND OF IRIDESCENT CLOUDS

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1. *Introduction*

Amongst the best-known phenomena of meteorological optics are the coronæ consisting of a central disc and one or more coloured concentric rings seen surrounding the sun or the moon, when viewed through thin clouds. The observed angular extension of the coronæ is variable, being generally between a degree and ten degrees. Closely related to these are the iridescent clouds exhibiting vivid colours, which are sometimes observed at distances of about 5° to 30° from the sun. The correlation between these two phenomena was recognized by Simpson (1912), who showed that iridescent clouds are nothing but parts of coronæ.

A mathematical theory of the corona was put forward by Airy, who assumed that the water droplets in the cloud act like opaque disks, and thus give rise to a diffraction pattern of bright and dark rings. Obviously, however, one is not justified in considering the water droplets as opaque, since actually they are transparent. Indeed, many of the experimental facts, as for example those observed by Barus (1907, 1908), Mecke (1920 *a*), and Mitra (1928) are not explained by the "opaque disk" theory. It is evident that the light transmitted through the droplets must be taken into account.

Mecke (1920 *b*) has tried to give a more complete theory, taking into account the rays that pass through the drops also. His method consists in dividing the effect of the droplet into three parts, due to diffraction, transmission and reflection respectively. The first part is taken to be the same as that due to an opaque disk of the same size. The transmission effect is calculated by tracing the refracted rays, and finding their divergence and phase change. A similar calculation is also made for the rays reflected from the surface of the droplet. All these parts are taken to be coherent, the amplitudes are added, and the resultant intensity is calculated.

It may be pointed out, however, that Mecke's application of the ideas of geometrical optics cannot be justified when the particles under consideration have a radius which is only a moderate multiple of the wave-length.

For, the transmitted rays themselves form a beam whose aperture is of the order of the diameter of the droplet, and hence comparable with the wave-length of light. Such a beam would give diffraction effects of its own, so that at the focal plane, where the geometric rays are supposed to come to a focus, the diffraction pattern would have a size about the same as that of the droplet itself. The ray-paths indicated by geometrical optics cease, therefore, to have any significance when one is dealing with droplets whose radius is only a small multiple of the wave-length. It is obvious, in these circumstances, that one must treat the problem on the basis of wave-optics alone, and not on the lines suggested by Mecke.

A theory of coronæ based exclusively on the ideas of the wave optics has been outlined by Balakrishnan (1941) taking into consideration the phase changes occurring in the passage of the light through the drops. The present author has used similar ideas in discussing the transmission of light through a cloud of transparent droplets, in a paper appearing earlier in these *Proceedings.* Both in that paper and in Balakrishnan's work, the aspect considered was the evaluation of the intensity in the forward direction, *viz.*, along the incident rays. The present paper concerns itself with the evaluation of the intensity in other directions, which is necessary to find the position and intensity of the rings observed in the corona.

2. *Derivation of the Formulæ*

It has been shown by the author (*loc. cit.*) that the amplitude of the wave diffracted by a droplet of radius a in a direction making an angle ϕ with the incident direction is

$$X = X_1 - X_2, \tag{1}$$

where

$$X_1 = \frac{2\pi a^2}{\lambda}\int_0^{\pi/2} J_0(\eta \sin\theta)\sin(\chi - \xi\cos\theta)\sin\theta\cos\theta\, d\theta, \tag{2}$$

$$X_2 = \frac{2\pi a^2}{\lambda}\cdot\frac{J_1(\eta)}{\eta}\cdot\sin\chi, \tag{3}$$

$$\xi = 4\pi(\mu - 1)\, a/\lambda, \text{ and } \eta = 2\pi a(\sin\phi)/\lambda, \tag{4}$$

the incident wave being represented by $\sin\chi$. The whole problem thus reduces to the evaluation of the integral in the expression for X_1. The integration is not simple for finite values of η. The author tried various methods of putting it in a form amenable for numerical computation. The best method was found to be one of partial integration, by which it could be put in the form of a series, of which the later terms are small compared with the first. The method consists in partially integrating $\sin(\chi - \xi\cos\theta)$, keeping $J_0(\eta \sin\theta)$ as a constant. We may then write the integral as

$$-\frac{1}{\xi^2}\int_0^{\pi/2} J_0(\eta\sin\theta)\sin(\chi-\xi\cos\theta)(\xi\cos\theta)\,d(\xi\cos\theta), \text{ which is}$$

$$=\frac{1}{\xi^2}\Big[J_0(\eta\sin\theta)\{\sin(\chi-\xi\cos\theta)-\xi\cos\theta\cos(\chi-\xi\cos\theta)\}\Big]_0^{\pi/2} - \int \text{involving } J_1$$

$$=\frac{1}{\xi^2}\Big[J_0(\eta)\sin\chi-\sin(\chi-\xi)+\xi\cos(\chi-\xi)\Big]-\int \text{involving } J_1.$$

