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Relative Curves, Theta Divisor, and Deligne Pairing

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1 Introduction

Let X be a compact Riemann surface of genus $g \geq 1$. Let J denote the component of the Picard group of X consisting of all line bundles of degree $g-1$ on X. Fix a line bundle α on X of degree $g-1$.

Fix an integer $m \geq 1$, and consider the following map from the $2m$-fold Cartesian product

$$\phi : X^{2m} \longrightarrow J \tag{1.1}$$

defined by $(x_1, x_2, \ldots, x_{2m}) \longmapsto \alpha \otimes \mathcal{O}(\sum_{i=1}^{m} x_i - x_{i+m})$. (We will use the same notation for the sheaf given by a divisor and the line bundle corresponding to it; $\mathcal{O}(D)_p$ will denote the fiber of the line bundle at p. The dual of a line bundle L will be denoted by L^{-1}.) On J there is a canonical theta divisor given by $\{\xi \in J \mid H^0(X, \xi) \neq 0\}$. We will use the notation Θ for the theta divisor as well as for the line bundle on J given by it.

Let $p_i : X^{2m} \longrightarrow X$, $1 \leq i \leq 2m$, be the projection onto the ith factor. For $1 \leq i < j \leq 2m$, let $D_{i,j} \subset X^{2m}$ be the divisor given by $(p_i \times p_j)^*\Delta$, where $\Delta \subset X \times X$ is the diagonal. The canonical bundle of X is denoted by K. Consider the following line bundle on X^{2m}:

$$\begin{aligned}\mathcal{M}_\alpha(m, m) := \bigotimes_{i=1}^{m} \left(p_i^*(K \otimes \alpha^{-1}) \otimes p_{i+m}^*(\alpha)\right) \bigotimes \left(\sum_{i \leq m}^{m<j} D_{i,j}\right) \\ \bigotimes \left(\sum_{i<j}^{j \leq m} -D_{i,j} - D_{i+m,j+m}\right).\end{aligned} \tag{1.2}$$

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In [R1, Theorem 11.1] the following result was proved: The pullback bundle $\phi^*\Theta$ on X^{2m} is isomorphic to $\mathcal{M}_\alpha(m, m)$. For the case $m = 1$, this result was proved in [K]. Moreover, if $H^0(X, \alpha) = 0$, then

$$\dim H^0(X^{2m}, \mathcal{M}_\alpha(m, m)) = 1$$

[R1, Theorem 11.2]. (More generally, $\dim H^0(X^{2m}, \mathcal{M}_\alpha(m, m)) = 1$ also for α a smooth point of the theta divisor [R3].) It was then shown in [R1] (see [R2] or [LB] for an exposition) how these results lead to a proof of the Fay trisecant identity for theta functions [F].

Our aim here is to generalize the above results on the pair (X, α) to any family of the form (X_T, L_T), where $X_T \longrightarrow T$ is a family of Riemann surfaces parametrized by T and $L_T \longrightarrow X_T$ is a line bundle of relative degree $g-1$. Our approach is not a routine extension of the earlier one: even in the special case where T is a point, we obtain a completely new proof of the earlier results in [R1] and [R2], which provides a new mathematical insight into the earlier results which were motivated by physics. Moreover, it turns out that for a family, the pullback of the theta divisor is not isomorphic to the obvious generalization of $\mathcal{M}_\alpha(m, m)$: they differ by the pullback of a line bundle on the parameter space. This line bundle is given by the restriction of the theta bundle to a subvariety of the relative Jacobian given by L_T. Of course, the restriction of this bundle for a pair (X, α) is trivial. (See Theorem 2.2 and Remark 2.3 for precise statements.)

In Section 3 we first prove that the space of sections of the line bundle which generalizes $\mathcal{M}_\alpha(m, m)$ to a family is canonically identified with the space of regular functions on the parameter space (Theorem 3.1). *Theorem 3.3, which is obtained using this result, can be interpreted as the relative version of the Fay trisecant identity.*

Our proof is based on a general construction of P. Deligne [D], which gives a bilinear map from pairs of line bundles on a family of curves to line bundles on the parameter space.

