THETA FUNCTIONS AND SZEGÖ KERNELS

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ABSTRACT. We study relations between two fundamental constructions associated to vector bundles on a smooth complex projective curve: the theta function (a section of a line bundle on the moduli space of vector bundles) and the Szegö kernel (a section of a vector bundle on the square of the curve). Two types of relations are demonstrated. First, we establish a higher–rank version of the prime form, describing the pullback of determinant line bundles by difference maps, and show the theta function pulls back to the determinant of the Szegö kernel. Next, we prove that the expansion of the Szegö kernel at the diagonal gives the logarithmic derivative of the theta function over the moduli space of bundles for a fixed, or moving, curve. In particular, we recover the identification of the space of connections on the theta line bundle with moduli space of flat vector bundles, when the curve is fixed. When the curve varies, we identify this space of connections with the moduli space of *extended connections*, which we introduce.

1. INTRODUCTION

Let X be a connected smooth projective curve over $\mathbb{C}$ of genus g. There are two fundamental constructions associated to vector bundles over X of rank n and Euler characteristic zero (hence degree $n(g-1)$). The first is the nonabelian theta function, which is a canonical section of the dual of the *determinant line bundle* on the corresponding moduli space. It is nonzero precisely for vector bundles E which have no sections, that is $\mathrm{H}^0(X,E) = 0 = \mathrm{H}^1(X,E)$, which form the complement of the canonical theta divisor Θ_X. The second starts with such a vector bundle E with no sections, and constructs a canonical "kernel function" on $X \times X$, the Szegö kernel of E. This kernel is the unique section of the vector bundle $E \boxtimes (E^* \otimes \Omega_X)$ over $X \times X$ (Ω_X is the holomorphic cotangent bundle of X), with only a first order pole along the diagonal and residue the identity endomorphism of E. The Szegö kernel in the case of line bundles has been extensively studied (see for example [19, 13, 25, 20]), as part of the study of abelian theta functions. In the higher rank case, it has appeared in the pioneering works of Fay [15], [14] and Takhtajan–Zograf [26] from an analytic point of view, where it is related to determinants of Laplace operators and analytic torsion. (Complex analytically, the role of the Szegö kernel is that of "reproducing kernel" for the holomorphic sections of E, or kernel for the inverse of the Dolbeault operator on E.) Our aim in this paper is to study the Szegö kernel algebraically, investigating some of its close connections with the theta function, thereby clarifying some relations between line bundles on moduli spaces and geometric objects on the curve itself.

1.1. **The nonabelian prime form.** We will focus on two different mechanisms linking our protagonists. The first and most direct link is given by difference maps. To a bundle E is naturally associated the family $E(y-x)$ of vector bundles on X, parametrized by $(x,y) \in X \times X$. This family is classified by a map δ_E from $X \times X$ to the moduli space of bundles. The pullback of the theta function by δ_E is a canonical section of the dual determinant line bundle of this family. In Theorem 4.3.3, we describe this line bundle canonically in terms of E. The identification, the "nonabelian prime form", holds for arbitrary families of curves and bundles on them (that is, over the moduli stack of curves and bundles, Corollary 4.3.5).

In the case of line bundles, this identification is given by the classical Klein prime form. Moreover, up to a scalar and multiplication by this prime form, the abelian Szegö kernel and the pullback of the theta function are in fact equal. (This is sometimes used as the *definition* of the Szegö kernel.) In Theorem 4.4.1 we provide a cohomological description of the nonabelian Szegö kernel and of the nonabelian prime form (for E with no sections). This makes it easy to compare to the (cohomologically defined) theta function: the prime form identifies the pullback of the theta function with the *determinant* of the Szegö kernel. (See also Remark 1.5 for related work.)

1.2. **Twisted cotangent bundles.** The second mechanism to relate the theta function and Szegö kernel is through a geometric description of the differential of the theta function, in terms of connection operators on the curve. More precisely, we identify the twisted forms of the cotangent bundle which carry the logarithmic differential of the theta function with spaces of kernel functions, as well as the corresponding sections. This is done by describing the behavior of the Szegö kernel along the generalized Theta divisor.

Let $\mathcal{L}$ denote a line bundle on a complex manifold M, and $\mathcal{C}onn_M(\mathcal{L})$ the sheaf of holomorphic connections on $\mathcal{L}$. Since the difference between any two (locally defined) connections on $\mathcal{L}$ is a holomorphic one–form, $\mathcal{C}onn_M(\mathcal{L})$ forms an affine bundle (torsor) for the cotangent bundle Ω^1_M. Such an affine bundle, equipped with a compatible symplectic form (which for $\mathcal{C}onn_M(\mathcal{L})$ arises as the curvature of a tautological connection), is known as a *twisted cotangent bundle* for M (see [2]). A meromorphic section s of $\mathcal{L}$ provides a meromorphic section of $\mathcal{C}onn_M(\mathcal{L})$, which is the coordinate–free form of the logarithmic differential $\mathrm{d}\log s$. Twisted cotangent bundles on moduli spaces of curves and of bundles have been studied in [27, 12, 3, 5]. It is known that the twisted cotangent bundle on $\mathfrak{M}_X(n)$ associated to the theta line bundle (which we denote $\mathcal{C}onn_X(\Theta)$) is identified with the moduli space $\mathcal{C}onn_X(n)$ of vector bundles with flat connection (with its natural symplectic structure).