Normally, the value of ξ is fairly large, and we can neglect all terms invloving higher powers of $1/\xi$ as compared with those involving $1/\xi$. Then, the integral appearing in X_1 is $=\frac{1}{\xi}\cos(\chi-\xi)$, so that

$$X = K\,[\cos\chi\cos(\xi)/\xi+\sin\chi\,\{\sin(\xi)/\xi - J_1(\eta)/\eta\}] \tag{5}$$

where $K = 2\pi a^2/\lambda$. Hence, the intensity is

$$I = K^2\,[\{\cos(\xi)/\xi\}^2+\{\sin(\xi)/\xi - J_1(\eta)/\eta\}^2]$$

$$= K^2\left[\frac{1}{\xi^2}+\frac{J_1^2(\eta)}{\eta^2}-\frac{2J_1(\eta)}{\eta}\cdot\frac{\sin\xi}{\xi}\right]. \tag{6}$$

This expression (6) is not very suitable when ξ is small, and also when η becomes comparable with ξ. In such a case, a second approximation can be worked out by expanding the integral for X_1 and summing up the most important terms (*vide* Appendix I) which gives

$$I = K^2\left[\frac{1}{\xi^2}+\frac{J_1^2(\eta)}{\eta^2}-2\frac{J_1(\eta)}{\eta}\cdot\frac{\sin(\xi+\eta^2/2\xi)}{\xi}\right]. \tag{7}$$

A third approximation can also be obtained, which leads to the expression for I as

$$I = K^2\,[S^2+\{C-J_1(\eta)/\eta\}^2],$$

where $$S=\frac{1}{\xi}\cos(\xi+\eta^2/2\xi)$$
$$+\frac{1}{\xi^2}\left[\sin(\xi+\eta^2/2\xi)-\frac{\eta^2}{\xi}\cos(\xi+\eta^2/2\xi)-\frac{\eta^6}{8\xi^3}\sin(\xi+\eta^2/2\xi)\right]$$

and $$C=\frac{1}{\xi}\sin(\xi+\eta^2/2\xi)$$
$$+\frac{1}{\xi^2}\left[\cos(\xi+\eta^2/2\xi)+\frac{\eta^2}{\xi}\sin(\xi+\eta^2/2\xi)-\frac{\eta^6}{8\xi^3}\cos(\xi+\eta^2/2\xi)\right].$$

3. *Discussion of the Expression for the Intensity*

For large values of ξ, the first approximation (6) is sufficiently accurate to evaluate the positions of the maxima. In order to determine how far it is different from the more accurate expression (7), the author actually computed the intensity distribution given by both of them for a few cases. It was found that the higher degree of approximation did not appreciably alter the positions of the maxima and the minima, but that it gave values for the actual intensities at these positions different from those given by (6). Therefore the simpler expression (6) could be used for a discussion of the *positions* of the bright and dark rings in the corona, and, in the following discussion, this alone has been used.

We shall first derive a few general properties of the function on the right hand side of (6). For drops of a particular size, ξ is a constant, and the quantity which varies with the angle of diffraction ϕ is η. Hence, the varying part of I is given by

$$I' = K^2 \left[J_1^2(\eta)/\eta^2 - 2J_1(\eta)/\eta \cdot \sin(\xi)/\xi \right] \tag{8}$$

Putting $\sin(\xi)/\xi$ as k, this can be written as

$$I' = K^2 \left[J_1^2(\eta)/\eta^2 - 2kJ_1(\eta)/\eta \right] \tag{9}$$

Differentiating with respect to η,

$$\frac{dI'}{d\eta} = 2K^2 \left[J_1(\eta)/\eta - k \right] \cdot J_2(\eta)/\eta. \tag{10}$$

Hence, the maxima and minima of I' occur for the values of η given by

$$J_1(\eta)/\eta = k \qquad (a)$$

$$J_2(\eta)/\eta = 0 \qquad (b)$$

Call the roots of the former $a_1, a_2, \ldots.$ and those of the later $b_1, b_2, \ldots.$

Differentiating (16) once again, we get

$$\left[d^2I'/d\eta^2 \right]_{\eta=a_m} = +2K^2J_2^2(\eta)/\eta^2 \tag{11}$$

which is always positive. Hence, at $\eta = a_m$, I is always a minimum.

Again, $[d^2I'/d\eta^2]_{\eta=0} = -K^2(1-2k)\,[d/d\eta\,\{J_2(\eta)/\eta\}]_{\eta=0}$

$$= -K^2(1-2k)\,(a\ +ve\ \text{quantity}) \tag{12}$$

This is positive or negative according as k is $>$ or $<$ 1/2. Hence, at $\eta = 0$, I is a maximum or minimum according as $k \gtrless 1/2$.

The other roots of (b), namely $b_1, b_2, \ldots.$ may correspond to a maximum or minimum. It is also easily seen that they correspond to the value of η for which $J_1(\eta)/\eta$ has a turning value.

4. *The Rings of the Corona*

We are now in a position to discuss the disposition of the rings in the corona, which correspond to the maxima and minima of intensity. In this discussion, we shall limit ourselves to the case when monochromatic light is used. The nature of the ring system with white light can be readily deduced from this.