2 Generalization to a family

Let T be a scheme of finite type over $\mathbb{C}$. Let $\pi : \mathcal{X} \longrightarrow T$ be a proper smooth family of geometrically connected curves of genus $g \geq 1$. Let $F : \mathcal{J} \longrightarrow T$ be the relative Jacobian of line bundles of degree $g-1$. Assume that there is a relative Poincaré bundle $\mathcal{P}$ on the fiber product $\mathcal{X} \times_T \mathcal{J}$. The relative theta divisor on $\mathcal{J}$ is denoted by $\bar{\Theta}$. Also assume that we are given a section $\bar{f} : T \longrightarrow \mathcal{J}$ of F, i.e., $F \circ \bar{f} = \mathrm{Id}$. The map $\bar{f}$ induces the map $f : \mathcal{X} \longrightarrow \mathcal{X} \times_T \mathcal{J}$ over T defined by $x \longmapsto (x, (\bar{f} \circ \pi)(x))$.

Example. Given a smooth family of curves, $\gamma : \mathcal{Z} \longrightarrow U$, we can construct a family of curves π satisfying the above conditions as follows. Let $\rho : \mathcal{J}' \longrightarrow U$ be the relative

Jacobian of line bundles of degree $g-1$. Using the map ρ we may pull back the family of curves on U to $\mathcal{J}'$. It is easy to check that the projection onto the second factor $\mathcal{Z}\times_U\mathcal{J}' \longrightarrow \mathcal{J}'$ gives this pullback family on $\mathcal{J}'$. Take T to be $\mathcal{Z}\times_U\mathcal{J}'$ and $\mathcal{X}$ to be the fiber product $\mathcal{Z}\times_U T$, and let π be the projection onto the second factor. It is easy to see that the relative Jacobian for this family of curves given by π admits a tautological section $\bar{f}$. (The evaluation of $\bar{f}$ at $z\times j\in\mathcal{Z}\times_U\mathcal{J}'=T$ is $j\times z\times j\in\mathcal{J}'\times_U T$.) We claim that there is a relative Poincaré bundle on the fiber product of $\mathcal{X}$ with the relative Jacobian, i.e., on $\mathcal{X}\times_T(\mathcal{J}'\times_U T)$. To prove the claim, first note that for any smooth family of curves there is a canonical Poincaré bundle on the fiber product of the family with the relative Jacobian of degree-$(g-2)$ line bundles. Indeed, the pullback of the theta bundle on the relative Jacobian of degree-$(g-1)$ line bundles using the obvious map from the above fiber product is the Poincaré bundle. But the family of curves given by π admits a natural section. Using this section, any two relative Jacobians (corresponding to different degrees) are naturally identified. This proves the above claim. Thus the family given by π satisfies all the above conditions.

The $2m$-fold fiber product of $\mathcal{X}$ with itself is denoted by $\mathcal{X}^{2m}$. The projection $\mathcal{X}^{2m}\longrightarrow\mathcal{X}$ to the ith factor is denoted by $\bar{p}_i$. Let ν_1 (resp. ν_2) be the projection of $\mathcal{Y}:=\mathcal{X}\times_T\mathcal{X}^{2m}$ onto $\mathcal{X}$ (resp. $\mathcal{X}^{2m}$). The projection ν_2 defines a family of curves on $\mathcal{X}^{2m}$. In fact, it is the pullback to $\mathcal{X}^{2m}$ of the family of curves on T by the obvious projection

$$q:\mathcal{X}^{2m}\longrightarrow T.$$

Let $\Delta\subset\mathcal{X}\times_T\mathcal{X}$ be the divisor given by the diagonal. For $1\le i\le 2m$, define the following divisor on $\mathcal{Y}$:

$$\mathcal{D}_i:=(\mathrm{Id}\times\bar{p}_i)^*\Delta.$$

Consider the following line bundle on $\mathcal{Y}$:

$$\mathcal{L}:=\left((f\circ\nu_1)^*\mathcal{P}\right)\otimes\mathcal{O}\left(\sum_{i=1}^{m}\mathcal{D}_i-\mathcal{D}_{i+m}\right). \tag{2.1}$$