In Proposition 5.3.1 we describe the sheaf of kernel functions of which the Szegö kernel is a section, and hence the behavior of the Szegö kernel along the theta divisor. In Theorem 5.4.1 we use this description to calculate the logarithmic derivative of the theta function and provide a new construction of the isomorphism $\mathcal{C}onn_X(\Theta) \cong \mathcal{C}onn_X(n)$. Specifically, the restriction of the Szegö kernel to 2Δ defines a canonical flat connection for every bundle E off of the theta divisor. The construction develops poles when E hits Θ_X, which match up precisely with the poles of $\mathrm{d}\log\theta$. It follows that the restriction of the Szegö kernel is identified with $\mathrm{d}\log\theta$. (This may be considered an algebraic analog of the calculation of the logarithmic derivative of the determinant of the Laplacian in [13], [14], [15] and [26].) It also follows that the Szegö kernel defines a Lagrangian family of flat connections. (In § 5.2, we work out an identity involving composition of Szegö kernels, which while not used elsewhere should have independent interest.)

Finally, in § 6 we let the curve vary as well as the bundle, describing twisted cotangent bundles over the moduli space $\mathfrak{M}_g(n)$ of curves and bundles on them. To do so we introduce the notion of *extended connections*, which form a twisted cotangent bundle $\mathcal{E}x\mathcal{C}onn_g(n)$ over the moduli space of curves and bundles. The restriction of the Szegö kernel to 3Δ gives rise to a canonical extended connection associated to bundles E off of the theta divisor. In Theorem 6.3.1, we give a (unique) identification of the twisted cotangent bundle $\mathcal{C}onn_g(\Theta)$ over this larger moduli space with $\mathcal{E}x\mathcal{C}onn_g(n)$ (due in a different form to [5]), and identify the logarithmic differential of the theta function with the restriction of the Szegö kernel.

1.3. **Organization of the paper.** In Sections § 2 and § 3 we provide some background information on theta functions, twisted cotangent bundles, kernel functions and connections. In Section § 4 we discuss the nonabelian prime form, the difference map, the cohomological description of the Szegö kernel, and the identification of its determinant with the theta function. In Section § 5 we study the relation of the Szegö kernel to the moduli space of bundles with connections, by identifying its behavior near the theta divisor. Finally in Section § 6 we introduce extended connections and extend the results of Section § 5 over the moduli space of curves and bundles.

Section § 4 and Sections § 5, § 6 can be read independently of each other.

1.4. **Further aspects.** In [6] we study different aspects of extended connections and their relation with projective structures on X. In particular, we introduce a quadratic map from extended connections to projective structures, which is a deformation of the quadratic part of the Hitchin map. Applying this map to the Szegö kernels, we map moduli spaces of bundles rationally to spaces of projective structures. We study these maps in detail for line bundles, relating them to several classical constructions and proving finiteness results. Extended connections and the quadratic map appear in [7] from classical limits of the heat operators for nonabelian theta functions.

1.5. **Remark.** After this paper was completed in early 2001, we became aware of the papers [17] and [23], which prove addition formulae for non-abelian theta functions, giving a different approach to the results of § 4 relating the Szegö kernel and theta function via the difference map.

1.6. **Notation.** Throughout this paper, X will denote a smooth connected complex projective curve (a compact connected Riemann surface) of genus at least two. All constructions will be algebraic over $\mathbb{C}$. The diagonal in $X \times X$ consisting of points of the form $(x\,, x)$ will be denoted by Δ. For vector bundles V, W on X, we will denote by $V \boxtimes W$ the vector bundle $p_1^* V \bigotimes p_2^* W$ over $X \times X$, where p_i is the projection to the ith factor. For a coherent sheaf F on $X \times X$, we will use the notation $F(n\Delta)$ for the sheaf $F \otimes \mathcal{O}_{X\times X}(n\Delta)$ of meromorphic sections with only nth order pole at the diagonal. The canonical line bundle (holomorphic cotangent bundle) of X will be denoted by Ω_X. For a holomorphic vector bundle E over X, its Serre dual, namely $E^* \otimes \Omega_X$, will be denoted by $E^\vee$. We will often use the same notation for a sheaf and its space of sections (which is implied should be clear from the context).

2. Background

2.1. **Theta characteristics.** A *theta characteristic* on X is a square root of the canonical bundle – that is, a line bundle $\Omega_X^{\frac{1}{2}}$ together with an isomorphism $(\Omega_X^{\frac{1}{2}})^{\otimes 2} = \Omega_X$. There are 2^{2g} possible choices of $\Omega_X^{\frac{1}{2}}$ on a connected smooth projective curve of genus g. The moduli space of curves with a choice of theta characteristic is $\mathfrak{M}_g^{spin}$, the moduli space of *spin curves*.

Tensoring by $\Omega_X^{-\frac{1}{2}}$ sends bundles of Euler characteristic 0 to bundles of degree zero. For a vector bundle E, we use the notation E_0 for $E \otimes \Omega_X^{-\frac{1}{2}}$ (so the operation $E \mapsto E_0$ is well–defined over the moduli space of spin curves.) Note that $(E_0)^* = (E^\vee)_0$, so that multiplication by $\Omega_X^{-\frac{1}{2}}$ converts Serre duality into sheaf duality.

The ratio of two theta characteristics is a line bundle κ equipped with an isomorphism $\kappa^{\otimes 2} = \mathcal{O}$. Thus κ carries a canonical flat connection, inducing the usual connection on $\mathcal{O}$. Using parallel transport for this connection, a canonical trivialization of $\kappa \boxtimes \kappa^*$ is obtained on any neighborhood of $\Delta \subset X \times X$ that contracts to Δ, in particular, on the infinitesimal neighborhood $n\Delta$ for any n. It follows, for example, that the line bundle

$$\mathcal{M}_1 \;=\; \Omega_X^{\frac{1}{2}} \boxtimes \Omega_X^{\frac{1}{2}}(\Delta)$$

on $X \times X$ when restricted to the infinitesimal neighborhood $n\Delta$ for any n, is in fact independent of the choice of theta characteristic. Our use of the ambiguous notation $\Omega_X^{\frac{1}{2}}$ is thus justified whenever we study kernel functions of the above form in a neighborhood of the diagonal. The choice of $\Omega_X^{\frac{1}{2}}$ will appear only when translating statements for Euler characteristic zero bundles to statements for degree zero bundles.

