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RESTRICTION OF THE POINCARÉ BUNDLE TO A CALABI-YAU HYPERSURFACE

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1. Introduction

Let X be a compact connected Riemann surface of genus g, where $g \geq 3$. Denote by $\mathcal{M}_\xi := \mathcal{M}(n, \xi)$ the moduli space of stable vector bundles over X of rank n and fixed determinant ξ. If the degree $\deg(\xi)$ and the rank n are coprime, then there is a universal family of vector bundles, $\mathcal{U}$, over X parametrized by $\mathcal{M}_\xi$. This family is unique up to tensoring by a line bundle that comes from $\mathcal{M}_\xi$. We fix one universal family over $X \times \mathcal{M}_\xi$ and call it the *Poincaré bundle*. For any $x \in X$, let $\mathcal{U}_x$ denote the vector bundle over $\mathcal{M}_\xi$ obtained by restricting $\mathcal{U}$ to $x \times \mathcal{M}_\xi$. It is known that $\mathcal{U}$ (see [BBN]) and $\mathcal{U}_x$ (see [NR] and [Ty]) are stable vector bundles with respect to any polarization on $X \times \mathcal{M}_\xi$ and $\mathcal{M}_\xi$ respectively.

A smooth anti-canonical divisor D on $\mathcal{M}_\xi$ is an example of a Calabi-Yau variety, i.e., it is connected and simply connected with trivial canonical line bundle. The Calabi-Yau varieties are of interest both in string theory and in algebraic geometry.

In this paper we consider the restrictions of $\mathcal{U}$ and $\mathcal{U}_x$ to $X \times D$ and $x \times D$ respectively, where $x \in X$ and D is a smooth anti-canonical divisor. Denote such restrictions by $\mathcal{U}_D$ and $(\mathcal{U}_D)_x$ respectively.

In Theorem 2.5 and Corollary 2.6 we prove the following :

If $n \geq 3$, then the vector bundle $(\mathcal{U}_D)_x$ is stable with respect to any polarization on D. Moreover, for the general Riemann surface X, the connected component of the moduli space of semistable sheaves over D, containing the point represented by $(\mathcal{U}_D)_x$, is isomorphic to the Riemann surface X.

Actually, we prove that for any point x of any X (not necessarily the general Riemann surface), the infinitesimal deformation map for the family $\mathcal{U}_D$ of vector bundles over D parametrized by X, is an isomorphism from $T_x X$ to $H^1(D, \operatorname{End}((\mathcal{U}_D)_x))$ [Theorem 2.5]. Therefore, X is an étale cover of the above component of the moduli space of semistable sheaves over D.

In Theorems 2.9 and 2.10 we establish the following property of $\mathcal{U}_D$ when rank $n \geq 3$.

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For any polarization on $X \times D$, the vector bundle $\mathcal{U}_D$ is stable. Moreover, the connected component $\mathcal{M}_X^0(\mathcal{U}_D)$ of the moduli space of semistable sheaves over $X \times D$, containing the point represented by $\mathcal{U}_D$, is isomorphic to the Jacobian of X.

Let $d = \deg(\xi)$, and let $X \times \mathcal{M}_\xi \times \mathrm{Pic}^0(X) \longrightarrow X \times \mathcal{M}(n,d)$ be the map defined by $(x, E, L) \longmapsto (x, E \otimes L)$.

Consider the restriction to $X \times D \times \mathrm{Pic}^0(X)$ of the pullback, using the above map, of a universal vector bundle over $X \times \mathcal{M}(n, \deg(\xi))$. Theorem 2.10 is proved by establishing that this vector bundle over $X \times D \times \mathrm{Pic}^0(X)$ gives a universal family of stable vector bundles over $X \times D$ parametrized by $\mathrm{Pic}^0(X)$.

The above results are analogous to those on $\mathcal{U}_x$ and $\mathcal{U}$ obtained in [NR] and [BBN] respectively.

It is known that the connected component, $\mathcal{M}_X^0(\mathcal{U})$, of the moduli space of semistable sheaves over $X \times \mathcal{M}_\xi$, containing the point represented by $\mathcal{U}$, is isomorphic to the Jacobian of X [BBN]. As M. S. Narasimhan pointed out to one of the authors, in order to be able to recover the Riemann surface X what is still needed is an isomorphism of $\mathcal{M}_X^0(\mathcal{U})$ with $\mathrm{Pic}^0(X)$ as polarized varieties. In the final section we produce a canonical polarization on $\mathcal{M}_X^0(\mathcal{U})$ and also on $\mathcal{M}_X^0(\mathcal{U}_D)$. Using Torelli's theorem we have (see Theorem 4.3 and Theorem 4.4)

Let X and X' be two compact connected Riemann surfaces of genus $g \geq 3$. If $\mathcal{M}_X^0(\mathcal{U}) \cong \mathcal{M}_{X'}^0(\mathcal{U})$ or $\mathcal{M}_X^0(\mathcal{U}_{D'}) \cong \mathcal{M}_{X'}^0(\mathcal{U}_{D'})$, as polarized varieties, then $X \cong X'$.

Actually, we give a general construction of a certain line bundle equipped with a Hermitian structure on a moduli space, $\mathcal{M}_Y$, of stable vector bundles over a compact Kähler manifold Y of arbitrary dimension. The curvature of the Hermitian connection on this line bundle has been computed in Theorem 4.1. If $Y = X \times \mathcal{M}_\xi$ or $X \times D$, and $\mathcal{M}_Y = \mathrm{Pic}^0(X)$, then the curvature form represents a nonzero multiple of the natural polarization on $\mathrm{Pic}^0(X)$ given by the cohomology class of a theta divisor.

We note that if the assumption $n \geq 4$ is imposed, then all the results proved here remain valid for $g = 2$.

Restriction of vector bundles to a Calabi-Yau hypersurface has been considered in [Th], [DT].

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2. Vector bundles over a smooth anti-canonical divisor on the moduli space

Let X be a compact connected Riemann surface of genus $g \geq 3$. Fix a line bundle ξ over X of degree d, and also fix an integer $n \geq 3$ coprime to d. Let $\mathcal{M}_\xi := \mathcal{M}(n, \xi)$

denote the moduli space of stable vector bundles E over X of rank n and with the determinant $\bigwedge^n E$ isomorphic to ξ.