The bundle $\mathcal{L}$ gives the (classifying) morphism

$$\Phi:\mathcal{X}^{2m}\longrightarrow\mathcal{J}. \tag{2.2}$$

We recall the definition of the classifying morphism: For $z\in\mathcal{X}^{2m}$ over $t\in T$, the image $\Phi(z)$ is the point on the Jacobian of the curve $\pi^{-1}(t)$ given by restriction of the line bundle $\mathcal{L}$ to $\pi^{-1}(t)\times z$. This map Φ is the obvious generalization of the map ϕ in (1.1).

There is a natural relative version of the divisor $D_{i,j}$ as a divisor $\mathcal{D}_{i,j}$ on $\mathcal{X}^{2m}$. Let $\mathcal{K}$ denote the relative canonical bundle on $\mathcal{X}$. Consider the bundle

$$\mathcal{F} := \bigotimes_{i=1}^{m} \left(\bar{p}_i^*(\mathcal{K} \otimes f^*\mathcal{P}^{-1}) \otimes \bar{p}_{i+m}^*(f^*\mathcal{P})\right) \tag{2.3}$$

on $\mathcal{X}^{2m}$. Let $\mathcal{P}'$ be another Poincaré bundle on $\mathcal{X} \times_T \mathcal{J}$, and let $\mathcal{F}'$ be the corresponding bundle on $\mathcal{X}^{2m}$.

Proposition 2.1. The two line bundles $\mathcal{F}$ and $\mathcal{F}'$ on $\mathcal{X}^{2m}$ are canonically isomorphic. □

Note that by "canonical isomorphism" we shall always mean that there is a given isomorphism which is compatible with base change.

Proof. There is a line bundle ξ on $\mathcal{J}$ such that $\mathcal{P}' = \mathcal{P} \otimes p_2^*\xi$, where $p_2 : \mathcal{X} \times_T \mathcal{J} \longrightarrow \mathcal{J}$ is the projection. So on $\mathcal{X}$ the bundle

$$f^*\mathcal{P}' = (f^*\mathcal{P}) \otimes ((p_2 \circ f)^*\xi) = (f^*\mathcal{P}) \otimes \pi^*(\bar{f}^*\xi),$$

where $q = \pi \circ \bar{p}_i$ is the projection of $\mathcal{X}^{2m}$ onto T defined earlier. For any $1 \le i \le 2m$, we have

$$\bar{p}_i^*(f^*\mathcal{P}') = (\bar{p}_i^*(f^*\mathcal{P})) \otimes q^*(\bar{f}^*\xi).$$

Now substituting, in the definition (2.3), the above equation between $\bar{p}_i^*(f^*\mathcal{P}')$ and $\bar{p}_i^*(f^*\mathcal{P}')$, and also the relation between the duals given by it, we get that the bundle $\mathcal{F}'$ is canonically isomorphic to $\mathcal{F}$. ■

Define the line bundle

$$\mathcal{M} := (\bar{f} \circ q)^*\bar{\Theta} \otimes \mathcal{F} \otimes \mathcal{O}\left(\sum_{i \le m}^{m<j} \mathcal{D}_{i,j}\right) \otimes \mathcal{O}\left(\sum_{i<j}^{j \le m} -\mathcal{D}_{i,j} - \mathcal{D}_{i+m,j+m}\right) \tag{2.4}$$

on $\mathcal{X}^{2m}$.

Theorem 2.2. The pullback bundle $\Phi^*\bar{\Theta}$ on $\mathcal{X}^{2m}$ is canonically isomorphic to $\mathcal{M}$. □

Remark. The bundle $\Phi^*\bar{\Theta}$ clearly does not depend on the choice of the Poincaré bundle. The bundle $\mathcal{M}$ also does not depend upon the Poincaré bundle, because of Proposition 2.1.

