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Green-Lazarsfeld Sets and the Logarithmic Dolbeault Complex for Higgs Line Bundles

Indranil Biswas

1 Introduction

Let X be an irreducible smooth projective variety over $\mathbb{C}$ of dimension n. Let $D = \sum_{i=1}^{d} D_i$ be a divisor of normal crossing with decomposition into irreducible components. Let $\Omega^*(\log D)$ denote the sheaf of logarithmic differential forms on X, which is a locally free $\mathcal{O}_X$ module with a structure of exterior algebra [D, p. 72, Definition 3.1].

Let $[D_i] \in H^2(X, \mathbb{Q})$ denote the Poincaré dual of the divisor D_i.

Fix rational numbers $\{\alpha_1, \ldots, \alpha_d\}$ with $0 < \alpha_i < 1$ such that the cohomology class

$$\omega := \sum_{i=1}^{d} \alpha_i.[D_i] \in H^2(X, \mathbb{Q})$$

is in the image of $H^2(X, \mathbb{Z})$. Lefschetz's $1-1$ theorem ensures that there are line bundles on X with rational first Chern class $-\omega$.

Let $P(X)$ denote a component of the Picard group of X consisting of line bundles with first Chern class $-\omega$.

Define $V := H^0(X, \Omega^1_X)$ to be the space of all holomorphic 1-forms. The product variety

$$\mathcal{M} := P(X) \times V \tag{1.1}$$

is a component of the moduli space of Higgs bundles of degree $-\omega$.

For any $(\xi, \theta) \in \mathcal{M}$ we have the following complex of locally free $\mathcal{O}_X$-coherent sheaves on X, which we will denote by $D.$:

$$D. : D_0 = \xi \xrightarrow{\wedge\theta} D_1 = \xi \otimes \Omega^1(\log D) \xrightarrow{\wedge\theta} D_2 = \xi \otimes \Omega^2(\log D) \xrightarrow{\wedge\theta} \cdots$$

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$$\cdots \xrightarrow{\wedge\theta} D_{n-1} = \xi \otimes \Omega^{n-1}(\log D) \xrightarrow{\wedge\theta} D_n = \xi \otimes \Omega^n(\log D) \longrightarrow 0, \tag{1.2}$$

where the map $\wedge\theta$ is taking a wedge product with the section θ. Let $\mathbb{H}^j(D_{\cdot})$ denote the jth hypercohomology of the complex. Following the terminology of [S1], we call the complex $D_{\cdot}$ the *logarithmic Dolbeault complex* and its hypercohomology the *logarithmic Dolbeault cohomology*.

For $j, m \geq 0$, define the Green-Lazarsfeld set

$$T^j_m := \{(\xi, \theta) \in \mathcal{M} \mid \dim \mathbb{H}^j(D_{\cdot}) \geq m\} \subset \mathcal{M}. \tag{1.3}$$

Let $\mathrm{Pic}^0(X)$ be the abelian variety consisting of isomorphism classes of topologically trivial line bundles, which is same as the kernel of the natural homomorphism of the Picard group of X to the Neron-Severi group of X.

Consider the product

$$\mathcal{G} := \mathrm{Pic}^0(X) \times V, \tag{1.4}$$

which has a structure of an algebraic group given by the group structures of $\mathrm{Pic}^0(X)$ and V. The group $\mathcal{G}$ acts on $\mathcal{M}$ in the following way: for $(\xi, \theta) \in \mathcal{G}$ and $(\xi', \theta') \in \mathcal{M}$,

$$(\xi, \theta) \circ (\xi', \theta') = (\xi \otimes \xi', \theta + \theta'). \tag{1.5}$$

Our aim here is to prove the following theorem.

Theorem 1.6. Any irreducible component of T^j_m (defined in (1.3)) is a translation of an algebraic subgroup of $\mathcal{G}$ by a point of $\mathcal{M}$ by the action defined in (1.5). □

The special case of the above theorem where D is empty and there is no Higgs field was proved in [GL]; where D is empty was first proved in [A] (alternative proofs were given in [S2], [B1]); where there is no Higgs field (D may be nonempty) was proved in [B2].