It is well known that there is a *universal vector bundle* $\mathcal{U}$ over $X \times \mathcal{M}_\xi$. A universal vector bundle $\mathcal{U}$ is called the *Poincaré bundle*. Let $Ad(\mathcal{U}) \subset \mathrm{End}(\mathcal{U})$ be the subbundle of rank n^2-1 defined by the trace zero endomorphisms. The vector bundle over $\mathcal{M}_\xi$ obtained by restricting $\mathcal{U}$ (respectively, $Ad(\mathcal{U})$) to $x \times \mathcal{M}_\xi$, where $x \in X$, will be denoted by $\mathcal{U}_x$ (respectively, $Ad(\mathcal{U})_x$).

We know that both the vector bundles $\mathcal{U}$ and $\mathcal{U}_x$ are stable (see [NR] and [BBN]). In general, restrictions of stable vector bundles need not be stable. In this section we will consider the restrictions of $\mathcal{U}_x$ and $\mathcal{U}$ to certain subvarieties of $\mathcal{M}_\xi$ and $X \times \mathcal{M}_\xi$ respectively, and prove that they are stable.

Denoting the canonical line bundle of $\mathcal{M}_\xi$ by $K_{\mathcal{M}_\xi}$, let D be a smooth divisor on $\mathcal{M}_\xi$ in the complete linear system for the anti-canonical line bundle $K^{-1}_{\mathcal{M}_\xi}$. Using the *Poincaré residue theorem*, the canonical line bundle of D is the trivial line bundle. The variety D is connected since $K^{-1}_{\mathcal{M}_\xi}$ is ample, and as $\mathcal{M}_\xi$ is also simply connected, D must be simply connected. In other words, any such divisor D is a *Calabi-Yau variety*.

The restriction of $\mathcal{U}$ to $X \times D$ will be denoted by $\mathcal{U}_D$. For any $x \in X$, the vector bundle over D obtained by restricting the vector bundle $\mathcal{U}_x$ to the subvariety $x \times D \subset x \times \mathcal{M}_\xi$, will be denoted by $(\mathcal{U}_D)_x$. Let $Ad(\mathcal{U}_D)$ denote the adjoint bundle of the vector bundle $\mathcal{U}_D$ over D. For any $x \in X$, denote by $Ad(\mathcal{U}_D)_x$ the vector bundle over D obtained by restricting $Ad(\mathcal{U}_x)$ to the subvariety $x \times D \subset x \times \mathcal{M}_\xi$.

The divisor D induces the following exact sequence of sheaves over $\mathcal{M}_\xi$

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}_\xi}(-D) \longrightarrow \mathcal{O}_{\mathcal{M}_\xi} \longrightarrow \mathcal{O}_D \longrightarrow 0\,. \tag{2.1}$$

Tensoring (2.1) with $Ad(\mathcal{U}_x)$ we have the sequence

$$0 \longrightarrow Ad(\mathcal{U}_x) \otimes \mathcal{O}_{\mathcal{M}_\xi}(-D) \longrightarrow Ad(\mathcal{U}_x) \xrightarrow{F} Ad(\mathcal{U}_D)_x \longrightarrow 0\,. \tag{2.2}$$

The following lemma plays a key rôle in the proofs of the results in this section.

Lemma 2.1. *$H^i(\mathcal{M}_\xi,\, Ad(\mathcal{U}_x) \otimes \mathcal{O}_{\mathcal{M}_\xi}(-D)) = 0$, for $i = 0, 1, 2$.*

Lemma 2.1 will be proved in Section 3. Meanwhile, in this section, Lemma 2.1 will be assumed to be valid.

The restriction homomorphism $F : Ad(\mathcal{U}_x) \longrightarrow Ad(\mathcal{U}_D)_x$ induces a homomorphism

$$F_i : \; H^i(\mathcal{M}_\xi,\, Ad(\mathcal{U}_x)) \longrightarrow H^i(D,\, Ad(\mathcal{U}_D)_x)\,.$$

We will now deduce from Lemma 2.1 that the two homomorphisms F_0 and F_1 are isomorphisms.

Proposition 2.2. *The homomorphism F_i is an isomorphism, for $i = 0, 1$.*

Proof. This proposition follows by considering the long exact sequence of cohomology for the exact sequence (2.2) and then using Lemma 2.1. □

Theorem 2 (page 392) in [NR] says that

$$H^0(\mathcal{M}_\xi,\, Ad(\mathcal{U}_x)) \,=\, 0 \quad \text{and} \quad H^1(\mathcal{M}_\xi,\, Ad(\mathcal{U}_x)) \,=\, T_xX\,.$$

Therefore, Proposition 2.2 has the following corollary.

Corollary 2.3. *For any $x \in X$,*

$$H^0(D,\, Ad(\mathcal{U}_D)_x) \;\;=\;\; 0\,.$$

Moreover, $H^1(D,\, Ad(\mathcal{U}_D)_x) \cong T_xX$ where T_xX is the holomorphic tangent space of X at x.

Since

$$H^i(D,\, \mathcal{O}_D) = \begin{cases} \mathbb{C} & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i = 1 \end{cases}$$

and

$$H^i(D,\, \mathrm{End}(\mathcal{U}_D)_x) \;\;=\;\; H^i(D,\, \mathcal{O}_D) \oplus H^i(D,\, Ad(\mathcal{U}_D)_x)\,,$$

we obtain from Corollary 2.3 that the following two equalities are valid

$$H^0(D,\, \mathrm{End}(\mathcal{U}_D)_x) \;\;\cong\;\; \mathbb{C} \tag{2.3}$$

$$H^1(D,\, \mathrm{End}(\mathcal{U}_D)_x) \;\;\cong\;\; T_xX\,. \tag{2.4}$$

Proposition 2.4. $\dim H^1(X \times D,\, \mathrm{End}(\mathcal{U}_D)) \,=\, g$. *Moreover, there is a canonical isomorphism $\alpha :\, H^1(X, \mathcal{O}) \,\longrightarrow\, H^1(X \times D, \mathrm{End}(\mathcal{U}_D))$.*

Proof. Let $p : X \times D \longrightarrow X$ be the projection onto the first factor, and let

$$0 \to H^1(X,\, \mathcal{R}^0_{p*}\mathrm{End}(\mathcal{U}_D)) \to H^1(X \times D,\, \mathrm{End}(\mathcal{U}_D)) \to H^0(X,\, \mathcal{R}^1_{p*}\mathrm{End}(\mathcal{U}_D)) \tag{2.5}$$

be the associated exact sequence obtained from the Leray spectral sequence. From (2.3) and (2.4) we obtain the following two canonical isomorphisms

$$\mathcal{R}^1_{p*}\mathrm{End}(\mathcal{U}_D) \;\;\cong\;\; TX$$

and

$$\mathcal{R}^0_{p*}\mathrm{End}(\mathcal{U}_D) \;\;\cong\;\; \mathcal{O}_X\,.$$

Now the proposition follows from the exact sequence (2.5). □

Since $\mathrm{Pic}(D) \cong \mathbb{Z}$, the stability condition of a vector bundle over D does not depend on the choice of a polarization. Moreover, the vector space parametrizing infinitesimal deformations of a vector bundle V over D, namely $H^1(D,\, \mathrm{End}(V))$, coincides with $H^1(D,\, \mathrm{Ad}(V))$.