Proof of Theorem 2.2. Let $\gamma : \mathcal{Z} \longrightarrow U$ be a smooth family of curves. Given two line bundles L and L' on $\mathcal{Z}$, the construction of Deligne [D] gives a line bundle on U; this bundle on U is denoted by $\langle L, L' \rangle$. The line bundle

$$d(L) := \bigotimes_i \left(\det R^i\gamma_*(L)\right)^{(-1)^i}$$

on U is called the *determinant* of L. (The sheaf $R^i\gamma_*(L)$ is an $\mathcal{O}_U$ coherent sheaf on U, and $\det R^i\gamma_*(L)$ is the determinant of this coherent sheaf. See [Ko, Chapter V, Section 6] for the definition of the determinant of a coherent sheaf.) We reproduce Lemma 6, Section 2, of [BM]:

$$\langle L, L'\rangle = d(L\otimes L')\otimes d(\mathcal{O}_{\mathcal{Z}})\otimes d(L)^{-1}\otimes d(L')^{-1}. \tag{2.6}$$

The line bundle given by the theta divisor on the Jacobian is the dual of the determinant of a Poincaré bundle. Since the determinant bundle is compatible with base change [KM], we have

$$\Phi^*\bar{\Theta} = d(\mathcal{L})^{-1}, \tag{2.7}$$

where $d(\mathcal{L})$ is the determinant bundle of $\mathcal{L}$ for the family given by ν_2.

Let ζ (resp. η) denote the bundle $(f\circ\nu_1)^*\mathcal{P}$ (resp. $\mathcal{O}(\sum_{i=1}^m \mathcal{D}_i - \mathcal{D}_{i+m})$) on $\mathcal{Y}$. From (2.6), we have

$$d(\mathcal{L}) = d(\zeta\otimes\eta) = \langle\zeta,\eta\rangle\otimes d(\mathcal{O}_{\mathcal{Y}})^{-1}\otimes d(\zeta)\otimes d(\eta). \tag{2.8}$$

Using the compatibility of the determinant bundle with base change, we have

$$d(\zeta) = (\bar{f}\circ q)^*\bar{\Theta}^{-1}. \tag{2.9}$$

From Proposition 5a, 5c, Section 2, of [BM], we have the following: Let L be a line bundle on a Riemann surface X, and $D = \sum_i a_i - \sum_j b_j$ a divisor on X. (The points $a_i, b_j \in X$ may not be all distinct.) Then

$$\langle L, \mathcal{O}(D)\rangle = (\otimes_i L_{a_i})\otimes(\otimes_j L_{b_j}^{-1}), \tag{2.10}$$

where L_{a_i} is the fiber of L over a_i.

For each i ($1\le i\le 2m$) there is a natural section $s_i : \mathcal{X}^{2m}\longrightarrow\mathcal{Y}$ of ν_2, and clearly, $\nu_1\circ s_i = \bar{p}_i$. Using (2.10), we have

$$\langle\zeta,\eta\rangle = \bigotimes_{i\le m}^{j>m}\left((s_i^*\zeta)\otimes s_j^*\zeta^{-1}\right) = \bigotimes_{i\le m}^{j>m}\left((f\circ\bar{p}_i)^*\zeta\right)\otimes\left((f\circ\bar{p}_j)^*\zeta^{-1}\right). \tag{2.11}$$

Consider the line bundles $\eta_1 := \mathcal{O}(\sum_{i=1}^m \mathcal{D}_i)$ and $\eta_2 := \mathcal{O}(\sum_{i=m}^{2m} -\mathcal{D}_i)$ on $\mathcal{Y}$, with $\eta = \eta_1\otimes\eta_2$. Applying (2.6) to η, we have

$$d(\eta) = \langle\eta_1,\eta_2\rangle\otimes d(\eta_1)\otimes d(\eta_2)\otimes d(\mathcal{O}_{\mathcal{Y}})^{-1}. \tag{2.12}$$

Let X be a compact Riemann surface X, and $D := \sum_{k=1}^{n} x_k$ a divisor on X. Repeatedly using (2.6) and (2.10), we get

$$d(\mathcal{O}_X(D)) = \left(\bigotimes_{1\le k<l\le n} \mathcal{O}(x_k)_{x_l}\right) \bigotimes_{r=1}^{n} \left(d(\mathcal{O}(x_r))\right) \bigotimes d(\mathcal{O})^{\otimes(1-n)}.$$