In [S2] it was proved that the translating element in $\mathcal{M}$ (in [S2] D is empty, so $\mathcal{M} = \mathrm{Pic}^0(X) \times V$) can be taken to be a torsion point. However, in the general case, the question of torsion has a negative answer. Indeed, when the Neron-Severi group of X is torsion-free, all the torsion points of the Picard group are contained in $\mathrm{Pic}^0(X)$. So the component $P(X)$ cannot have a torsion point unless D is empty. Hence, the translating point of $\mathcal{M}$ cannot be chosen to be a torsion point.

The proof of Theorem A is an extension of the proof in [B2], where the key point was that using the "covering lemma" of Y. Kawamata, the space $P(X)$ can be identified with the moduli space of equivariant line bundles of degree zero on a suitable variety equipped with an action of a finite group. In Section 3 we observe that the above identification extends to an identification between Higgs line bundles and equivariant Higgs line bundles. Moreover, the logarithmic Dolbeault complex is the invariant part of the Dolbeault complex of the corresponding Higgs bundle equipped with a group action.

2 Equivariant Higgs line bundles

Let Y be an irreducible smooth projective variety over $\mathbb{C}$ of dimension n. The group of automorphisms of Y is denoted by $\mathrm{Aut}(Y)$. Let G be a finite subgroup of $\mathrm{Aut}(Y)$. The action of G on Y is denoted by ρ.

We will assume that the quotient of Y by the action of G is a smooth projective variety.

We recall the definition of an orbifold line bundle [B2].

Definition 2.1. An *orbifold line bundle* on Y is a line bundle L on Y together with a lift of action of G, which means that G acts on the total space of L, and for any $g \in G$, the action of g on L is an isomorphism between L and $\rho(g^{-1})^*L$. Two orbifold bundles are said to be isomorphic if there is an isomorphism between them which commutes with the actions of G.

Define $W := H^0(Y, \Omega^1_Y)$ to be the space of all holomorphic 1-forms. The group G has a natural action on W; let

$$W^G \subset W$$

denote the space of invariants.

Definition 2.2. An *orbifold Higgs line bundle* is a pair of the form (L, ϕ) where L is an orbifold line bundle and $\phi \in W^G$.

For a pair (L, ϕ) as in (2.2), we have the following Dolbeault complex on Y (as in (1.2)) :

$$\begin{aligned} C. : C_0 = L \xrightarrow{\wedge\phi} C_1 = L \otimes \Omega^1 \xrightarrow{\wedge\phi} C_2 = L \otimes \Omega^2 \xrightarrow{\wedge\phi} \cdots \\ \cdots \xrightarrow{\wedge\phi} C_{n-1} = L \otimes \Omega^{n-1} \xrightarrow{\wedge\phi} C_n = L \otimes \Omega^n \longrightarrow 0. \end{aligned} \tag{2.3}$$

Since ϕ is invariant under the action of G, the action of G on L induces an action on the hypercohomology $\mathbb{H}^j(C.)$. Let

$$\mathbb{H}^j(C.)^G \subset \mathbb{H}^j(C.)$$

denote the space of invariants.

The moduli space of all isomorphism classes of orbifold line bundles such that the underlying bundle is topologically trivial (an element of $\mathrm{Pic}^0(Y)$) is a smooth projective variety equipped with a group operation. We will denote this group by $P_G(Y)$, which is clearly nonempty since the pullback of any element of $\mathrm{Pic}^0(Y/G)$ is an element of $P_G(Y)$. Let $P^0_G(Y)$ be an irreducible component of $P_G(Y)$, and let $\mathrm{Pic}^0_G(Y)$ denote the component of $P_G(Y)$ which contains the identity element.

Consider the product variety

$$\mathcal{M}_H(Y) := P^0_G(Y) \times W^G, \tag{2.4}$$

which is a component of the moduli space of orbifold Higgs line bundles. Imitating the definition of Green-Lazarsfeld set (1.3), we define, for $j, m \geq 0$,

$$S^j_m := \{(L, \phi) \in \mathcal{M}_H(Y) \mid \dim \mathbb{H}^j(C.)^G \geq m\} \subset \mathcal{M}_H(Y). \tag{2.5}$$

The product group $\mathrm{Pic}^0_G(Y) \times W^G$ acts on $\mathcal{M}_H(Y)$ as in (1.5). We will prove the following simple proposition, which is actually a corollary of Theorem 3.1 (a) of [S2].