The diffraction in the forward direction and the colours exhibited by the central portion of the corona have already been discussed in the previous paper. It is now only necessary to consider the outer rings of the corona. First of all, it can be shown that the thickness or density of the fog has no influence on the position of the rings in the corona. The intensity of the light diffracted by a single droplet at an angle ϕ is given by (6). But the light reaching it, and the light diffracted have both to travel through the thickness of the fog. This total path will not differ appreciably from the total thickness l of the fog, if the angle ϕ is not very large. Hence, from the equation (32) of the paper already cited, the intensity of the light diffracted per unit volume of the fog in the direction ϕ is

$$I = NK^2 \left[\frac{1}{\xi^2} + \frac{J_1^2(\eta)}{\eta^2} - \frac{2J_1(\eta)}{\eta} \cdot \frac{\sin \xi}{\xi}\right] \exp\left\{-4\pi a^2 Nl\left(\frac{1}{2} - \frac{\sin \xi}{\xi}\right)\right\}, \tag{13}$$

where N is the number of droplets per unit volume. If the size of the droplet is fixed, ξ is a constant, and the position of the ring system is determined only by the expression within the square brackets. Neither N, nor l enters this expression, so that the ring system must be uninfluenced by the thickness of the fog. This has already been verified by Barus (1912) and also by Mecke (1920 *a*).

Taking, therefore, the expression within the square brackets, we have already shown that the position of its maxima and minima are given by the

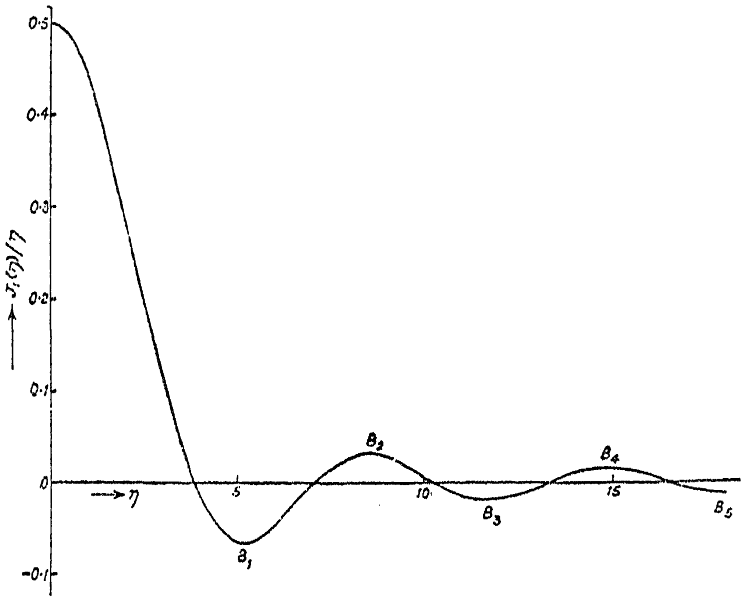

FIG. 1. Graph of $J_1(\eta)/\eta$ against η

roots of the equations (*a*) and (*b*). In the following discussion we shall limit ourselves to the first two or three rings, since, in general, the outer ones would be faint. Also, from Appendix III, it will be seen that the series for C and S are not convergent if $\eta > \xi$, so that this discussion cannot be extended to values of $\eta > \xi$. This corresponds to $\sin\phi > 0 \cdot 667$, or $\phi > 41° 49'$. Since the coronæ usually observed do not extend beyond about 20°, this does not present any serious difficulty.

A careful analysis of the case shows that the existence and the number of the roots of (a) and (b) depend very much on the value of $k = \sin(\xi)/\xi$, and that there are four distinct cases. Denote by B_1, B_2, ... the maximum and minimum values of $J_1(\eta)/\eta$ which correspond to $\eta = b_1, b_2, b_3, \ldots$. The course of the curve $J_1(\eta)/\eta$ is shown in Fig. 1, and the values of the B's and the b's are tabulated in Table I below.

TABLE I

$b_1 =$ 5·14	$b_2 =$ 8·42	$b_3 =$ 11·62	$b_4 =$ 14·78	$b_5 =$ 17·96
$B_1 = -0 \cdot 0662$	$B_2 = +0 \cdot 0322$	$B_3 = -0 \cdot 200$	$B_4 = +0 \cdot 0140$	$B_5 = -0 \cdot 0105$

The four cases are as follow:

Case I: k is $+ve$, and $1/2 > k > B_2$.

In this case, equation (a) has only one root, a_1, between zero and b_1, which corresponds to a minimum value of I. Consequently the distribution of maxima and minima are as shown in Table II (a).

Case II: k is $+ve$, and $k < B_2$.

Equation (a) will have 3 roots at $\eta = a_1$, a_2 and a_3, where $a_1 < b_1$, and a_2 and a_3 are on either side of b_2. The arrangement of the maxima and minima is tabulated in Table II (b).

Case III: k is $-ve$, and $|k| < |B_1|$

In this case, equation (a) has two roots, a_1 and a_2, on either side of b_1, which give rise to the arrangement listed in Table II (c).

TABLE II

(a) Case I ..	Max.	0		b_1				b_3	
	Min.		a_1			b_2			b_5
(b) Case II ..	Max.	0		b_1		b_2		b_3	
	Min.		a_1		a_2		a_3		b_4
(c) Case III ..	Max.	0		b_1		b_2			b_4
	Min.		a_1		a_2			b_3	
(d) Case IV ..	Max.	0				b_2			b_4
	Min.			b_1				b_3	

Case IV: k is $-ve$, and $|k| > |B_1|$

Here, (a) has no roots at all, and the maxima and minima occur at $\eta = b_1, b_2, \ldots$ as shown in II (d).