The tensor power $(\Omega_X^{\frac{1}{2}})^{\otimes k}$ (for any $k \in \mathbb{Z}$) will be denoted by $\Omega_X^{\frac{k}{2}}$. Note that the line bundle $\Omega_X^{\frac{1}{2}}$ is Serre self–dual, $\Omega_X^{\frac{1}{2}} = (\Omega_X^{\frac{1}{2}})^\vee$.

2.2. **Generalized theta functions and determinant bundles.** We recall some facts about theta functions, determinant bundles and the canonical theta divisor (see for example [12, 11, 24, 8]).

Let $\pi : \mathfrak{X} \to S$ be a smooth family of projective curves over a connected base S, with fibers X_s for $s \in S$. To a coherent sheaf E on $\mathfrak{X}$ flat over S, one assigns a line bundle $d(E)$ over the base S, the determinant of the cohomology of E, $d(E) = \det R^{\cdot}\pi_* E$ (EGA III,7). More precisely, this line bundle is calculated as follows. Locally on S there exists a complex

$$\mathcal{K}^{\cdot} = \{\mathcal{K}^0 \xrightarrow{\partial} \mathcal{K}^1\}$$

of finite rank free $\mathcal{O}_S$–modules, with the property that for any coherent sheaf $\mathcal{G}$ on S there is a natural isomorphism

$$R^i\pi_*(E \otimes \pi^*\mathcal{G}) \cong \mathrm{H}^i(\mathcal{K}^{\cdot} \otimes \mathcal{G}).$$

Such $\mathcal{K}^{\cdot}$ is determined uniquely up to unique quasi-isomorphism. The determinant line bundle $d(E)$ is then defined as $d(E) = \det \mathcal{K}^0 \otimes (\det \mathcal{K}^1)^*$. The determinant functor d descends to a well–defined homomorphism from the K–group of coherent sheaves to the Picard group of S (so that the determinant of the virtual bundle $E - F$ is $d(E)d(F)^*$). For a point $s \in S$, the fiber of $d(E)$ over s is identified with the one dimensional vector space

$$\wedge^{\mathrm{top}}\mathrm{H}^0(X_s, E|_{X_s}) \otimes \wedge^{\mathrm{top}}\mathrm{H}^1(X_s, E|_{X_s})^* .$$

Now suppose that the Euler characteristic of $E|_{X_s}$ vanishes for some (hence every) s so that $\mathcal{K}^0$ and $\mathcal{K}^1$ are of the same rank. Then the determinant $\det \partial$ gives a canonical section θ of the dual bundle $d(E)^*$, known as the theta function of E. The theta function vanishes precisely for $s \in S$ with $\mathrm{H}^0(X_s, E|_{X_s}) \neq 0$ (hence $\mathrm{H}^1(X_s, E|_{X_s}) \neq 0$).

2.2.1. *Moduli spaces.* Let $\mathfrak{M}_X(n)$ denote the moduli space of semistable vector bundles over X of rank n and Euler characteristic 0. So the degree of a vector bundle in $\mathfrak{M}_X(n)$ is $n(g-1)$. It is known that $\mathfrak{M}_X(n)$ is an irreducible normal projective variety. Moreover, the strictly semistable locus is of codimension greater than one (unless both n and g are 2). Its complement $\mathfrak{M}^s_X(n)$ defined by all stable vector bundles is smooth.

We will repeatedly take advantage of the principle that functions, line bundles and twisted cotangent bundles are not affected by subvarieties of codimension at least 2. In other words, the pullback defined by the inclusion of of the complement of a subvariety of codimension at least two is an isomorphism. Thus, we will be able to ignore the strictly semistable locus of $\mathfrak{M}_X(n)$ and assume all vector bundles involved are stable.

We will always exclude the special case where both n and g are two. Although there is no universal vector bundle over $X \times \mathfrak{M}_X(n)$, it exists locally (in analytic or étale topologies) on the stable locus $\mathfrak{M}^s_X(n) \subset \mathfrak{M}_X(n)$. Moreover, all automorphisms of a stable vector bundle are scalars. It follows that there is a universal vector bundle over $X \times X \times \mathfrak{M}^s_X(n)$ of rank n^2 such that for any $E \in \mathfrak{M}^s_X(n)$, the restriction of the vector bundle to $X \times X \times \{E\}$ is $E \boxtimes E^*$.

Let $\mathfrak{M}_X(n)_0$ denote the moduli space of semistable vector bundles of rank n and degree 0. If we fix a theta characteristic $\Omega_X^{\frac{1}{2}}$, then there is an isomorphism

$$\mathfrak{M}_X(n) \longrightarrow \mathfrak{M}_X(n)_0, \qquad E \mapsto E_0 = E \otimes \Omega_X^{-\frac{1}{2}}$$

since tensoring by a line bundle preserves semistability. Thus over the moduli space of spin curves, the moduli space of degree zero and Euler characteristic zero bundles are canonically identified.

The cotangent bundle to $\mathfrak{M}_X(n)$ (at stable points E) is identified with the space $\mathrm{H}^0(X,\, \Omega_X \otimes \mathrm{End}\, E)$ of endomorphism–valued one–forms, or *Higgs fields*, on E.