Proposition 2.6. Any irreducible component of S^j_m (defined in (2.5)) is a translation of an algebraic subgroup of $\mathrm{Pic}^0_G(Y) \times W^G$ by a point of $\mathcal{M}_H(Y)$ for the above action. □

Proof. Let

$$\mathcal{M}(Y) := \{(L, \theta) \mid L \in \mathrm{Pic}^0(Y), \theta \in H^0(Y, \Omega^1_Y)\}$$

be the moduli space of rank one Higgs bundles on Y. The group G acts on $\mathcal{M}(Y)$ in the following natural way: for $g \in G$ and $(L, \theta) \in \mathcal{M}(Y)$,

$$g \circ (L, \theta) = (\rho(g^{-1})^* L, \rho(g^{-1})^* \theta).$$

Let $\Gamma \subset \mathcal{M}(Y)$ be the fixed point set for this action (fixed by the entire G), which is an algebraic subgroup of $\mathcal{M}(Y)$.

Let $\mathcal{M}_G = P_G(Y) \times W^G$ be the moduli of orbifold Higgs line bundles. There is a natural projection

$$f : \mathcal{M}_G \longrightarrow \mathcal{M}(Y)$$

given by forgetting the group action.

The image of f coincides with Γ. The map f identifies the component $\mathcal{M}_H(Y)$ (of $\mathcal{M}_G$) with Γ. Indeed, the group $\mathrm{Hom}(G, \mathbb{C}^*)$ (the group of characters of G), denoted by $\hat{G}$, acts freely on $\mathcal{M}_G$, with the quotient being Γ. Also $\hat{G}$ acts freely on $P_G(Y)/\mathrm{Pic}^0_G(Y)$, the group of components of $P_G(Y)$. So f identifies the component $\mathcal{M}_H(Y)$ with Γ. (The details of this argument can be found in [B2].) From the semicontinuity considerations, the image $f(S^j_m)$ is clearly an algebraic subvariety of $\mathcal{M}(Y)$. Also, clearly this image is closed under the action of $\mathbb{C}^*$ on $\mathcal{M}(Y)$. So, from Theorem 3.1 (a) of [S2] (p. 366), it follows that any irreducible component of $f(S^j_m)$ is a translation of a subgroup of $\mathcal{M}(Y)$. This completes the proof. ■

In the next section, we will use Proposition 2.6 in the proof of Theorem 1.6.

3 Relation between Dolbeault complexes

We continue with the notation of Section 1. So X is a connected smooth projective variety over $\mathbb{C}$ of dimension n, and D is a divisor of normal crossing on X. This means that D is a reduced effective divisor, each irreducible component D_i of D is smooth, and they intersect transversally.

Assume that the number $\alpha_i = m_i/N$, where N is a fixed positive integer and $m_i \in \mathbb{N}$, with $0 < m_i < N$.

The "covering lemma" of Y. Kawamata (Theorem 1.1.1 of [KMM], Theorem 17 of [K]) says that there is a connected smooth projective variety Y and a finite Galois morphism

$$\pi : Y \longrightarrow X$$

with Galois group $G = \mathrm{Gal}(\mathrm{Rat}(Y)/\mathrm{Rat}(X))$ such that $\tilde{D} := (\pi^* D)_{\mathrm{red}}$ is a divisor of normal crossing on Y and $\pi^* D_i = k_i N.(\pi^* D_i)_{\mathrm{red}}$, $1 \le i \le d$, where k_i are positive integers.

Define $\tilde{D}_i := (\pi^* D_i)_{\mathrm{red}}$; so we have $\pi^* D_i = k_i N \tilde{D}_i$. Since the divisor $\tilde{D}_i$ is invariant under the action of G, for any $k \in \mathbb{Z}$, the line bundle $\mathcal{O}_Y(k\tilde{D}_i)$ has an orbifold structure.

For a line bundle ξ on X, the pullback bundle $\pi^*\xi$ has an obvious orbifold structure. Define

$$L := \pi^*(\xi) \otimes \mathcal{O}_Y\left(\sum_{i=1}^{d} k_i m_i \tilde{D}_i\right). \tag{3.1}$$

This line bundle L has an orbifold structure.