The dotted lines are intended to show that the maxima within them from a sort of gradual transition from the case before to the next case. This point will become clear if the intersection points of the graph $y = k$ and $y = J_1(\eta)/\eta$ are studied. In case I, there is no intersection point beyond b_1; but as k decreases, case II becomes operative. At first, a_2 and a_3 are close together, so that the maximum at b_2 will not be prominent. The reverse happens in case III, the first maximum becoming less and less pronounced, until it becomes a minimum in case IV.

It is thus clear that the nature of the ring system depends very much on the value of k, so that even a small fluctuation in the radius of the droplets produces a large change in the disposition of the rings. It is therefore interesting to see how the rings behave as the particle size steadily increases. Suppose that the wavelength λ of the light is a constant. Let us begin with the value $a/\lambda = 2\cdot25$ for which $\xi = 3\pi$. This is a case midway between cases II and III. There will be maxima at $\eta = b_1, b_2$, etc., and the rings will appear to be close together. As ξ increases, k becomes negative, case III becomes operative, and the first ring becomes fainter and fainter until it disappears for $|k| > |B_1|$. The whole ring system will now appear to have expanded. This continues for some time, after which k numerically decreases, and the changes take place in the reverse direction, the rings coming close together at $\xi = 4\pi$. Thereafter k is $+ve$, and the second ring becomes more and more indistinct until it disappears when k is $> B_2$. This continues for some time, after which the second ring reappears, the initial stage being reached once more at $\xi = 5\pi$. This finishes one cycle of changes. Another cycle, exactly similar to the above takes place in the range of values 5π to 7π of ξ, except that, in this cycle, case IV cannot be realised at all, since $|\sin(\xi)/\xi|$ is never $> |B_1|$. This continues upto $\xi = 9\pi$, after which case I cannot occur. For droplets of larger radius, *i.e.*, $a/\lambda > 7$, the expanding and contracting of the rings will become less and less noticeable, until, for very large drops, the ring system will correspond to the case of an opaque disk.

In this limiting case of very large drops, $1/\xi^2$ and $\sin(\xi)/\xi$ become intrinsically small, so that (10) reduces to

$$I = K^2 J_1^2(\eta)/\eta^2 \qquad (14)$$

which is the same as the expression for an opaque disk. An interesting point arises in this connection. As shown by equation (34) of the previous paper by the author, the decrease in the forward intensity due to a large droplet is the same as that for an opaque disk of the same size. Thus, for large droplets, the total energy in the corona must be equal to $2\pi a^2$, or double the amount in the corona due to an opaque disk. However, the expression (14) shows that the corona is identical with that given by an opaque disk of the same size. The discrepancy can be explained by the reason that our theory is not valid for large angles of diffraction. The theory only predicts that for small angles of diffraction, the corona is identical with that produced by an opaque disk. There should be an equal amount of energy diffracted at larger angles, of which no account is taken by the theory.

5. *Numerical Computation*

In order to bring out the facts discussed above in a clear manner, the intensity distribution in the corona has been calculated for a few cases, using

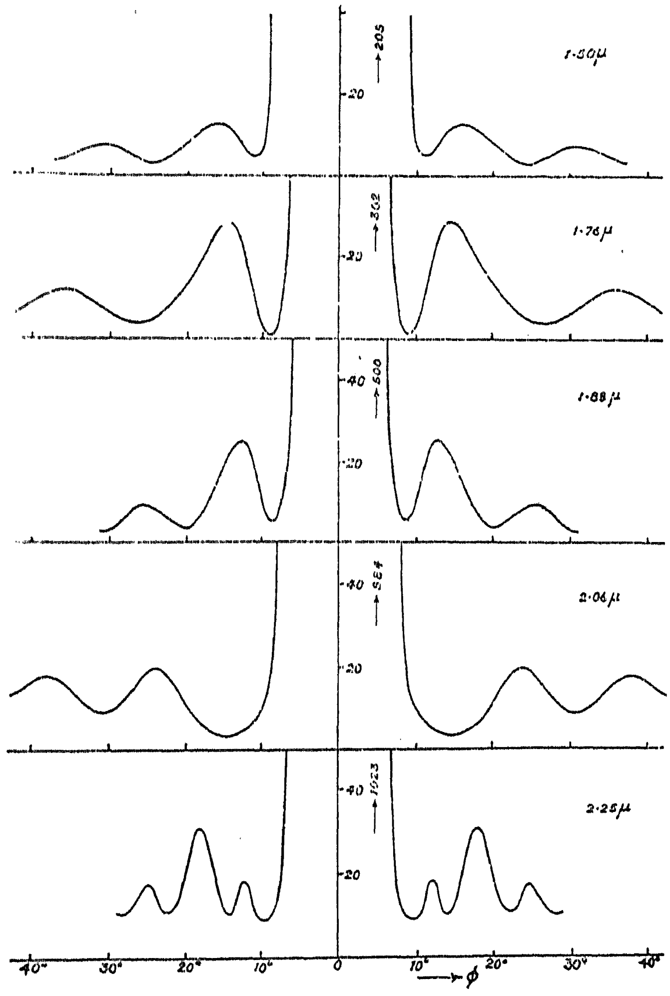

Fig. 2. Intensity Distribution in the Corona

the more accurate expression (7). The wave-length of light was assumed to be 5000 Å.U., and the computation was done for five values of the radius of the drop. In Fig. 2, the results have been plotted with the angle of diffraction as abscissæ, and the intensity as ordinate. The curves clearly show how the nature of the ring system is altered by changes in the size of drops. They also exhibit the contraction and expansion of the rings as the size of the droplets is increased.