2.2.2. *Line bundles on moduli.* The moduli space $\mathfrak{M}_X(n)$ carries a universal determinant line bundle. This is seen as follows. If E is a vector bundle over $X \times S$ for some scheme S and L a line bundle over S, then it is easy to see from the projection formula that

$$d(E \otimes \pi_S^* L) \;\cong\; d(E) \otimes L^{\chi} \tag{1}$$

where π_S is the projection of $X \times S$ to S and χ is the Euler characteristic of the restriction of E to the fibers of π_S (which is a locally constant function on S). The determinant line bundle over $\mathfrak{M}_X(n)$ is constructed using the local universal families; the independence of the choices of local families is ensured by (1).

The determinant line bundle on $\mathfrak{M}_X(n)$ has a concrete geometric description. The subvariety

$$\Theta_X \;:=\; \{V \in \mathfrak{M}_X(n) \,|\, \mathrm{H}^0(X,\, V) \;\neq\; 0\}$$

is a divisor, the *generalized theta divisor*, that gives the ample generator of $\mathrm{Pic}(\mathfrak{M}_X(n))$ [11]. Note that for any E in $\mathfrak{M}_X(n)$, we have $\mathrm{H}^0(X,\, E) \;=\; 0$ if and only if $\mathrm{H}^1(X,\, E) \;=\; 0$. This condition also guarantees that E is semistable. Indeed, if a subbundle F of E violates the semistability condition, then $\mathrm{h}^0(F) - \mathrm{h}^1(F) > 0$, thus contradicting the condition that $\mathrm{h}^0(E) = 0$ (as $\mathrm{H}^0(X, F) \subset \mathrm{H}^0(X, E)$). The line bundle over $\mathfrak{M}_X(n)$ defined by the divisor Θ_X will also be denoted by Θ_X, and is canonically identified with the determinant line bundle.

It is known that the smooth locus of the theta divisor Θ_X is precisely the subvariety Θ_X° of vector bundles E with $\mathrm{h}^0(E) = \mathrm{h}^1(E) = 1$. The singular locus $\Theta_X^{\mathrm{sing}} \subset \Theta_X$ is of codimension at least 2 and consists of all vector bundles E with $\mathrm{h}^0(E) = \mathrm{h}^1(E) > 1$.

2.3. **Twisted cotangent bundles.** (Our reference for twisted cotangent bundles is [2]; see also [27] for Ω–torsors.)

Let $\mathcal{L}$ denote a holomorphic line bundle over a complex manifold M, and $\mathcal{C}onn_M(\mathcal{L})$ the sheaf of holomorphic connections on $\mathcal{L}$. The difference between any two connections is a (scalar–valued) one–form on M. More precisely, $\mathcal{C}onn_M(\mathcal{L})$ is an affine bundle for the cotangent sheaf Ω_M of M (an Ω_M–torsor). Recall that an affine bundle, or torsor, is the relative version of the notion of affine space. Namely given a holomorphic vector bundle V over a variety M, an affine bundle for V over M is a morphism $\pi : A \to M$, which locally admits a section, equipped with a simply transitive action of the sheaf of sections of V on the sections of A. (We use the terms affine space for a sheaf, affine bundle and torsor interchangeably.)

It is convenient to describe this torsor using the Atiyah exact sequence and the bundle of one–jets. The Atiyah bundle of $\mathcal{L}$ is the sheaf $\mathcal{A}_{\mathcal{L}} = \mathcal{D}_{\leq 1}(\mathcal{L}, \mathcal{L})$ of differential operators of order at most one acting on sections of $\mathcal{L}$. It sits in an extension

$$0 \to \mathcal{O} \to \mathcal{A}_{\mathcal{L}} \to \mathcal{T}_M \to 0$$

of the tangent sheaf by the structure sheaf. The dual sequence is an extension

$$0 \to \Omega_M \to \mathcal{A}_{\mathcal{L}}^* \to \mathcal{O} \to 0$$

The bundle of one–jets $J^1\mathcal{L}$ of sections of $\mathcal{L}$, which is an extension

$$0 \to \Omega_M \otimes \mathcal{L} \to J^1\mathcal{L} \to \mathcal{L} \to 0$$

of $\mathcal{L}$ by the sheaf of $\mathcal{L}$–valued differentials, is obtained from the dual to the Atiyah sequence by tensoring by $\mathcal{L}$.

The sheaf $\mathcal{C}onn_M(\mathcal{L})$ of connections on $\mathcal{L}$ is naturally identified with the sheaf of splittings of any of the above three sequences. In particular, $\mathcal{C}onn_M(\mathcal{L})$ may be identified with the inverse image of the section $1 \in \Gamma(\mathcal{O})$ in $\mathcal{A}_{\mathcal{L}}^*$. From this description the structure of affine space over the sheaf Ω_M of differentials is evident.

Consider the projectivization $\mathbb{P}\mathcal{A}_{\mathcal{L}}^* = \mathbb{P}J^1\mathcal{L}$, which is a projective space bundle containing a projective subbundle $\mathbb{P}\Omega_M$. The complementary affine space bundle is naturally identified with $\mathcal{C}onn_M(\mathcal{L})$. A nonzero meromorphic section s of the line bundle $\mathcal{L}$

gives rise to a one–jet $J^1 s$, which by projectivization gives rise to a meromorphic section of $\mathcal{C}onn_M(\mathcal{L}) \subset \mathbb{P}J^1\mathcal{L}$, which has poles whenever s vanishes or has a pole. This is called the *logarithmic differential* and it is denoted by $\mathrm{d}\log s$. The resulting connection on $\mathcal{L}$ has logarithmic singularities along the divisor of s, and can be characterized as the unique meromorphic connection for which s is a flat section.