Similarly, we have

$$d(\mathcal{O}_X(-D)) = \left(\bigotimes_{1\le k<l\le n} \mathcal{O}(x_k)_{x_l}\right) \bigotimes_{r=1}^{n} \left(d(\mathcal{O}(-x_r))\right) \bigotimes d(\mathcal{O})^{\otimes(1-n)}.$$

For $a \in X$, the two exact sequences on X

$$0 \longrightarrow \mathcal{O}_X(-a) \longrightarrow \mathcal{O}_X \longrightarrow \mathbb{C} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(a) \longrightarrow (K^{-1})_a \longrightarrow 0$$

show that $d(\mathcal{O}(-a)) = d(\mathcal{O})$ and $d(\mathcal{O}(a)) = d(\mathcal{O}) \otimes (K^{-1})_a$.

From the above it now follows that

$$d(\eta_1) = \mathcal{O}\left(\sum_{i,j}^{i<j\le m} \mathcal{D}_{i,j}\right) \bigotimes_{i=1}^{m} \left(\bar{p}_i^* \mathcal{K}^{-1}\right) \bigotimes d(\mathcal{O}_y)$$

$$d(\eta_2) = \mathcal{O}\left(\sum_{i,j}^{m<i<j} \mathcal{D}_{i,j}\right) \bigotimes d(\mathcal{O}_y). \tag{2.13}$$

From (2.10) and the bilinearity of the pairing (Proposition 5a, Section 2, of [BM]), we have

$$\langle \eta_1, \eta_2 \rangle = \mathcal{O}\left(\sum_{i\le m}^{m<j} -\mathcal{D}_{i,j}\right). \tag{2.14}$$

Substituting in (2.8) the expressions of the different factors obtained in (2.9), (2.11), (2.12), (2.13), and (2.14) and noting (2.7), we get the identification of the pullback bundle $\Phi^*\bar{\Theta}$ with $\mathcal{M}$. ∎

Remark 2.3. The bundle $(\bar{f} \circ q)^*\bar{\Theta}$ being the pullback of a line bundle on T, if T is a single point, then it is trivializable. Hence, Theorem 2.2 implies Theorem 11.1 of [R1] mentioned in the introduction. In the complement of the theta divisor $\bar{\Theta}$, the line bundle $\bar{\Theta}$ has a canonical trivialization. Hence, if the image of the map $\bar{f} \circ q$ does not intersect $\bar{\Theta}$, then the line bundle $(\bar{f} \circ q)^*\bar{\Theta}$ has a canonical trivialization. In that case, Theorem 2.2 and (2.4) imply that

$$\Phi^*\bar{\Theta} = \mathcal{F} \otimes \mathcal{O}\left(\sum_{i\le m}^{m<j} \mathcal{D}_{i,j}\right) \otimes \mathcal{O}\left(\sum_{i<j}^{j\le m} -\mathcal{D}_{i,j} - \mathcal{D}_{i+m,j+m}\right).$$

3 Sections of $\Phi^*\overline{\Theta}$

Let $\mathcal{Z} \longrightarrow U$ be a smooth family of curves, and let $\mathcal{J}_{\mathcal{Z}} \longrightarrow U$ be the corresponding family of Jacobians. Consider the projection

$$\pi : \mathcal{X} := \mathcal{Z} \times_U \mathcal{J}_{\mathcal{Z}} \longrightarrow \mathcal{J}_{\mathcal{Z}} =: T \tag{3.1}$$

onto the second factor, and assume the family of curves given by π satisfies the hypotheses of Section 2. (This is equivalent to assuming that there is a relative Poincaré bundle for the family of curves given by π.) This assumption is satisfied if the family $\mathcal{Z} \longrightarrow U$ is a pointed family of curves. We continue with the notation of Section 2.