6. *Comparison with Experiment*

The theory developed in this paper explains most of the observations made by previous workers in the field. Even very early measurements, such as those of Kamtz (Pernter, 1922) showed that the opaque disk theory is only asymptotic in its application, being more and more correct, the larger the size of the drop, as is to be expected from the theory. Mecke (1920 *a*) and Mitra (1928) have observed that the colour sequence in the outer rings, with a white source, had no special order, and that it was very different even with droplets of slight difference in radii. This is easily understood, since our theory shows that, with monochromatic light, the distribution of the rings is highly susceptible to even small changes in a/λ. The case with monochromatic light has also been verified by Mitra, who found marked changes in the position of the rings as the size of the drop changes.

Mecke has given some values of the relative radii of the rings in different cases. If ϕ_1, ϕ_2, ϕ_3 are the angular radii of the first, second and third dark rings in monochromatic light, then $\sin\phi_1 : \sin\phi_2 : \sin\phi_3$ will give the relative values of η for these. They have been tabulated in Table IV of Mecke's paper (1920 *a*). An examination of it will show that the ratio, $\sin\phi_2/\sin\phi_1$, call it ρ_1 varies from $1 \cdot 8$ to nearly $2 \cdot 4$, but that mostly, it is in the neighbourhood of $2 \cdot 2$. The value of $\sin\phi_3/\sin\phi_1$, call it ρ_2, lies between $2 \cdot 9$ and $3 \cdot 4$. Mecke has also found that ρ_1 varies periodically with the radius of the droplet, and that it becomes less than $1 \cdot 83$, the normal value given by the opaque disk theory, only for a small range in each cycle. All these are easily explained by our theory.

Taking one cycle, we start with $\xi = 3\pi$, for which $k = 0$, and $\rho_1 = 1 \cdot 83$, and $\rho_2 = 2 \cdot 8$, which are the normal values. As ξ increases, ρ_1 will slightly diminish and ρ_2 will increase, as may be seen from Fig. 1. This will occur only for a short while, for the maximum at b_1 will soon be too faint, and the ratios will become $2 \cdot 3$ and $3 \cdot 6$ respectively. This holds for case IV also. If we take case II, the ratios will be in the neighbourhood of $2 \cdot 2$ and $3 \cdot 0$, and on passing to case I, they become larger than this. This explains the range of values of ρ_1 and ρ_2 as given by Mecke's experiments as also the fact that ρ_1 is seldom less than $1 \cdot 83$, since it occurs only for a short range after the transition from case II to case III.

It is with great pleasure that I take this opportunity of expressing my gratitude to Prof. Sir C. V. Raman for suggesting the problem, and for the keen and continued interest that he took in the investigation.

Summary

The commonly accepted theory of coronæ, based on the idea that water droplets act as opaque disks, is not only theoretically unsound, but is not also in accord with experimental facts. The theory due to Mecke is also not satisfactory as it based partly on geometrical optics and partly on the theory of diffraction. In this paper, a new theory of the phenomenon is developed, using only the principles of wave-optics, and taking into consideration the portions of the wave-front transmitted through the droplets. The integrals so obtained are integrated by a suitable method, and expressions are obtained for the intensity distribution in the corona. A discussion of these expressions shows that the theory satisfactorily explains most of the phenomena exhibited by coronæ, such as the extreme susceptibility of the ring system to even small changes in the radius of the drops, and the oscillation of the ring system as the radius steadily increases.

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APPENDIX I

Evaluation of the Integral in the expression for X_1

The integral to be evaluated is

$$A = \int_0^{\pi/2} J_0 (\eta \sin \theta) \sin (\chi - \xi \cos \theta) \sin \theta \cos \theta \, d\theta$$

The integration is to be done in such a way that the resulting function is readily adapted for numerical computation when ξ is large, *i.e.*, when terms in $1/\xi^2$, $1/\xi^3$, etc., can be neglected compared to those in $1/\xi$.

For the purpose of integration, it is convenient to split the integral into two, and write

$$A = \sin \chi \int_0^{\pi/2} J_0 (\eta \sin \theta) \cos (\xi \cos \theta) \sin \theta \cos \theta \, d\theta$$

$$- \cos \chi \int_0^{\pi/2} J_0 (\eta \sin \theta) \sin (\xi \cos \theta) \sin \theta \cos \theta d\theta$$

$$= C \sin \chi - S \cos \chi \qquad \text{(say)}$$

Both C and S can be expanded in the form of a series.