2.3.1. *Symplectic structure.* Let

$$\pi : \mathcal{C}onn_M(\mathcal{L}) \longrightarrow M$$

be the projection map. The pullback bundle $\pi^*\mathcal{L}$ over $\mathcal{C}onn_M(\mathcal{L})$ has a tautological connection $\nabla_{\mathcal{L}}$. To see this first observe that over $\mathcal{C}onn_M(\mathcal{L})$ we have a tautological section of $\pi^* \mathrm{Hom}(\mathcal{L}, J^1\mathcal{L})$. There is a natural inclusion of $\pi^*J^1\mathcal{L}$ in $J^1\pi^*\mathcal{L}$ defined by the pullback operation. Combining these two remarks, we obtain a section of $\mathrm{Hom}(\pi^*\mathcal{L}, \pi^*J^1\mathcal{L})$. The connection on $\pi^*\mathcal{L}$ is defined by this section.

Let $\omega_{\mathcal{L}}$ be the curvature of ∇_L, which is a holomorphic 2-form on $\mathcal{C}onn_M(\mathcal{L})$. It is easy to see that ω_L is a symplectic form. Indeed, if we choose a local trivialization of $\mathcal{L}$, the resulting identification $\mathcal{C}onn_M(\mathcal{L}) \to \Omega^1_M$ takes ω_L to the canonical symplectic form on the cotangent bundle. It follows that the form $\omega_{\mathcal{L}}$ is changed by the pullback of $d\alpha$ under the action of a local section α of Ω^1_M on $\mathcal{C}onn_M(\mathcal{L})$. In particular, the symplectic form is preserved precisely by the action of *closed* one–forms on $\mathcal{C}onn_M(\mathcal{L})$. It follows that $\mathcal{C}onn_M(\mathcal{L})$ is a *twisted cotangent bundle*:

2.3.2. *Definition.* A twisted cotangent bundle on M is an Ω^1_M–torsor A, equipped with a symplectic form ω_A which transforms under the translation action of a local section α of Ω^1_M as

$$\alpha : \omega_A \mapsto \omega_A + df^*\alpha$$

where f denotes the projection of A to M.

2.4. **Rigidity.** Twisted cotangent bundles on the moduli spaces of curves and bundles enjoy a strong rigidity property. The automorphisms of a twisted cotangent bundle on M are given by closed one–forms on M, and the automorphisms of an Ω^1_M–torsor are given by all one–forms on M. However, as we explain below, the moduli spaces in question carry no one–forms at all. It follows that any two isomorphisms between Ω^1_M–torsors on these spaces automatically agree. This will enable us to compare our descriptions of Ω^1_M–torsors on moduli spaces with those of [27, 12, 5].

2.4.1. *Moduli space of curves.* It is not hard to verify that the moduli space $\mathfrak{M}_g$ does not carry any closed one–forms α for $g > 2$. The first cohomology of $\mathfrak{M}_g$ vanishes ([18]), so that α must in fact be exact as the periods of a one-form must be zero. But $\mathfrak{M}_g$ does not admit any nonconstant holomorphic functions if $g > 2$. This follows from the existence of the Satake compactification of $\mathfrak{M}_g$, which is a projective variety in which the complement of $\mathfrak{M}_g$ has codimension 2, for $g > 2$. Therefore the form α must vanish identically. (Note that $\mathcal{M}_2$ is affine, and hence it admits nonzero exact holomorphic one-forms.)

In fact, a stronger statement holds: $\mathfrak{M}_g$ does not carry *any* nonzero holomorphic one–forms, provided $g > 2$. This follows from the fact, explained to us by E. Looijenga in [21], that all two–forms on $\mathfrak{M}_g$ vanish (hence all one–forms must be closed, and therefore zero.)

2.4.2. *Moduli of vector bundles.* For a fixed line bundle $\xi \in \mathrm{Pic}^{g-1}(X)$, let $\mathfrak{M}_X(n,\xi) \subset \mathfrak{M}_X(n)$ be the moduli space of semistable vector bundles E with $\det E \cong \xi$. The nonexistence of holomorphic one–forms on $\mathfrak{M}_X(n,\xi)$ follows from unirationality of the moduli space. It also follows immediately from the usual method of computing cohomologies using the Hecke transformation.

A similar statement holds for the moduli space of vector bundles *without fixed determinant* if we vary X as well. Namely, all the one–forms on $\mathfrak{M}_X(n)$ are pullbacks of one–forms from $\mathrm{Pic}^{g-1}(X)$, which are in turn identified with $\mathrm{H}^0(X,\Omega_X)$. Since there are no holomorphic relative forms over the universal curve $\mathfrak{X} \to \mathfrak{M}_g$, it follows that there is no nonzero one–form on the universal moduli space $\mathfrak{M}(n)$ over $\mathfrak{M}_g$.

3. Connections and kernels

3.1. **Kernel functions.** Let V, W be two holomorphic vector bundles over X, and let $\mathrm{Diff}^n(V,W)$ be the sheaf of differential operators of order n from V to W. There is an elegant description of $\mathrm{Diff}^n(V,W)$ in terms of "integral kernels" on $X \times X$, due to Grothendieck and Sato (see [5, 16]). (This is a coordinate–free reformulation of the Cauchy integral formula, describing differentiation in terms of residues.) Namely,

$$\mathrm{Diff}^n(V,W) \cong \frac{W \boxtimes V^\vee((n+1)\Delta)}{W \boxtimes V^\vee} \cong W \boxtimes (V^* \otimes \Omega_X)((n+1)\Delta))\big|_{(n+1)\Delta}. \tag{2}$$

Here we consider nth order differential operators as a bimodule over $\mathcal{O}_X$ (via left and right multiplication), and hence as a coherent torsion sheaf on $X \times X$, supported on $(n+1)\Delta$. In other words, we consider kernel functions of the form $\psi(z,w)dw$ on $X \times X$, with poles of order $\leq n+1$ allowed on the diagonal, and quotient out by regular sections. These act as differential operators by sending $\psi(z,w)dw$ to the operator δ_ψ defined by

$$\delta_\psi(f(z)) = \mathrm{Res}_{z=w}\langle \psi(z,w), f(z)\rangle dw$$

where $f(z)$ is a local section of V and $\langle -,-\rangle$ is the contraction of V^* with V. So, $\langle \psi(z,w), f(z)\rangle dw$ is a local section of $W \boxtimes \Omega_X((n+1)\Delta))$ and hence its residue is a local section of W.