Theorem 2.5. *For any $x \in X$, the vector bundle $(\mathcal{U}_D)_x$ over D is stable. Moreover, the infinitesimal deformation map*

$$T_xX \longrightarrow H^1(D, \mathrm{Ad}(\mathcal{U}_D)_x)$$

for the family $\mathcal{U}_D$ of vector bundles over D parametrized by X, is an isomorphism.

Proof. We will first prove that $(\mathcal{U}_D)_x$ is polystable. The proof of the polystability of $(\mathcal{U}_D)_x$ given below is similar to the proof of [BBN, Proposition 2.4].

For any $x \in X$, let

$$\pi : Y \longrightarrow X$$

be a spectral cover which is unramified over x. (See [H] for the definition of a spectral cover; also can be found in [BNR].)

The associated rational map

$$\overline{\pi} : \mathrm{Prym}(Y) \longrightarrow \mathcal{M}_\xi$$

is a generically finite dominant rational map; $\mathrm{Prym}(Y) \subset \mathrm{Pic}^c(Y)$ is the Prym variety for the covering π, where $c = \deg(\xi) + (g-1)(n^2-1)$. The subvariety of $\mathrm{Prym}(Y)$ where $\overline{\pi}$ is not defined – we will denote this subvariety by A – is of codimension at least 3 and the subvariety of $\mathcal{M}_\xi$ consisting of the complement of the image of $\overline{\pi}$ – we will denote this subvariety by B – is of codimension at least 2.

The subvariety $D \cap B$ of D is of codimension at least 2. Indeed, if it is not true, i.e., $D \cap B$ is a divisor on D, then the fact that $\mathrm{Pic}(D) \cong \mathbb{Z}$ would imply that $D \cap B$ is actually an ample divisor on D. This in turn would imply that there is a nonconstant holomorphic function f on $D - (D \cap B)$. On the other hand, since the codimension of A in $\mathrm{Prym}(Y)$ is at least 3, the pullback of the function f to $\overline{\pi}^{-1}(D - (D \cap B))$, being a nonconstant function, would yield the required contradiction.

Using the criterion of [BBN, page 7, Lemma 2.1] for a vector bundle to be (semi)stable, expressed in terms of the extension of a pullback, the following criterion for the polystability of $(\mathcal{U}_D)_x$ is obtained. In order to prove that the vector bundle $(\mathcal{U}_D)_x$ is polystable, it is enough to show that $(\overline{\pi}|_{\pi^{-1}(D)})^*(\mathcal{U}_D)_x$ on $\overline{\pi}^{-1}(D)$ is polystable with respect to the restriction of the natural polarization on $\mathrm{Prym}(Y)$ given by the restriction of the first Chern class of a theta divisor on $\mathrm{Pic}^c(Y)$.

Now since π is unramified over x, the vector bundle $\overline{\pi}^*(\mathcal{U}_D)_x$ decomposes as a direct sum of line bundles of same degree. More precisely, this vector bundle coincides with the restriction of $\bigoplus_{y \in \pi^{-1}(x)} L_y$, where L_y is the line bundle over $\mathrm{Prym}(Y)$ obtained by restricting a Poincaré bundle over $Y \times \mathrm{Pic}^c(Y)$ to $y \times \mathrm{Prym}(Y)$. So $(\overline{\pi}|_{\pi^{-1}(D)})^*(\mathcal{U}_D)_x$ must be polystable, and hence $(\mathcal{U}_D)_x$ is polystable.

The Corollary 2.3 says that $(\mathcal{U}_D)_x$ is simple. Therefore $(\mathcal{U}_D)_x$ is stable.

Let

$$T_xX \longrightarrow H^1(\mathcal{M}_\xi, \mathrm{Ad}(\mathcal{U})_x) \tag{2.6}$$

be the infinitesimal deformation map for the family $\mathcal{U}$ of vector bundles over $\mathcal{M}_\xi$ parametrized by X. The infinitesimal deformation map

$$T_xX \longrightarrow H^1(D, \mathrm{Ad}(\mathcal{U}_D)_x)$$

for the vector bundle $\mathcal{U}_D$ over $X \times D$, is simply the composition of the homomorphism in (2.6) followed by the homomorphism F_1 in Proposition 2.2. In [NR] it has been proved that infinitesimal deformation map in (2.6) is actually an isomorphism. Since from Proposition 2.2 the homomorphism F_1 is an isomorphism, the infinitesimal deformation map for the family of vector bundles $\mathcal{U}_D$ over D, parametrized by is an isomorphism. This completes the proof of the theorem. □

Let $\mathcal{M}^0((\mathcal{U}_D)_x)$ denote the connected component of the moduli space of semistable sheaves over D with the same numerical invariants as $(\mathcal{U}_D)_x$ and containing the point that represents the stable vector bundle $(\mathcal{U}_D)_x$.

Corollary 2.6. *For a general Riemann surface X, the moduli space $\mathcal{M}^0((\mathcal{U}_D)_x)$ is canonically isomorphic to X .*

Proof. Consider the map

$$\beta : \quad X \longrightarrow \mathcal{M}^0((\mathcal{U}_D)_x)$$

defined by $x \longmapsto (\mathcal{U}_D)_x$. Theorem 2.5 implies that β is an étale covering map from X to $\mathcal{M}^0((\mathcal{U}_D)_x)$. The general Riemann surface of genus $g \geq 2$ does not admit a nontrivial étale covering map to another Riemann surface. By this we mean that the subset of the moduli space of Riemann surfaces of genus g representing all Riemann surfaces having a nontrivial étale covering map to another Riemann surface, is a subvariety of strictly lower dimension. Thus, for a general Riemann surface X, the map β is an isomorphism. □

Remark 2.7. In view of Corollary 2.6, the moduli space $\mathcal{M}^0((\mathcal{U}_D)_x)$ constitutes an explicit example of a complete family of semistable sheaves on a higher dimensional variety such that all the members of the family are locally free and also do not have the numerical invariants of a projectively flat vector bundle.