Theorem 3.1. The space of sections $H^0(\mathcal{X}^{2m}, \Phi^*\bar{\Theta})$ is canonically isomorphic to $H^0(U, \mathcal{O})$, or equivalently in view of Theorem 2.2, $H^0(\mathcal{X}^{2m}, \mathcal{M}) = H^0(U, \mathcal{O})$. □

Before proving the theorem, we want to establish a special case of it. Assume that U is a single point, so that $X := \mathcal{Z}$ is a smooth curve. Let $J := \mathrm{Pic}^{g-1}(X)$ be the Jacobian, with the theta divisor on it denoted by Θ. In this special case $T = J$, $\mathcal{X}^{2m} = X^{2m} \times J$, and Theorem 3.1 implies the following proposition.

Proposition 3.2. $H^0(X^{2m} \times J, \Phi^*\Theta) = \mathbb{C}$. □

Proof of Proposition 3.2. We have $H^0(J, \Theta) = \mathbb{C}$, and the space of sections of any translate of Θ is also $\mathbb{C}$. Now fix any $\hat{x} := (x_1, \ldots, x_{2m}) \in X^{2m}$. The restriction of $\Phi^*\Theta$ to $\hat{x} \times J$ is the translate of Θ by $\sum_{i=1}^{m} x_i - x_{i+m}$. Hence, the space of sections of the restriction of $\Phi^*\Theta$ to $\hat{x} \times J$ is parametrized by $\mathbb{C}$. Thus $H^0(X^{2m} \times J, \Phi^*\Theta) = H^0(J, \mathcal{O}) = \mathbb{C}$. ■

Proof of Theorem 3.1. Clearly, $H^0(\mathcal{X}^{2m}, \mathcal{O})$ is contained in $H^0(\mathcal{X}^{2m}, \Phi^*\bar{\Theta})$, since the divisor $\bar{\Theta}$, and hence $\Phi^*\bar{\Theta}$ is an effective divisor. Since the obvious projection of $\mathcal{X}^{2m}$ onto U is a dominant map, $H^0(U, \mathcal{O})$ is contained in $H^0(\mathcal{X}^{2m}, \mathcal{O})$ by the pullback homomorphism. So we get that $H^0(U, \mathcal{O})$ is naturally contained in $H^0(\mathcal{X}^{2m}, \Phi^*\bar{\Theta})$. In order to complete the proof, we must show that any section of $\Phi^*\bar{\Theta}$ is given by a function on U. Take any section $s \in H^0(\mathcal{X}^{2m}, \Phi^*\bar{\Theta})$. For a point $u \in U$, let X_u denote the curve over u, and let J_u denote the Jacobian of degree-$(g-1)$ line bundles on X_u. The fiber of the projection of $\mathcal{X}^{2m}$ to U is $X_u^{2m} \times J_u$. Applying Proposition 3.2 to the restriction of s to $X_u^{2m} \times J_u$, we get that the restriction is given by a complex number s_u. This implies that the section s is given by a regular function which is determined by the condition that its evaluation to u is s_u. ■

Let ω denote the natural section given by the constant function 1 of the relative theta divisor $\bar{\Theta}$ on $\mathcal{J}$. The pullback section $\bar{f}^*\omega \in H^0(T, \bar{f}^*\bar{\Theta})$ is not identically zero. Indeed, for any $u \in U$, the restriction of $\bar{f}^*\bar{\Theta}$ to the Jacobian $\mathcal{J}_u$ over the curve $\mathcal{X}_u$ over u is the theta line bundle, and the restriction of the section $\bar{f}^*\omega$ to $\mathcal{J}_u$ is the section given by the constant function 1. Hence, the section $\bar{f}^*\omega$ is not the zero section.

Let $(\bar{f} \circ q)^*\omega$ denote the pullback section of $(\bar{f} \circ q)^*\bar{\Theta}$ on $\mathcal{X}^{2m}$. From the above observation that $\bar{f}^*\omega$ is nonzero, we get that $(\bar{f} \circ q)^*\omega$ is not the zero section.