3.1.1. *The de Rham kernel.* Let $d : \mathcal{O}_X \longrightarrow \Omega_X$ be the de Rham differential. Using (2), this operator defines a holomorphic section $\mu(d)$ of $\Omega_X \boxtimes \Omega_X(2\Delta)$ over 2Δ. Since the symbol of the differential operator d is the section of $\mathcal{O}_X$ defined by the constant function 1, the restriction of $\mu(d)$ to Δ is the constant function 1 (considered as a section of $\Omega_X \boxtimes \Omega_X(2\Delta)|_\Delta = \mathcal{O}_\Delta$, by Poincaré adjunction formula).

More generally, consider the line bundle

$$\mathcal{M}_\nu = \Omega_X^{\frac{\nu}{2}} \boxtimes \Omega_X^{\frac{\nu}{2}}(\nu\Delta)$$

over $X \times X$, where $\nu \in \mathbb{Z}$ is an integer. By § 2.1, it follows that the restriction $\mathcal{M}_\nu|_{n\Delta}$ is independent of the choice of theta characteristic for any integers ν, n.

We will throughout use the convention that the normal bundle of Δ in $X \times X$ is identified with the tangent bundle $T\Delta$ using the projection to the first factor. In other words, if p_1 denotes the projection to the first factor of $X \times X$, then the isomorphism is the composition of the isomorphism of the normal bundle with p_1^*TX defined by the differential of p_1 with the natural identification of $T\Delta$ with TX. Note that if we use the projection to the second factor, then the isomorphisms differ by a sign.

Let $\sigma : X \times X \to X \times X$ be the involution defined by $(x, y) \mapsto (y, x)$. Note that the adjunction formula identifies the restriction of the line bundle $\mathcal{O}_{X\times X}(\Delta)$ to Δ with the normal bundle N of Δ in $X \times X$. Using the ordering of the product $X \times X$, the normal bundle N gets identified with the tangent bundle of the divisor Δ. The tangent bundle $T\Delta$ is identified with TX using the projection p_2. On the other hand, projecting the line subbundle $(p_2^*TX)|_\Delta \subset (T(X \times X))|_\Delta$ to the quotient N of $(T(X \times X))|_\Delta$ we get an isomorphism of $(p_2^*TX)|_\Delta$ with N. Note that the isomorphism of N with $T\Delta$ changes by multiplication with -1 if the two factors in $X \times X$ are interchanged (that is, if we use p_1 instead of p_2).

Consequently, the restriction of $\mathcal{M}_\nu$ to Δ is identified with $\Omega_X^{\otimes\mu} \otimes N^{\otimes\mu}$. Using the above identification of $T\Delta$ with N the line bundle $\Omega_X^{\otimes\mu} \otimes N^{\otimes\mu}$ is trivialized. It is easy to see that for any ν there is a unique trivialization μ_ν of $\mathcal{M}_\nu|_{2\Delta}$ such that

(1) $\mathcal{M}_\nu|_\Delta \cong \mathcal{O}_X$ (that is, $\mu_\nu|_\Delta = 1$);
(2) the trivialization is symmetric, respecting the identification $\mathcal{M}_\nu \cong \sigma^*\mathcal{M}_\nu$ (in other words, $\sigma^*\mu_\nu = (-1)^\nu \mu_\nu$).

The appearance of the factor $(-1)^\nu$ is due to the above remark that the isomorphism of N with $T\Delta$ changes sign as the two factors are interchanged. In particular, $\mu_\nu = \mu_1^{\otimes\nu}$ and $\mu_2 = \mu(d)$, where $\mu(d)$ is the section of $\mathcal{M}_2$ over 2Δ defined by the de Rham differential d.

3.2. **Connections.** Let E denote a vector bundle over X of rank n, and $\mathcal{C}onn_X(E)$ the sheaf of holomorphic connections on E. Note that $\mathcal{C}onn_X(E)$ will have no global sections unless E has degree zero.

A connection ∇ on E may be described as a first–order differential operator $\nabla : E \to E \otimes \Omega_X$ whose symbol is the identity automorphism of X. (This condition on symbols is equivalent to the Leibniz identity.) It follows from the differential operators–kernels dictionary in (2) that giving a holomorphic connection on E is equivalent to giving a section of

$$\mathcal{M}_2(E) := (E \otimes \Omega_X) \boxtimes (E^* \otimes \Omega_X)(2\Delta)$$

over 2Δ whose restriction to Δ is the identity section of

$$\mathcal{M}_2(E)|_\Delta \;\cong\; \operatorname{End} E.$$

We also see that the difference between any two connections is a section of $\Omega_X \otimes \operatorname{End} E$. Thus $\mathcal{C}onn_X(E)$ is an affine space for the space $\mathrm{H}^0(X,\, \Omega_X \otimes \operatorname{End} E)$ of Higgs fields on E.

More generally, let

$$\mathcal{M}_\nu(E) = (E \boxtimes E^*) \otimes \mathcal{M}_\nu = (E \otimes \Omega_X^{\frac{\nu}{2}}) \boxtimes (E^* \otimes \Omega_X^{\frac{\nu}{2}})(\nu\Delta).$$

On 2Δ, we obtain an extension

$$0 \to \operatorname{End} E \otimes \Omega_X \to \mathcal{M}_\nu(E)|_{2\Delta} \to \operatorname{End} E \to 0.$$

Since $\mathcal{M}_\nu$ is trivialized on 2Δ (by the section μ_ν in § 3.1.1), we may identify the sheaves $\mathcal{M}_\nu(E)|_{2\Delta}$ above for all integers ν. Consequently, $\mathcal{C}onn_X(E)$ coincides with the space of sections of $\mathcal{M}_\nu(E)|_{2\Delta}$ lifting $\mathrm{Id} \in \operatorname{End} E$, and the difference between any two such sections is naturally a Higgs field.