Remark 2.8. Let $X_T \longrightarrow T$ be a smooth family of irreducible projective curves of genus g, with $g \geq 3$. Suppose we are also given a family of Poincaré bundles, and a family of smooth anti-canonical divisors, $D_T \longrightarrow T$, parametrized by T, for the family of curves. Let $M_T \longrightarrow T$ denote the relative moduli space of stable vector bundles for the family $D_T \longrightarrow T$. Let $p : X_T \longrightarrow M_T$ denote the morphism which sends any point x in the fiber X_t over $t \in T$ to the stable vector bundle over the anti-canonical divisor D_t obtained from Theorem 2.5 (by simply restricting the Poincaré bundle to $x \times D_t$). Suppose that T is irreducible, and the general member in the family of curves X_T does not have any nontrivial automorphism. Therefore, over an open subset U of T, the

map p is an isomorphism. Since p is a morphism over T, it must be an isomorphism. Consequently, the assertion in Corollary 2.6 extends to all Riemann surfaces.

We will now consider the vector bundle $\mathcal{U}_D$ over $X \times D$. Actually, the results (and proofs) are similar to those for $\mathcal{U}$ given in [BBN].

Theorem 2.9. *The restriction of the Poincaré bundle $\mathcal{U}$ to $X \times D$ is stable with respect to any polarization on $X \times D$.*

Proof. Since $H^1(D, \mathbb{Z}) = 0$, any polarization η on $X \times D$ is the form

$$\eta \;=\; a\lambda + b\gamma, \quad a, b > 0\,,$$

where λ and γ are polarizations on X and D respectively.

Recall Theorem 2.5 which says that the vector bundle $(\mathcal{U}_D)_x$ is stable for any $x \in X$. Furthermore, by definition, for any $\{d\} \in D$, the vector bundle $(\mathcal{U}_D)|_{X\times\{d\}}$ over X is stable. Hence by Lemma 2.2 in [BBN], the vector bundle $\mathcal{U}_D$ is stable with respect to any polarization on $X \times D$. □

Let $\mathcal{M}^0(\mathcal{U}_D)$ denote the connected component of the moduli space of semistable sheaves over $X \times D$ with the same numerical invariants as $\mathcal{U}_D$ and containing the point representing the stable vector bundle $\mathcal{U}_D$.

Theorem 2.10. $\mathcal{M}^0(\mathcal{U}_D) \cong \mathrm{Pic}^0(X)$.

Proof. Let $p : X \times D \longrightarrow X$ denote the projection onto the first factor. Consider the morphism

$$\delta \;:\; \mathrm{Pic}^0(X) \;\longrightarrow\; \mathcal{M}^0(\mathcal{U}_D)$$

defined by $L \longmapsto \mathcal{U}_D \otimes p^*L$. As in Lemma 3.4 of [BBN, page 12], we have that δ is injective; we will omit the details. Moreover, the Zariski tangent space of $\mathcal{M}^0(\mathcal{U}_D)$ at the point $\mathcal{U}_D \otimes p^*L$ is naturally isomorphic to $H^1(X \times D, \mathrm{End}(\mathcal{U}_D))$, which has dimension g by Proposition 2.4. From the Zariski's Main Theorem and the fact that $\mathrm{Pic}^0(X)$ is complete we have that δ is actually an isomorphism. □

Remark 2.7 is also valid for the moduli space $\mathcal{M}^0(\mathcal{U}_D)$.

The following section will be devoted to the proof of Lemma 2.1.

3. Proof of the main lemma

As D is linearly equivalent to $K^{-1}_{\mathcal{M}_\xi}$, we have

$$\mathcal{O}_{\mathcal{M}_\xi}(-D) \;\cong\; K_{\mathcal{M}_\xi}\,. \tag{3.1}$$

So Lemma 2.1 is equivalent to the following statement :

$$H^i(\mathcal{M}_\xi,\, Ad(\mathcal{U})_x \otimes K_{\mathcal{M}_\xi}) \;=\; 0$$

for $i \leq 2$.

The ample generator of $\mathrm{Pic}(\mathcal{M}_\xi) \cong \mathbb{Z}$ will be denoted by Θ. We will recall a relationship between the canonical line bundle and a determinant bundle over $\mathcal{M}_\xi$.

Define $b := \mathrm{g.c.d}\,(n,d)$, $n' := \frac{n}{b}$ and $\chi' := \frac{d+n(1-g)}{b}$.

Take a point $y \in X - x$. Since the anti-canonical line bundle $K^{-1}_{\mathcal{M}_\xi}$ is isomorphic to 2Θ [Ra], we conclude that

$$K_{\mathcal{M}_\xi} \;=\; (\det(\mathcal{U}))^{-2n'} \otimes (\wedge^n \mathcal{U}_y)^{-2\chi'}\,, \tag{3.2}$$

where $\det(\mathcal{U})$ is the *determinant line bundle* over $\mathcal{M}_\xi$, whose fiber over a point represented by a vector bundle E over X is canonically isomorphic to following line (cf. [KM]) :

$$\bigwedge\nolimits^{\mathrm{top}} H^0(X,\, E)^* \bigotimes \bigwedge\nolimits^{\mathrm{top}} H^1(X,\, E)\,.$$

The determinant bundle depends upon the choice of the universal vector bundle, but the expression of Θ in terms of the determinant bundle is valid even in the non-coprime situation where there is no Poincaré bundle.

Proof of Lemma 2.1. The proof is an application of the idea initiated in [NR] of computing cohomologies using the Hecke transformation. We will not deal with the issues of codimension computation needed for the application of Hartog type results on cohomology, since they have already been resolved in [NR].

Consider the projective bundle

$$p \;:\; \mathbb{P}(\mathcal{U}_x) \;\longrightarrow\; \mathcal{M}_\xi$$

consisting of all hyperplanes in $\mathcal{U}_x$. Now $\mathbb{P}_x := \mathbb{P}(\mathcal{U}_x)$ parametrizes a natural family of vector bundles over X of rank n and degree $d-1$. Let $\mathcal{V}$ denote this vector bundle over $X \times \mathbb{P}_x$. The restriction of $\mathcal{V}$ to $X \times \{H\}$, where $H \subset E_x$ is a hyperplane representing a point in $\mathbb{P}_x$ over $E \in \mathcal{M}_\xi$, fits in the following exact sequence

$$0 \longrightarrow \mathcal{V}|_{X\times\{H\}} \longrightarrow E \longrightarrow \frac{E_x}{H} \longrightarrow 0\,, \tag{3.3}$$

where E_x/H is supported at x. The general member of the family of vector bundles over X, defined by $\mathcal{V}$, is stable.