The line bundle $\mathcal{O}(\sum_{i\leq m}^{m<j} \mathcal{D}_{i,j}) \otimes \mathcal{O}(\sum_{i<j}^{j\leq m} -\mathcal{D}_{i,j} - \mathcal{D}_{i+m,j+m})$ (see (2.4)) on $\mathcal{X}^{2m}$ has a meromorphic section given by the constant function 1. We will denote this section by β.

In Theorem 3.1, we saw that $H^0(\mathcal{X}^{2m}, \mathcal{M}) = H^0(U, \mathcal{O})$. Let $s \in H^0(\mathcal{X}^{2m}, \mathcal{M})$ denote the section corresponding to $1 \in H^0(U, \mathcal{O})$. Then the quotient

$$N(m) := \frac{s}{\beta \otimes (\bar{f} \circ q)^*\omega} \tag{3.2}$$

is a meromorphic section of the bundle $\mathcal{F}$ (defined in (2.3)) on $\mathcal{X}^{2m}$. The poles of $N(m)$ are the divisor $\sum_{i\leq m}^{m<j} \mathcal{D}_{i,j}$ (in the notation of Section 2) union with the divisor defined by the vanishing of $(\bar{f} \circ q)^*\omega$.

If $m = 1$, then, from (2.3) and (3.2), $N(1)$ is the meromorphic section of the line bundle $\bar{p}_1^*(\mathcal{K} \otimes f^*\mathcal{P}^*) \otimes \bar{p}_2^*(f^*\mathcal{P})$ on $\mathcal{X}^2$ obtained above.

Getting back to general m, let

$$p_{i,j} : \mathcal{X}^{2m} \longrightarrow \mathcal{X}^2 \tag{3.3}$$

denote the projection along the (i, j)th factor for any two indices i, j with $i \leq m$ and $j > m$. Define

$$S(i, j) := p_{i,j}^* N(1) \tag{3.4}$$

to be the pullback meromorphic section on $\mathcal{X}^{2m}$.

Form the matrix $(S(i, j))$; its formal determinant

$$\Gamma := \det(S(i, j)) \tag{3.5}$$

gives a meromorphic section of the bundle $\mathcal{F}$ defined in (2.3). The poles of the section Γ are the union of the divisor $\sum_{i\leq m}^{m<j} \mathcal{D}_{i,j}$ with the divisor defined by the vanishing of $(\bar{f} \circ q)^*\omega$. Thus the poles of the two meromorphic sections Γ and $N(m)$ coincide.

Let Ψ be the meromorphic function on $\mathcal{X}^{2m}$ that satisfies the condition

$$\Psi.\Gamma = N(m).$$

We shall investigate the function Ψ.

For a point $u \in U$, let X_u denote the curve over u, and let J_u denote the Jacobian of degree-$(g-1)$ line bundles on X_u. Take $\alpha \in J_u$ such that $H^0(X_u, \alpha) = 0$, i.e., α lies outside the theta divisor in J_u. From this condition and the Künneth formula, it follows that the restriction of the line bundle $\mathcal{F}$ to the subvariety $X_u^{2m} \times \alpha$ (of $\mathcal{X}^{2m}$) does not admit any nonzero section. Since the section $(\bar{f} \circ q)^*\omega$ is nowhere zero on $X_u^{2m} \times \alpha$, the observation that the poles of the two meromorphic sections Γ and $N(m)$ coincide implies that the function Ψ must be constant on $X_u^{2m} \times \alpha$. Since all pairs u and α satisfying the above condition form a Zariski open dense set in T, we get that Ψ must be a pullback of a meromorphic function on T. In other words,

$$\Psi = \psi \circ q$$

where ψ is a meromorphic function on T.