On the other hand, given any orbifold line bundle L on Y, the group G acts on the direct image $\pi_* L$, and the invariant subsheaf

$$(\pi_* L)^G \subset \pi_* L \tag{3.2}$$

is a locally free $\mathcal{O}_X$-coherent sheaf of rank one.

The two constructions (3.1) and (3.2) are inverses of each other and give a one-to-one identification of orbifold line bundles on Y with line bundles on X. (The details can be found in [B2].)

If ξ in (3.1) is in $P(X)$, then

$$c_1(L) = \pi^* c_1(\xi) + \sum_{i=1}^{d} k_i m_i [\tilde{D}_i] = \pi^* c_1(\xi) + \sum_{i=1}^{d} \frac{m_i}{N} [k_i N \tilde{D}_i].$$

But by definition, $[k_i N \tilde{D}_i] = \pi^*[D_i]$. Hence,

$$c_1(L) = \pi^* c_1(\xi) + \sum_{i=1}^{d} \frac{m_i}{N} \pi^*[D_i] = 0.$$

So the above identification between orbifold line bundles on Y and $\mathrm{Pic}(X)$ identifies $P(X)$ with a component of $P_G(Y)$. We will call this component $P^0_G(Y)$.

Note that the action of G on Y induces an action of G on the direct image $\pi_* \Omega^i_Y$.

Lemma 3.3. Let $\xi \in P(X)$, and let L be defined by (3.1). Then the invariant direct image sheaf $(\pi_*(L \otimes \Omega^i_Y))^G$ is canonically isomorphic to $\xi \otimes \Omega^i_X(\log D)$ for any $i \geq 0$. □

Proof. First we note that $\pi^* \Omega^i_X(\log D) = \Omega^i_Y(\log \tilde{D})$ (recall that $\tilde{D} = (\pi^* D)_{\mathrm{red}}$) for any $i \geq 0$. Since all the m_i in (3.1) are strictly positive, we have the inclusion

$$\pi^*(\xi \otimes \Omega^i_X(\log D)) \subset L \otimes \Omega^i_Y \tag{3.4}$$

of $\mathcal{O}_Y$ coherent sheaves. So we have the composition homomorphism

$$\gamma : \xi \otimes \Omega^i_X(\log D) \longrightarrow \pi_* \pi^*(\xi \otimes \Omega^i_X(\log D)) \longrightarrow \pi_*(L \otimes \Omega^i_Y) \tag{3.5}$$

induced by (3.4). Clearly, γ commutes with the action of G, with the action of G on $\xi \otimes \Omega^i_X(\log D)$ being trivial. So γ induces a homomorphism of $\mathcal{O}_X$-coherent sheaves

$$\bar{\gamma} : \xi \otimes \Omega^i_X(\log D) \longrightarrow (\pi_*(L \otimes \Omega^i_Y))^G. \tag{3.6}$$

We will prove that $\bar{\gamma}$ is an isomorphism, which will complete the proof of the lemma. But before that, we make the following simple observation.

Lemma 3.7. The invariant direct image sheaf $(\pi_* \Omega^i_Y)^G$ is canonically isomorphic to Ω^i_X. □

Proof. The homomorphism $\pi^* \Omega^i_X \longrightarrow \Omega^i_Y$ given by the differential of π induces a homomorphism

$$\Omega^i_X \longrightarrow \pi_* \Omega^i_Y,$$

which in turn gives the following homomorphism of bundles on X

$$\nu : \Omega_X^i \longrightarrow (\pi_* \Omega_Y^i)^G. \tag{3.8}$$

(Note that since π is a finite and flat morphism, $\pi_*\Omega_Y^i$ is actually locally free, and moreover the sheaf of its invariants is also locally free.)

Let $D' \subset X$ denote the (reduced) divisor over which the map π is ramified.

The homomorphism ν gives a nonzero section, denoted by s, of the line bundle $(\overset{\text{top}}{\wedge}\, \Omega_X^i)^* \otimes (\overset{\text{top}}{\wedge}\, \pi_*\Omega_Y^i)^G$. The homomorphism ν fails to be an isomorphism exactly over the reduced divisor $(\mathrm{div}(s))_{\mathrm{red}}$.