Thus, from a Zariski open subset of U of $\mathbb{P}_x$ there is a natural projection

$$q \;:\; U \;\longrightarrow\; \mathcal{M}(n, \xi(-x))$$

to the moduli space of stable vector bundles of rank n and determinant $\xi(-x) := \xi \otimes \mathcal{O}_X(-x)$. The key point in [NR] is that the relative tangent bundle over U for the projection p actually coincides with the relative cotangent bundle for the projection q [Ty, page 85].

Considering the exact sequence (3.3), from the definition of the determinant bundle it follows immediately that we have the following isomorphism

$$\det(\mathcal{V}) \;=\; p^*\det(\mathcal{U}) \otimes \mathcal{O}_{\mathbb{P}_x}(1)\,, \tag{3.4}$$

where $\mathcal{O}_{\mathbb{P}_x}(1)$ is the tautological line bundle for the vector bundle $\mathcal{U}_x$.

Let L denote the line bundle over $\mathbb{P}_x$ whose fiber over the point $(E, H) \in \mathbb{P}_x$ (as in (3.3)) is the kernel of the homomorphism of the fibers

$$(\mathcal{V}|_{X\times\{H\}})_x \longrightarrow E_x$$

in the exact sequence (3.3). It is easy to check that $\mathcal{O}_{\mathbb{P}_x}(1) = L$. Indeed, this is a consequence of the exactness of the sequence (3.3) and the fact that the line bundle $\bigwedge^n \mathcal{V}|_{x\times\mathbb{P}_x}$ over $x \times \mathbb{P}_x$ is isomorphic to $\bigwedge^n \mathcal{U}_x$. The last isomorphism is a consequence of the fact that the Picard group, $\mathrm{Pic}(\mathbb{P}_x)$, of $\mathbb{P}_x$ is discrete and for $z \neq x$, the line bundle $\bigwedge^n \mathcal{V}|_{z\times\mathbb{P}_x}$ is canonically isomorphic to $\bigwedge^n \mathcal{U}_z$.

Since $p_* T_p^{\mathrm{rel}} = Ad(\mathcal{U})_x$, where T_p^{rel} is the relative tangent bundle for the projection p, and furthermore the higher direct images of T_p^{rel} vanish, from the Leray spectral sequence for the projection p we conclude that the following

$$\begin{aligned} H^i(\mathcal{M}_\xi,\, Ad(\mathcal{U})_x \otimes K_{\mathcal{M}_\xi}) &\cong H^i(U,\, T_q^{*,\mathrm{rel}} \otimes p^* K_{\mathcal{M}_\xi}) \\ &\cong H^i(U,\, T_q^{*,\mathrm{rel}} \otimes (\det(\mathcal{V}) \otimes L^*)^{-2n'} \otimes (\wedge^n \mathcal{V}_y)^{-2\chi'}) \end{aligned} \tag{3.5}$$

is valid for any $i \geq 0$, where $T_q^{*,\mathrm{rel}}$ is the relative cotangent bundle for the projection q. Indeed, since $x \neq y$, the vector bundle $\mathcal{V}_y$ over $\mathbb{P}_x$ is canonically isomorphic to $p^*\mathcal{U}_y$, and furthermore, as we already noted earlier, the relative tangent bundle T_p^{rel} is isomorphic to $T_q^{*,\mathrm{rel}}$.

Since $K_{\mathcal{M}_\xi}$ is negative ample, and the restriction of to a fiber of q is a nonconstant morphism, the restriction of $p^* K_{\mathcal{M}_\xi}$ to a fiber of q has strictly negative degree. Furthermore,

$$H^i(\mathbb{CP}(r),\, \Omega^1(k)) = 0$$

if $i \neq r$ and $k < 0$ (see [Bo] and [SS, page 71, Theorem 4.3]). Consequently, if $n \geq 4$, then the following equality

$$\mathcal{R}^i q_*(T_q^{*,\mathrm{rel}} \otimes (\det(\mathcal{V}) \otimes L^*)^{-2n'} \otimes (\wedge^n \mathcal{V}_y)^{-2\chi'}) = 0 \tag{3.6}$$

is valid for $i \leq 2$.

Combining the equality (3.6) with the Leray spectral sequence applied to the vector bundle $T_q^{*,\mathrm{rel}} \otimes (\det(\mathcal{V}) \otimes L^*)^{-2n'} \otimes (\wedge^n \mathcal{V}_y)^{-2\chi'}$, for the projection q, we conclude that

$$H^i(U,\, T_q^{*,\mathrm{rel}} \otimes (\det(\mathcal{V}) \otimes L^*)^{-2n'} \otimes (\wedge^n \mathcal{V}_y)^{-2\chi'}) = 0$$

for $i \leq 2$ whenevr $n \geq 4$. Now the equality (3.5) yields that if $n \geq 4$, then

$$H^i(\mathcal{M}_\xi,\, Ad(\mathcal{U})_x \otimes K_{\mathcal{M}_\xi}) = 0$$

for $i \leq 2$. This completes the proof of the lemma when $n \geq 4$.

We now consider the remaining case, namely $n = 3$.

First note that in this case the left-hand side of (3.6) vanishes for $i = 0$ and $i = 1$. Thus the proof of lemma will be completed once we are able to establish the following equality

$$H^0\Big(\mathcal{M}(n,\xi(-x)),\, \mathcal{R}^2q_*(T_q^{*,\rm rel}\otimes(\det(\mathcal{V})\otimes L^*)^{-2n'}\otimes(\wedge^n\mathcal{V}_y)^{-2\chi'})\Big) \;=\; 0\,. \tag{3.7}$$

Using a special case of Serre duality, which says that for a line bundle ζ over a smooth projective surface S

$$H^2(S,\,\Omega^1_S\otimes\zeta) \;=\; H^0(S,\,\Omega^1_S\otimes\zeta^*)^*\,,$$

the following isomorphism is obtained

$$\begin{aligned}&\mathcal{R}^2q_*(T_q^{*,\rm rel}\otimes(\det(\mathcal{V})\otimes L^*)^{-2n'}\otimes(\wedge^n\mathcal{V}_y)^{-2\chi'})\\ \cong\;&\Big(\mathcal{R}^0q_*(T_q^{*,\rm rel}\otimes(\det(\mathcal{V})\otimes L^*)^{2n'}\otimes(\wedge^n\mathcal{V}_y)^{2\chi'})\Big)^*\,.\end{aligned} \tag{3.8}$$