Put $\xi \cos \theta = x$, and $\eta \sin \theta = y$.

$$\text{Then, } \frac{dy}{dx} = - \frac{\eta \cos \theta}{\xi \sin \theta} = - \frac{x}{y} \left(\frac{\eta}{\xi} \right)^2,$$

Also, the limits of integration are given by

$\theta = 0$, *i.e.*, $x = \xi$, $y = 0$

$\theta = \pi/2$, *i.e.*, $x = 0$, $y = \eta$.

$$\text{Hence, } \qquad C = \frac{1}{\xi^2} \int_0^{\xi} J_0 (y) \cos x \, x \, dx$$

Partially integrating with respect to x,

$$C = \frac{1}{\xi^2} \Big[(\cos x + x \sin x) J_0 (y) \Big]_{x=0}^{x=\xi} + \frac{1}{\xi^2} \int_0^{\xi} (\cos x + x \sin x) J_1 (y) \, dy$$

Substituting for dy in terms of dx,

$$C = \frac{1}{\xi^2}\Big[(\cos\xi + \xi\sin\xi) - J_0(\eta)\Big] - \frac{\eta^2}{\xi^4}\int_0^{\xi} (\cos x + x\sin x)\cdot x\cdot J_1(y)/y.dx$$

Again, partially integrating with respect to x, the second term becomes

$$= -\frac{\eta^2}{\xi^4}\Big[(3\cos x + 3x\sin x - x^2\cos x)\cdot J_1(y)/y\Big]_0^{\xi}$$

$$-\frac{\eta^2}{\xi^4}\int_0^{\xi}(3\cos x + 3x\sin x - x^2\cos x)\cdot\frac{J_2(y)}{y}\,dy$$

Since $\underset{y\to 0}{Lt}\ \dfrac{J_1(y)}{y} = \dfrac{1}{2}$, this is

$$= -\frac{\eta^2}{\xi^4}\Big[\tfrac{1}{2}(3\cos\xi + 3\xi\sin\xi - \xi^2\cos\xi) - \frac{3J_1(y)}{y}\Big]$$

$$+\frac{\eta^4}{\xi^6}\int_0^{\xi}(3\cos x + 3x\sin x - x^2\cos x)\cdot x\cdot J_2(y)/y^2\cdot dx.$$

Continuing this process, we can finally put C in the form

$$C = \frac{1}{\xi^2}(\cos\xi + \xi\sin\xi)$$

$$-\frac{\eta^2}{2\xi^4}(3\cos\xi + 3\xi\sin\xi - \xi^2\cos\xi)$$

$$+\frac{\eta^4}{2^2\,2!\,\xi^6}(15\cos\xi + 15\xi\sin\xi - 6\xi^2\cos\xi - \xi^3\sin\xi)$$

$$-\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(-1)^p\frac{\eta^{2p-2}}{2^{p-1}(p-1)!\,\xi^{2p}}(a_{p,p}\cos\xi + a_{p,p-1}\xi\sin\xi$$

$$- a_{p,p-2}\xi^2\cos\xi - \cdots)$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-\frac{1}{\xi^2}\Big[J_0(\eta) - \frac{3\eta^2}{\xi^2}\frac{J_1(\eta)}{\eta} + \frac{15\eta^4}{\xi^4}\frac{J_2(\eta)}{\eta^2} - \ldots$$

$$+(-1)^{p-1}a_{p,p}\left(\frac{\eta}{\xi}\right)^{2p-2}\frac{J_{p-1}(\eta)}{\eta^{p-1}} + \cdots\Big]$$

where $a_{p,q} = \dfrac{1}{2^q(q)!}\cdot\dfrac{(p+q)!}{(p-q)!}$ (*Vide* Appendix II)

In an exactly similar manner, S can also be put in the form of a series as below:

$$S = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}\eta^{2p-2}}{2^{p-1}(p-1)!\,\xi^{2p}} (a_{p,\,p} \sin\xi - a_{p,\,p-1}\,\xi\cos\xi - a_{p,\,p-2}\,\xi^2 \sin\xi + a_{p,\,p-3}\,\xi^3 \cos\xi + \ldots).$$

The above two series give the complete expansion of the integrals C and S. Ordinarily, for the range of particle sizes in which we are interested, ξ is large, and only terms in $1/\xi$ need be taken. Then,

$$A = \sin\chi \cdot \sin\xi/\xi + \cos\chi \cdot \cos\xi/\xi, \text{ so that}$$

$$I = K^2[1/\xi^2 + J_1^2(\eta)/\eta^2 - 2J_1(\eta)/\eta \cdot \sin\xi/\xi].$$

This first approximation is not sufficiently accurate if η is comparable with ξ. Then, the last terms inside the brackets in the right hand side of C must be taken into account. C then becomes

$$C = \frac{\sin\xi}{\xi}\left[1 - \frac{1}{2^2(2)!}\cdot\frac{\eta^4}{\xi^2} + \frac{1}{2^4(4)!}\cdot\frac{\eta^8}{\xi^4} - \cdots\right] + \frac{\cos\xi}{\xi}\left[\frac{\eta^2}{2\xi} - \frac{1}{2^3(3)!}\cdot\frac{\eta^6}{\xi^3} + \cdots\right] = \frac{1}{\xi}\cdot\sin\left(\xi + \frac{\eta^2}{2\xi}\right).$$

Similarly, to a second approximation, $S = \frac{1}{\xi}\cos\left(\xi + \frac{\eta^2}{2\xi}\right)$, so that

$$I = K^2\left[\frac{1}{\xi^2} + \frac{J_1^2(\eta)}{\eta^2} - 2\,\frac{J_1(\eta)}{\eta}\cdot\frac{\sin(\xi + \eta^2/2\xi)}{\xi}\right].$$