The homomorphism ν is clearly an isomorphism at any point $x \in X - D'$. So $(\mathrm{div}(s))_{\mathrm{red}} \subset D'$.

Take a smooth point x of the divisor D'. From the following observation we conclude that ν is an isomorphism at x.

Let U and U' be two copies of the unit disk in $\mathbb{C}$, and let $p : U' \longrightarrow U$ be the map defined by $z \longmapsto z^n$, where $n \geq 1$. Then the space of differential forms on U' invariant under the deck transformations (of p) is generated by the form $z^{n-1}.dz$, which also happens to be a pullback of a differential form on U. Take any $y \in \pi^{-1}(x)$. Then there is a neighborhood (in Y) of y of the form $U' \times T$ and a neighborhood (in X) of x of the form $U \times T$ such that the map π is of the form $p \times \mathrm{Id}$ on $U' \times T$.

The above conclusion that ν is an isomorphism at x together with the earlier observation that $(\mathrm{div}(s))_{\mathrm{red}} \subset D'$ imply that the divisor $\mathrm{div}(s)$ must be the empty divisor. This completes the proof of Lemma 3.7. ■

We continue with the proof of Lemma 3.3. From Lemma 3.7, it can be easily deduced that the homomorphism $\bar{\gamma}$ is an isomorphism outside the divisor D.

Let $x \in D$ be a smooth point of the divisor. Then by a local checking on the unit disk (as done in the proof of Lemma 3.7), we get that $\bar{\gamma}$ is an isomorphism at x. This implies that the subvariety of X where $\bar{\gamma}$ fails to be an isomorphism must be of codimension at least two. On the other hand, by a previous argument, $\bar{\gamma}$ fails to be an isomorphism on a divisor (possibly empty). Hence, $\bar{\gamma}$ must be an isomorphism everywhere. This completes the proof of Lemma 3.3. ■

A computation similar to that above can also be found in [EV].

Proof of Theorem 1.6. Let $F : \mathcal{M} \longrightarrow \mathcal{M}_H(Y)$ be the morphism defined by $(\xi, \theta) \longmapsto (L, \pi^*\theta)$, where ξ and L are related by (3.1). From the identification of $P(X)$ with $P_G^0(Y)$ and from Lemma 3.7, we conclude that F is an isomorphism.

Take ϕ in (2.3) to be $\pi^*\theta$. Since π is a finite and flat morphism, $\mathbb{H}^j(C.)$ is identified with the jth hypercohomology of the following complex of sheaves on X:

$$\pi_* L \xrightarrow{\wedge\phi} \pi_*(L \otimes \Omega^1_Y) \xrightarrow{\wedge\phi} \cdots \quad \cdots \xrightarrow{\wedge\phi} \pi_*(L \otimes \Omega^{n-1}_Y) \xrightarrow{\wedge\phi} \pi_*(L \otimes \Omega^n_Y) \longrightarrow 0.$$

For any G module V, the inclusion $V^G \longrightarrow V$ of the invariants has a natural splitting given by the kernel of

$$\sum_{g\in G} g \subset \mathrm{End}(V).$$

Using this, it can be shown that the invariant subspace $(\mathbb{H}^j(C'.))^G$ is identified with the jth hypercohomology of the following complex:

$$C'. : (\pi_* L)^G \xrightarrow{\wedge\phi} (\pi_*(L \otimes \Omega^1_Y))^G \xrightarrow{\wedge\phi} \cdots \quad \cdots \xrightarrow{\wedge\phi} (\pi_*(L \otimes \Omega^{n-1}_Y))^G \xrightarrow{\wedge\phi} (\pi_*(L \otimes \Omega^n_Y))^G \longrightarrow 0.$$

But from Lemma 3.3 and 3.7, the above complex $C'.$ is identified with the complex $D.$ defined in (1.2). Now Theorem 1.6 follows from Proposition 2.6. ∎

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Université de Grenoble 1, Institut Fourier, 100 rue des Maths, B.P. 74, 38402 Saint-Martin-d'Hères Cedex, FRANCE; biswas@puccini.ujf-grenoble.fr