Let $Ad(\overline{\mathcal{U}})$ denote the universal adjoint vector bundle over the smooth locus of $X\times\mathcal{M}(n,\xi(-x))$. We observe that although there may not be any universal vector bundle over the smooth locus of $X\times\mathcal{M}(n,\xi(-x))$, the universal adjoint always exists. The restriction of $Ad(\overline{\mathcal{U}})$ to $x\times\mathcal{M}(n,\xi(-x))$ will be denoted by $Ad(\overline{\mathcal{U}})_x$. Let

$$\overline{\Theta} \;\in\; \mathrm{Pic}(\mathcal{M}(n,\xi(-x)))$$

be the positive generator.

Now $\mathcal{R}^0q_*(T_q^{*,\rm rel}\otimes(\det(\mathcal{V})\otimes L^*)^{2n'}\otimes(\wedge^n\mathcal{V}_y)^{2\chi'}) \;\cong\; Ad(\overline{\mathcal{U}})_x\bigotimes\overline{\Theta}^{\otimes k}$, where $k>0$. Since $Ad(\overline{\mathcal{U}})_x$ is semistable of degree zero, and $Ad(\overline{\mathcal{U}})_x = Ad(\overline{\mathcal{U}})_x^*$, we conclude that

$$H^0\Big(\mathcal{M}(n,\xi(-x)),\, \Big(\mathcal{R}^0q_*(T_q^{*,\rm rel}\otimes(\det(\mathcal{V})\otimes L^*)^{2n'}\otimes(\wedge^n\mathcal{V}_y)^{2\chi'})\Big)^*\Big) \;=\; 0\,.$$

Now (3.8) implies (3.7), and this completes the proof of the Lemma 2.1. □

If $n\geq 4$, then Lemma 2.1 is also valid for $g=2$. Consequently, all the results in Section 2 remain valid for $g=2$ if the condition $n\geq 4$ is imposed. We note that the special situation where $n=3$ and $g=2$ has been left out in Theorem 2 of [NR].

4. A determinant line bundle over the moduli space of vector bundles over a Kähler manifold

Let Y be a compact connected Kähler manifold of complex dimension d. Fix cohomology classes $c_i\in H^{2i}(Y,\mathbb{Q})$, $1\leq i\leq r$. Let $\mathcal{M}$ be a moduli space of stable vector bundle E of rank r over Y with $c_i(E)=c_i$ and fix the Hilbert polynomial. It is known that in general there is no universal vector bundle over $Y\times\mathcal{M}$ [Ra], [Le].

Fix a point $y_0\in Y$. Let $i: y_0\longrightarrow Y$ be the inclusion of y_0 and let $q: Y\longrightarrow y_0$ be the projection. There is a natural vector bundle W over $Y\times\mathcal{M}$ of rank r^2 with the following property : for any $m\in\mathcal{M}$ if E is the vector bundle over Y represented by m, then the restriction of W to $Y\times m$ is isomorphic to $\mathrm{Hom}((i\circ q)^*E, E)$. The vector bundle W is constructed using the observation that for any analytic subset $U\subseteq\mathcal{M}$

such that there is a universal vector bundle over $Y \times U$, any two universal vector bundles over $Y \times U$ differ by tensoring with a line bundle pulled back from U. This observation is a simple consequence of the projection formula and the fact that any automorphism of a stable vector bundle is a scalar multiplication.

Fix a Kähler metric ω on Y. From a celebrated theorem of Yau and Uhlenbeck [UY] (due to Donaldson for projective manifolds [Do] and due to Narasimhan and Seshadri [NS] for Riemann surfaces) for any stable vector bundle E over Y there is a unique Hermitian-Einstein connection on E. Any two Hermitian-Einstein metrics differ by multiplication with a constant scalar. This implies that the above vector bundle W has a natural Hermitian metric induced by any of the Hermitian-Einstein metrics over the vector bundles represented in $\mathcal{M}$.

Given any vector bundle V over $Y \times \mathcal{M}$, by a general construction of determinant line bundle [KM], [BGS] we have a line bundle $\det(V)$ over $\mathcal{M}$ whose fiber over $m \in \mathcal{M}$ is canonically identified with the line

$$\bigotimes_{i=0}^{d} \bigwedge^{\rm top} \Big(H^i(Y,\, V|_{Y\times m}) \Big)^{(-1)^i} ,$$

where $\Big(\bigwedge^{\rm top} F\Big)^{-1}$ means $\bigwedge^{\rm top} F^*$. This construction of [KM], [BGS] gives a homomorphism from the Grothendieck K-group $K(Y \times \mathcal{M})$ to the group of holomorphic line bundles over $\mathcal{M}$.

Consider the element

$$(W \,-\, \mathcal{O}^{\oplus r^2})^{\otimes (d+1)} \quad \in \quad K(Y \times \mathcal{M})\,, \tag{4.1}$$

where $\mathcal{O}^{\oplus r^2}$ is the trivial vector bundle on rank r^2 over $Y \times \mathcal{M}$. Let

$$\mathcal{L} \;:=\; \det\Big((W \,-\, \mathcal{O}^{\oplus r^2})^{\otimes (d+1)}\Big) \;\longrightarrow \mathcal{M} \tag{4.2}$$

be the determinant line bundle over $\mathcal{M}$.

By a general construction of [BGS] ([Q] when Y is a Riemann surface), using the earlier obtained natural metric on W and the Kähler metric ω on Y as the input, we have a Hermitian metric on $\mathcal{L}$ which is known as the *Quillen metric*. Let Ω denote the first Chern form for the Hermitian connection on $\mathcal{L}$. Our next step will be to identity the form Ω, and in particular, to calculate the first Chern class of $\mathcal{L}$.

Let $\psi \in H^1(Y, \mathbb{Q}) \bigotimes H^1(\mathcal{M}, \mathbb{Q})$ be the Künneth component of $c_1(W) \in H^2(Y \times \mathcal{M}, \mathbb{Q})$. Denoting the projection of $Y \times \mathcal{M}$ onto the i-th factor by p_i, consider

$$\gamma \;:=\; (\psi \cup \psi \cup p_1^*(c_1)^{d-1}) \cap [Y] \;\in\; H^2(\mathcal{M}, \mathbb{Q}) \tag{4.3}$$

where c_1, as defined earlier, is the first Chern class of a typical vector bundle represented in $\mathcal{M}$ and $\cap[Y]$ is the cap product with the oriented generator of $H_{2d}(Y, \mathbb{Q})$.