3.2.1. *Remark.* Recall that, in Grothendieck's formulation, a connection ∇ on E is the data of identifications of the fibers of E at infinitesimally nearby points. More precisely, a connection on E is an isomorphism between the two pullbacks p_1^*E and p_2^*E on the first–order infinitesimal neighborhood 2Δ of the diagonal, which restricts to the identity endomorphism of E on the diagonal. Note that since

$$\underline{Hom}_{\mathcal{O}_{X\times X}}(p_2^*E, p_1^*E) = E \boxtimes E^*,$$

this is equivalent to the data of a section, which we denote k_∇, of $E \boxtimes E^*$ on 2Δ, whose restriction to Δ is the identity. This is the case $\nu = 0$ of the above construction.

3.2.2. *The Atiyah bundle.* Another related point of view on connections is due to Atiyah, [1]. For a vector bundle E over X, the symbol map sends $\mathcal{D}_{\leq 1}(E, E)$, the sheaf of differential operators of order 1, to $\mathrm{Hom}(E \otimes \Omega_X, E)$. The *Atiyah bundle* $\mathrm{At}(E)$ of E consists of differential operators with *scalar* image. So $\mathrm{At}(E)$ is the inverse image of the scalars $\mathcal{T} \subset \mathrm{Hom}(E \otimes \Omega_X, E)$ in $\mathcal{D}_{\leq 1}(E, E)$. Thus, $\mathrm{At}(E)$ fits in an exact sequence

$$0 \longrightarrow \mathrm{End}\, V \longrightarrow \mathrm{At}(E) \longrightarrow \mathcal{T}_X \longrightarrow 0\,.$$

In the language of kernels, the Atiyah bundle appears as follows:

$$\mathrm{At(E)} = \{\psi \in \Gamma(\mathrm{E} \boxtimes \mathrm{E}^\vee(2\Delta)|_{2\Delta}) \bigm| \psi|_\Delta \in \Omega_\mathrm{X} \cdot \mathrm{Id}_\mathrm{E}\}\,.$$

(Here we use the identification $E \boxtimes E^\vee(2\Delta)|_\Delta \cong \Omega_X \otimes \mathrm{End}\, E$ of symbols.) The sheaf $\mathrm{At(E)} \otimes \Omega_\mathrm{X}$ is naturally identified with the subsheaf of $\mathcal{M}_2(E)|_{2\Delta}$ consisting of all sections whose restriction to the diagonal is contained in the subsheaf $\mathcal{O} \cdot \mathrm{Id} \subset \mathrm{End}\,\mathrm{E}$. Hence again we see that a holomorphic connection on E is simply a splitting of the Atiyah sequence.

3.3. **Twisted connections.** Let E denote a rank n vector bundle on X, and consider the sheaf

$$\mathcal{M}(E) = E \boxtimes E^\vee(\Delta)$$

on $X \times X$. For an arbitrary choice of theta characteristic, there is a canonical identification

$$\mathcal{M}(E) = \mathcal{M}_1(E_0).$$

The affine space $\mathcal{C}onn_X(E_0)$ of connections on $E_0 = E \otimes \Omega_X^{-\frac{1}{2}}$ depends only on E, not on the choice of $\Omega_X^{\frac{1}{2}}$: it is identified with (sections of) the affine bundle

$$\mathcal{C}onn_X(E_0) = \{s \in \mathcal{M}(E)|_{2\Delta} \bigm|\, s|_\Delta = \mathrm{Id}_E\}.$$

This can also be seen as follows: if κ is a line bundle with a given flat connection, then there is a canonical isomorphism between $\mathcal{C}onn_X(F)$ and $\mathcal{C}onn_X(F \otimes \kappa)$. Since the ratio of two theta characteristics is such a κ (§ 2.1), we see that the affine bundles $\mathcal{C}onn_X(E_0)$ are independent of the choice.

We will refer to elements of $\mathcal{C}onn_X(E_0)$ as *twisted connections* on E. In particular $\mathcal{C}onn_X(E_0)$ will have no global sections unless E has Euler characteristic zero (so that E_0 has degree zero.)

3.4. **The Szegö kernel.** There is a canonical kernel function associated to vector bundles E off of the theta divisor: the nonabelian Szegö kernel. (The proof of the uniqueness and existence below is straightforward — a stronger version is given in Proposition 5.1.1.)

3.4.1. *Definition.* Let $E \in \mathfrak{M}_X(n) \setminus \Theta_X$ be a vector bundle with $\mathrm{h}^0(E) = \mathrm{h}^1(E) = 0$. The Szegö kernel of E is the unique section $\mathfrak{s}_E \in \mathrm{H}^0(X \times X, \mathcal{M}(E))$ with $\mathfrak{s}_E|_\Delta = \mathrm{Id}_E$.

Equivalently, we will also consider the Szegö kernel as a meromorphic section of $E \boxtimes E^\vee$ with a pole of order exactly one on the diagonal. The that case, the above condition $\mathfrak{s}_E|_\Delta = \mathrm{Id}$ translates into the condition that the residue of the meromorphic section is the identity automorphism of E.