For u and α as above, choose a point $x \in X_u$. Consider the element

$$\hat{x} := \{x, x, \ldots, x; \alpha\} \in \mathcal{X}^{2m}.$$

We want to prove that $\Gamma(\hat{x}) = N(m)(\hat{x})$. To prove this, first note that it is enough to show the following: The two meromorphic sections of the line bundle

$$\mathcal{F} \otimes \mathcal{O}\left(\sum_{i \le m}^{m<j} \mathcal{D}_{i,j}\right) \otimes \mathcal{O}\left(\sum_{i<j}^{j \le m} -\mathcal{D}_{i,j} - \mathcal{D}_{i+m,j+m}\right) = \mathcal{M} \otimes (\bar{f} \circ q)^* \bar{\Theta}^{-1}$$

(the equality follows from the Theorem 2.2), namely,

$$\Gamma \otimes \beta \qquad \text{and} \qquad N(m) \otimes \beta \tag{3.6}$$

(recall that β is the canonical meromorphic section of $\mathcal{O}(\sum_{i \le m}^{m<j} \mathcal{D}_{i,j}) \otimes \mathcal{O}(\sum_{i<j}^{j \le m} -\mathcal{D}_{i,j} - \mathcal{D}_{i+m,j+m})$) are holomorphic in a neighborhood of $\hat{x}$, and the two holomorphic sections (around $\hat{x}$) actually coincide at $\hat{x}$.

We claim that the fiber of the line bundle $\mathcal{M} \otimes (\bar{f} \circ q)^* \bar{\Theta}^{-1}$ at $\hat{x}$ is canonically isomorphic to $\mathbb{C}$. To prove this claim, we first observe that for any pair of indices (i, j) with $i \neq j$, the fiber $\mathcal{O}(\mathcal{D}_{i,j})_{\hat{x}}$ (resp. $\mathcal{O}(-\mathcal{D}_{i,j})_{\hat{x}}$) (see Section 2 for the definition of $\mathcal{D}_{i,j}$) is canonically identified with the fiber $(K_{X_u}^{-1})_x$ (resp. $(K_{X_u})_x$), where K_{X_u} is the canonical bundle of the curve X_u. Indeed, the Poincaré adjunction formula [GH, page 146] identifies the fiber $\mathcal{O}(-\mathcal{D}_{i,j})_{\hat{x}}$ with the fiber of the conormal bundle of the diagonal in $X_u \times X_u$ at (x, x). It is easy to see that the conormal bundle of the diagonal is canonically identified with the canonical bundle $K_{X_u}^{-1}$. Recall the definition of $\mathcal{F}$ in (2.3). For any (i, j), we have

$$(\bar{p}_i^*(f^*\mathcal{P}))_{\hat{x}} = (\bar{p}_j^*(f^*\mathcal{P}))_{\hat{x}}. \tag{3.7}$$

The above observation, the equality (3.7), and the isomorphism given by Theorem 2.2 combine together to imply the above claim.

Next we want show that both the sections $\Gamma \otimes \beta$ and $N(m) \otimes \beta$ in (3.6) are actually 1 (in the above identification of the fiber $(\mathcal{M} \otimes (\bar{f} \circ q)^* \bar{\Theta}^{-1})_{\hat{x}}$ with $\mathbb{C}$). The restriction of the line bundle $\mathcal{L}$ (defined in (2.1)) to the subvariety $X_u \times \hat{x} \in \mathcal{X} \times \mathcal{X}^{2m}$ is the restriction of $f^*\mathcal{P}$ to X_u. This implies that

$$\Phi(\hat{x}) = (\bar{f} \circ q)(\hat{x}).$$

So the two fibers

$$((\bar{f} \circ q)^* \bar{\Theta})_{\hat{x}} \qquad \text{and} \qquad (\Phi^* \bar{\Theta})_{\hat{x}}$$

are naturally identified, and moreover in this identification the evaluations of the sections $s(\hat{x})$ and $((\bar{f} \circ q)^* \omega)(\hat{x})$ coincide. (Recall that $(\bar{f} \circ q)^* \omega$ is a meromorphic section of $(\bar{f} \circ q)^* \bar{\Theta}$.) This implies that evaluation of the two sections $\Gamma \otimes \beta$ and $N(m) \otimes \beta$ at $\hat{x}$ are 1, i.e., that $\psi(\hat{x}) = 1$. Thus we have proved the following theorem.

Theorem 3.3. The two meromorphic sections Γ and $N(m)$ of the line bundle $\mathcal{F}$ on $\mathcal{X}^{2m}$ coincide. □

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