Let $\overline{\psi}$ be the differential 2-form on $Y \times \mathcal{M}$ obtained by taking the $(1,1)$-Künneth component of the first Chern form $c_1(W)$ for the Hermitian connection on W. Also, let $\bar{c}$ denote the (unique) harmonic two form on Y representing the cohomology class

c_1. Clearly the following two form

$$\Gamma \ := \ \int_Y \overline{\psi} \cup \overline{\psi} \cup p_1^*(\bar{c})^{d-1}$$

on $\mathcal{M}$ is closed, and it represents the cohomology class γ defined in (4.3).

For any $y \in Y$, let $i_y : \mathcal{M} \longrightarrow Y \times \mathcal{M}$ be the inclusion defined by $m \longmapsto (y, m)$. Denote by $c^{0,2}(W)$ the function from Y to the space of two forms on $\mathcal{M}$ which sends any $y \in Y$ to $i_y^* c_1(W)$, the pullback of the first Chern form of W. The volume form ω^d on Y, for the Kähler form ω, will denoted by dV. Now we are in a position the describe the curvature form Ω.

Theorem 4.1. *The first Chern form Ω for the Hermitian connection on $\mathcal{L}$ has the following expression:*

$$\Omega \ = \ \binom{d+1}{2}\Gamma \ + \ (d+1)\int_Y c^{0,2}(W)dV\,.$$

Furthermore, the first Chern class of $\mathcal{L}$ is $\binom{d+1}{2}\gamma$.

Proof. The main theorem of [BGS] (Theorem 0.1) expresses the first Chern form of $\mathcal{L}$ in terms of the Chern character form of W and the Todd forms of the relative tangent bundle for the projection p_2 of $Y \times \mathcal{M}$ onto $\mathcal{M}$. Since the virtual rank of $W - \mathcal{O}^{\oplus r^2}$ is zero, the lowest degree term of the Chern character of $(W - \mathcal{O}^{\oplus r^2})^{\otimes(d+1)}$ is of degree $2(d+1)$. In other words, if $ch_i(V) \in H^{2i}(Y \times \mathcal{M}, \mathbb{Q})$ denotes the component of the Chern character $ch(V)$ of $V \in K(Y \times \mathcal{M})$, then $ch_i\Big(W - \mathcal{O}^{\oplus r^2}\Big) = 0$ for $i \leq d$. Thus there is no contribution of the Todd forms of the tangent bundle of Y in the expression of the first Chern form of $\mathcal{L}$ according to Theorem 0.1 of [BGS]. The first part of the theorem now follows easily.

The second part of the theorem can be proved using the Grothendieck-Riemann-Roch theorem. The relevant observation is that the Künneth component of the first Chern class $c_1(W)$ in $H^0(Y, \mathbb{Q}) \bigotimes H^2(\mathcal{M}, \mathbb{Q})$ vanishes. The second part of the theorem now follows from the earlier observation that there is no contribution of the Todd classes of Y in the Grothendieck-Riemann-Roch formula for the first Chern class of $\mathcal{L}$.

The second part of the theorem can also be deduced directly from the first part. The form $i_{y_0}^* c_1(W)$ on $\mathcal{M}$ vanishes identically. Hence by the homotopy invariance of the pullback of a cohomology class, any $i_y^* c_1(W)$ is an exact form. This implies that $\int_Y c^{0,2}(W)dV$ is a exact form. Now the second part of the theorem follows from the first part. □

If either $H^1(Y, \mathbb{Q}) = 0$ or $c_1^{d-1} = 0$, then from Theorem 4.1 it follows that $c_1(\mathcal{L}) = 0$. We will give examples where $\mathcal{L}$ is a nontrivial line bundle.

Let A be an abelian variety of dimension g. For $c \in H^2(A, \mathbb{Z}) \cap H^{1,1}(A)$ let $\mathrm{Pic}^c(A)$ denote the component of $\mathrm{Pic}(A)$ consisting of line bundles L with $c_1(L) = c$. The vector

space $H^1(\mathrm{Pic}^c(A), \mathbb{Q})$ is canonically identified with $H^1(A, \mathbb{Q})^*$. Indeed, by fixing a point of $\mathrm{Pic}^c(A)$ it gets identified with $\mathrm{Pic}^0(A)$ Since

$$\mathrm{Pic}^0(A) \,=\, \mathrm{Hom}(H_1(A, \mathbb{Z}), U(1))$$

the vector space $H^1(\mathrm{Pic}^0(A), \mathbb{Q})$ is canonically identified with $H^1(A, \mathbb{Q})^*$. It is known that the Künneth component in $H^1(A, \mathbb{Q}) \bigotimes H^1(\mathrm{Pic}^c(A), \mathbb{Q})$ of the first Chern class of a universal line bundle over $A \times \mathrm{Pic}^c(A)$ is the above mentioned isomorphism.

Let $a \in H^2(A, \mathbb{Q})^*$ be the Poincaré dual of $c^{d-1} \in H^{2d-2}(A, \mathbb{Q})$. The earlier mentioned isomorphism between $H^1(\mathrm{Pic}^c(A), \mathbb{Q})$ and $H^1(A, \mathbb{Q})^*$ identifies $H^2(A, \mathbb{Q})^*$ with $H^2(\mathrm{Pic}^c(A), \mathbb{Q})$. Let

$$\bar{a} \quad \in \quad H^2(\mathrm{Pic}^c(A), \mathbb{Q})$$

be the element corresponding to the element a of $H^2(A, \mathbb{Q})^*$.

Using the earlier remark on the Künneth component of the first Chern class of a universal line bundle over $A \times \mathrm{Pic}^c(A)$, from Theorem 4.1 it is easily deduced that in this situation the following equality

$$c_1(\mathcal{L}) \;\; = \;\; -\binom{d+1}{2}\bar{a}$$

is valid. If c is a principal polarization on A then $\bar{a}$ is a principal polarization on $\mathrm{Pic}^c(A)$.

If ω is a translation invariant Kähler form on A, then the first Chern form for the Hermitian connection on $\mathcal{L}$ coincides with $-\binom{d+1}{2}\overline{\omega}$, where $\overline{\omega}$ is the unique translation invariant closed form on $\mathrm{Pic}^c(A)$ representing the cohomology class $\bar{a}$.