If a further approximation were required, it can be obtained by taking all the penultimate terms inside the brackets in the right hand side of C and S. They can also be summed, and lead to the result that, to a third approximation,

$$C = \frac{1}{\xi}\cdot\sin(\xi + \eta^2/2\xi) + \frac{1}{\xi^2}\left[\cos(\xi + \eta^2/2\xi) + \frac{\eta^2}{\xi}\sin(\xi + \eta^2/2\xi) - \frac{\eta^6}{8\xi^3}\cos(\xi + \eta^2/2\xi)\right] \text{ and}$$

$$S = \frac{1}{\xi}\cos(\xi + \eta^2/2\xi) + \frac{1}{\xi^2}\left[\sin(\xi + \eta^2/2\xi) - \frac{\eta^2}{\xi}\cos(\xi + \eta^2/2\xi) - \frac{\eta^6}{8\xi^3}\sin(\xi + \eta^2/2\xi)\right].$$

Any desired degree of approximation can thus be obtained by taking further terms in the expansion, but one more than the second degree will not be needed practically.

APPENDIX II

To prove that $a_{p,q} = (p+q)!/2^q(q)!(p-q)!$

The quantities $a_{p,q}$ depend on the coefficients of the terms in the expansion of the definite integrals, $\int_0^{\xi} x^p \cos x\,dx$, and $\int_0^{\xi} x^p \sin x\,dx$. If we call these respectively C_p and S_p, then

$$C_p = p_0\,\xi^p \sin\xi + p_1\,\xi^{p-1}\cos\xi - p_2\,\xi^{p-2}\sin\xi - \ldots .$$
$$S_p = -p_0\,\xi^p \cos\xi + p_1\,\xi^{p-1}\sin\xi + p_2\,\xi^{p-2}\cos\xi - \ldots .$$

where two terms are $+ve$, the next two are $-ve$ and so on. The series on the right hand side has $(p+1)$ terms each, the last coefficient being p_p. The values of the coefficients are easily seen to be

$$p_q = (p)!/(p-q)!$$

It is to be noted that $p_p = p_{p-1}$.

From the method of partial integration used in Appendix I to determine the series for C, it can be verified that the laws of formation of the numbers $a_{p,q}$ are given by the following:

$$a_{p,0} = p_0\,a_{p-1,0}$$
$$a_{p,1} = (p-1)_0\,a_{p-1,1} + p_1\,a_{p-1,0}$$
$$a_{p,2} = (p-2)_0\,a_{p-1,2} + (p-1)_1\,a_{p-1,1} + p_2\,a_{p-1,0}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{p,q} = (p-q)_0\,a_{p-1,q} + (p-q+1)_1\,a_{p-1,q-1} + \ldots + p_q\,a_{p-1,0}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{p,p-1} = 1_0\,a_{p-1,q-1} + 2_1\,a_{p-1,p-2} + \ldots + p_{p-1}\,a_{p-1,0}$$

and $a_{p,p} = a_{p,p-1}$.

From these substituting the values of p_q, we get,

(1) $a_{p,0} = a_{p-1,0}$, so that $a_{p,0} = 1$, since $a_{1,0} = 1$.

(2) $a_{p,1} = a_{p-1,1} + p\,a_{p-1,0} = a_{p-1,1} + p$. Also, $a_{1,1} = 1$, so that

$$a_{p,1} = \sum_1^p n.$$

(3) $a_{p,2} = a_{p-1,2} + (p-1)\,a_{p-1,1} + p(p-1)\,a_{p-1,0}$

$$= a_{p-1,2} + (p-1)\sum_1^{p-1} n + p(p-1)$$

$$= a_{p-1,2} + (p-1)\sum_1^{p} n$$

Hence, $a_{p,2} = a_{2,2} + \sum_3^p (m-1)\sum_1^m n.$

But, $a_{2,2} = a_{2,1} = \sum_1^2 n = (2-1)\sum_1^2 n$, so that

$$a_{p,2} = \sum_1^p (m-1)\sum_1^m n.$$

(4) $a_{p,3} = a_{p-1,3} + (p-2)\,a_{p-1,2} + (p-1)(p-2)\,a_{p-1,1} + p(p-1)(p-2)\,a_{p-1,0}$

$$= a_{p-1,3} + (p-2)\sum_1^{p-1}(m-1)\sum_1^m n + (p-1)(p-2)\sum_1^{p-1} n + p(p-1)(p-2)$$

$$= a_{p-1,3} + (p-2)\sum_1^p (m-1)\sum_1^m n.$$

Hence, $a_{p,3} = a_{3,3} + \sum_4^p (r-2)\sum_1^r (m-1)\sum_1^m n$.