Now we will construct a second example where $\mathcal{L}$ is nontrivial. Actually this particular example led us to the construction of $\mathcal{L}$.

Let X be a compact connected Riemann surface of genus $g \geq 2$, Let $\mathcal{M}_\xi$ and $\mathcal{U}$ as before. Fix a Poincaré line bundle $\mathcal{P} \longrightarrow X \times J$, where $J = \mathrm{Pic}^0(X)$.

It has been proved in [BBN] that the vector bundle $\mathcal{U}$ is stable with respect to any polarization on $X \times \mathcal{M}_\xi$. Furthermore, the vector bundle

$$p_{12}^*\mathcal{U} \otimes p_{13}^*\mathcal{P} \quad \longrightarrow \quad X \times \mathcal{M}_\xi \times J\,,$$

where p_{ij} is the projection of $X \times \mathcal{M}_\xi \times J$ onto the product of the i-th and the j-th factor, is a complete family of vector bundles over $X \times \mathcal{M}_\xi$. In other words, J is a component of the moduli space of stable vector bundles over $X \times \mathcal{M}_\xi$.

Set $Y = X \times \mathcal{M}_\xi$ and in place of $\mathcal{M}$ substitute J realized as a component of the moduli space in the above fashion. Note that the Jacobian J has a natural polarization which is defined using the cap product on $H_1(X, \mathbb{Q})$. This polarization on J will be denoted by Θ.

Proposition 4.2. *The first Chern class $c_1(\mathcal{L})$ coincides with $l\Theta$ where l is an integer not equal to zero.*

Proof. Since $\mathrm{Pic}(\mathcal{M}_\xi) \cong \mathbb{Z}$, any polarization on $X \times \mathcal{M}_\xi$ is of the form

$$a[X] + b\Theta_{\mathcal{M}_\xi},$$

where $\Theta_{\mathcal{M}_\xi}$ (respectively, $[X]$) is the positive generator of the cyclic group $\mathrm{Pic}(\mathcal{M}_\xi)$ (respectively, $H^2(X, \mathbb{Z})$), and a, b are strictly positive integers.

Let β denote the Künneth component in $H^1(X, \mathbb{Q}) \bigotimes H^1(J, \mathbb{Q})$ of the first Chern class $c_1(\mathcal{P}) \in H^2(X \times J, \mathbb{Q})$.

Let q_2 denote the projection of $X \times \mathcal{M}_\xi \times J$ onto the factor $\mathcal{M}_\xi$. For a fixed $x \in X$, the first Chern class $c_1(\mathcal{U}|_{x\times\mathcal{M}_\xi}) \in H^2(\mathcal{M}_\xi, \mathbb{Q})$ will be denoted by δ.

We now note that from the second part of Theorem 4.1 the following equality easily obtained

$$c_1(\mathcal{L}) \;=\; \binom{d+1}{2} \int_{X\times\mathcal{M}_\xi} q_2^*\delta^{d-1} \cup p_{12}^*\beta^2\,, \tag{4.4}$$

where $d = \dim X \times \mathcal{M}_\xi = 1 + (r^2-1)(g-1)$ and $\int_{X\times\mathcal{M}_\xi}$ is the Gysin map from $H^i(X \times \mathcal{M}_\xi \times J, \mathbb{C})$ to $H^{i-2d}(J, \mathbb{C})$ defined by integration of forms along $X \times \mathcal{M}_\xi$.

The right-hand side of the equality (4.4) is equal to

$$\binom{d+1}{2}\Big(\int_{\mathcal{M}_\xi} \delta^{d-1}\Big) \int_X \beta^2\,,$$

where $\int_X$ is the Gysin map for the projection of $X \times J$ onto J. Using the isomorphism between $H^2(\mathcal{M}_\xi, \mathbb{Z})$ and $\mathbb{Z}$, the cohomology class δ is $mr+1$, where $m \in \mathbb{Z}$ [Ra]. This implies that the integer $\int_{\mathcal{M}_\xi} \delta^{d-1}$ is nonzero.

Finally, $\int_X \beta^2$ coincides with $-\Theta$. This is because the $(1,1)$-component β coincides with the cohomology class on $X \times J$ defined by the natural identification between $H^1(X, \mathbb{Q})$ and $H^1(J\,\mathbb{Q})$. This completes the proof of the proposition. □

Since Θ is a principal polarization on J, it can be recovered in a unique way from any given integral multiple of Θ. We equip the moduli space $\mathcal{M}_X^0(\mathcal{U})$ with the polarization defined by the positive one among $\pm c_1(\mathcal{L})$. Now Proposition 4.2 and the Torelli theorem for Riemann surfaces combine together to give the following theorem.

Theorem 4.3. *Let X and X' be two compact connected Riemann surfaces of genus $g \geq 2$. There is an isomorphism between $\mathcal{M}_X^0(\mathcal{U})$ and $\mathcal{M}_{X'}^0(\mathcal{U})$ preserving their polarizations if and only if $X \cong X'$.*

Now we set $Y = X \times D$, where D, as before, is a smooth anti-canonical divisor on $\mathcal{M}_\xi$, and set $\mathcal{M} = \mathcal{M}^0(\mathcal{U}_D)$, defined in Section 2. Assume that $n, g \geq 3$. From Theorem 2.10 we know that $\mathcal{M}^0(\mathcal{U}_D) = \mathrm{Pic}^0(X)$.

Theorem 4.4. *The first Chern class $c_1(\mathcal{L})$, where $\mathcal{L}$ is the line bundle over $\mathcal{M}^0(\mathcal{U}_D)$ defined in (4.2), coincides with a nonzero multiple of Θ on $\mathrm{Pic}^0(X)$. Moreover, let X and X' be two compact connected Riemann surfaces of genus $g \geq 3$. If $\mathcal{M}_X^0(\mathcal{U}) \cong \mathcal{M}_{X'}^0(\mathcal{U})$, as polarized varieties with polarizations obtained from $c_1(\mathcal{L})$, then $X \cong X'$.*

Proof. We have $H^1(D, \mathbb{Q}) = 0$ and $\mathrm{Pic}(D) = \mathbb{Z}$. The restriction of the cohomology class δ (defined in the proof of Proposition 4.2) on $\mathcal{M}_\xi$ to the subvariety D is nonzero. Given this situation, the proof of the theorem is exactly identical to the combination of the proofs of Proposition 4.2 and Theorem 4.3. □

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