But, $a_{3,3} = a_{3,2} = \sum_1^3 (m-1)\sum_1^m n = (3-2)\sum_1^3 (m-1)\sum_1^m n$, so that

$$a_{p,3} = \sum_1^p (r-2)\sum_1^r (m-1)\sum_1^m n.$$

Proceeding in this way, it is easily seen that

$$a_{p,q} = \sum_1^p (t-q+1)\sum_1^t (s-q+2)\cdots(r-2)\sum_1^r (m-1)\sum_1^m n.$$

The expression on the right hand side can be summed up by using the well-known summation:

$$\sum_{x=1}^n (a+x)(a+x+1)\cdots(a+x+r-1) = \frac{1}{(r+1)}(a+n)(a+n+1)\cdots(a+n+r).$$

We then get, $\sum_1^m n = m(m+1)/2 = (m+1)!/2\,(m-1)!$

$$\sum_1^r (m-1)\sum_1^m n = \sum_1^r (m-1)\,m\,(m+1)/2 = (r+2)!/2^2\,(2)!\,(r-2)!, \text{ etc.}$$

Thus, $a_{p,0} = 1$; $a_{p,1} = (p+1)!/(2)\cdot(p-1)!$;

$a_{p,2} = (p-2)!/2^2\,(2)!\,(p-2)!$; $a_{p,3} = (p+3)!/2^2\,(3)!\,(p-3)!$;

and so on, so that

$$a_{p,q} = (p+q)!/2^q\,(q)!\,(p-q)!$$

It is to be noted that this relation automatically satisfies the equation $a_{p,p} = a_{p,p-1}$.

APPENDIX III

A Note on the Convergency of the Series for C and S

It can be shown that both the integrals C and S, expressed in the form of the series we have chosen, are convergent if $\eta < \xi$. Taking first C, we can express it as follows:

$$C = \sin\xi\,\{a_{1,0}/\xi - (a_{2,1}\eta^2/2 + a_{3,0}\eta^4/2^2(2)!)/\xi^3 + (a_{3,2}\eta^4/2^2(2)! + a_{4,1}\eta^6/2^3(3)! + a_{5,0}\eta^8/2^4(4)!)/\xi^5 - \cdots\cdots\}$$

$$+ \cos\xi\,\{(a_{1,1} + a_{2,0}\eta^2/2)/\xi^2 - (a_{2,2}\eta^2/2 + a_{3,1}\eta^4/2^2(2)! + a_{4,0}\eta^6/2^3(3)!)/\xi^4 + \cdots\cdots\}$$

$$- \frac{1}{\xi^2}\left\{J_0(\eta) - a_{2,2}\frac{\eta^2}{\xi^2}\frac{J_1(\eta)}{\eta} + a_{3,3}\frac{\eta^4}{\xi^4}\frac{J_2(\eta)}{\eta^2} - \cdots\cdots\right\}$$

$$= A\sin\xi + B\cos\xi - J/\xi^2 \text{ (say)}$$

The series S can also be expressed similarly as

$$S = A\cos\xi - B\sin\xi.$$

It will now be shown that A, B and J are convergent if $\eta < \xi$. Taking A, if u_p be its p^{th} term, then

$$\left|\frac{u_{p+1}}{u_p}\right| = \frac{1}{\xi^2}\left[\frac{a_{p+1,p}\,\eta^{2p}/2^p(p)! + a_{p+2,p-1}\,\eta^{2p+2}/2^{p+1}(p+1)! + \cdots}{a_{p,p-1}\,\eta^{2p-2}/2^{p-1}(p-1)! + a_{p+1,p-2}\,\eta^{2p}/2^p(p)! + \cdots}\right]$$

which can be put in the form

$$\left|\frac{u_{p+1}}{u_p}\right| = \frac{\eta^2}{\xi^2}\left[\frac{\dfrac{p(p+\frac{1}{2})}{p^2} + \dfrac{p(p+\frac{1}{2})}{p(p+1)}\,\dfrac{\eta^2}{(3)!} + \dfrac{p(p+\frac{1}{2})(p-1)}{p(p+1)(p+2)}\cdot\dfrac{\eta^4}{(5)!} + \cdots}{1 + \dfrac{p-1}{p}\cdot\dfrac{\eta^2}{(3)!} + \dfrac{(p-1)(p-2)}{p(p+1)}\cdot\dfrac{\eta^4}{(5)!} + \cdots}\right]$$

Hence, $\underset{p\to\infty}{Lt}\left|\dfrac{u_{p+1}}{u_p}\right| = \dfrac{\eta^2}{\xi^2}$. If $\eta < \xi$, this is < 1, and the series is convergent. A similar test shows that for B also the same cnodition holds good.

Taking J, if v_p be its pth term, then

$$\left|\frac{v_{p+1}}{v_p}\right| = \frac{\eta}{\xi^2}\cdot\frac{a_{p+1,p+1}}{a_{p,p}}\,\frac{J_p(\eta)}{J_{p-1}(\eta)}.$$

Substituting for the a's, and also using the relation that, when p is large, $J_{p+1}(\eta)/J_p(\eta) = \eta/2(p+1)$, we get

$$\left|\frac{v_{p+1}}{v_p}\right| = \frac{2p+1}{2p} \cdot \frac{\eta^2}{\xi^2}.$$

Hence, $\underset{p\to\infty}{\mathrm{L}t}\left|\frac{v_{p+1}}{v_p}\right| = \frac{\eta^2}{\xi^2}$, so that J is convergent if $\eta < \xi$.

Thus, the expansion of the integral for X_1 in the form of the series C and S is convergent, and hence, valid, if $\eta < \xi$.