4. Nonabelian Prime Forms and the Szegö kernel

4.1. **The Prime Form.** Let $\mathrm{Jac} = \mathfrak{M}_X(1)_0$ denote the Jacobian variety of X. Consider the difference map

$$\delta\,:\, X \times X \to \mathrm{Jac}, \qquad (x,y) \mapsto y - x\,.$$

Its image is a surface in the Jacobian referred to as $X - X$. This map classifies a natural line bundle on $X \times (X \times X)$. The line bundle on $X \times (X \times X)$ in question, which we will denote by $\mathcal{O}(\mathfrak{y} - \mathfrak{x})$, is defined as follows:

$$\mathcal{O}(\mathfrak{y} - \mathfrak{x}) \;:=\; \mathcal{O}(\Delta_{02} - \Delta_{01})\,.$$

It may also be characterized by the following conditions:

(1) for any point $(x,y) \in X \times X$, the restriction of $\mathcal{O}(\mathfrak{y} - \mathfrak{x})$ to $X \times (x,y)$ is isomorphic to the line bundle $\mathcal{O}_X(y-x)$ over X;
(2) for a point $z \in X$, the restriction of $\mathcal{O}(\mathfrak{y} - \mathfrak{x})$ to $z \times X \times X$ is isomorphic to $\mathcal{O}_X(-z) \boxtimes \mathcal{O}_X(z)$.

Here and in what follows, we use the following notations for the cube $X \times X \times X$: we label the three copies of X by $0, 1, 2$ and let $\Delta_{ij} \subset X \times X \times X$ denote the diagonal where the points labelled i and j collide. Let $p_i : X \times X \longrightarrow X$, $i = 1, 2$ denote the projection to the i-th factor.

Let $\mathrm{Pic}^d(X)$ denote the Picard variety of degree d line bundles on X. For any line bundle $\mathcal{L} \in \mathrm{Pic}^d(X)$, we translate the difference map by $\mathcal{L}$, giving rise to

$$\delta_{\mathcal{L}} : X \times X \to \mathrm{Pic}^d(X)\,.$$

This map classifies the line bundle $\mathcal{L} \otimes \mathcal{O}(\mathfrak{y} - \mathfrak{x})$ on $X \times (X \times X)$. The inverse image of $\{\mathcal{L}\}$ is the diagonal $\Delta \subset X \times X$.

Consider now the variety $\mathrm{Pic}^{g-1}(X) = \mathfrak{M}_X(1)$ of line bundles over X of degree $g-1$. For $\mathcal{L} \in \mathrm{Pic}^{g-1}(X)$, it is well–known (see e.g. [25]) that the pullback $\delta_{\mathcal{L}}^*\Theta_X$ of the theta line bundle on $\mathrm{Pic}^{g-1}(X)$ is isomorphic to

$$\mathcal{M}(\mathcal{L}) = \mathcal{L} \boxtimes \mathcal{L}^\vee(\Delta).$$

However, while the restriction $\mathcal{M}(\mathcal{L})|_\Delta$ is canonically trivial, the restriction of the pullback $\delta_{\mathcal{L}}^*\Theta_X|_\Delta$ is naturally identified with the trivial line bundle $\mathcal{O} \otimes \Theta_X|_{\mathcal{L}}$ (since the values of theta functions at $\mathcal{L}$ are not numbers). Hence we must tensor $\mathcal{M}(\mathcal{L})$ by the complex line $\Theta_X|_{\mathcal{L}}$ to make the isomorphism canonical:

4.1.1. *Definition.* The unique isomorphism

$$E_{\mathcal{L}} : \mathcal{M}(\mathcal{L}) \otimes \Theta_X|_{\mathcal{L}} \longrightarrow \delta_{\mathcal{L}}^* \Theta_X$$

of line bundles on $X \times X$, which restricts to the identity on the diagonal, is known as the *prime form* of $\mathcal{L}$.

4.1.2. *Translation to degree zero and the classical prime form.* Let us pick a canonical basis $A_1, \ldots, A_g, B_1, \ldots, B_g$ in $\mathrm{H}_1(X, \mathbb{Z})$. This choice determines a theta characteristic $\Omega_X^{\frac{1}{2}}$ on X, known as Riemann's constant. It is characterized by the property that the $\Omega_X^{\frac{1}{2}}$ translate of the divisor Θ_0 of Riemann's theta function on Jac (defined using the polarization A_i, B_i) is the canonical theta divisor $\Theta_X \subset \mathrm{Pic}^{g-1}(X)$ (the Abel–Jacobi image of $\mathrm{Sym}^{g-1} X$).

The classical Klein prime form of X (for detailed discussions see [13], [22]) is a section of

$$\Omega_X^{-\frac{1}{2}} \boxtimes \Omega_X^{-\frac{1}{2}},$$

with values in $\delta^* \mathcal{O}(\Theta_0)$, and with divisor the diagonal. Dividing by this section defines an isomorphism $E : \Omega_X^{\frac{1}{2}} \boxtimes \Omega_X^{\frac{1}{2}}(\Delta) \to \delta^* \mathcal{O}(\Theta_0)$. To recover the isomorphisms $E_{\mathcal{L}}$ of Definition 4.1.1, note that $\delta^* \Theta_0 = \delta^*_{\Omega_X^{-\frac{1}{2}}} \Theta_X$. Since the ratio between the pullbacks of Θ_X by $\delta_{\mathcal{L}}$ and $\delta_{\Omega_X^{\frac{1}{2}}}$ is $\mathcal{L}_0 \boxtimes (\mathcal{L}_0)^* = \mathcal{L}_0 \boxtimes \mathcal{L}_0^\vee$, dividing by the Klein prime form may be used in defining the isomorphisms $E_{\mathcal{L}}$ for all $\mathcal{L}$.

4.2. **Abelian Szegö kernels and theta functions.** Suppose $\mathcal{L} \in \mathrm{Pic}^{g-1}(X) \setminus \Theta_X$ is a line bundle with $\mathrm{h}^0(\mathcal{L}) = \mathrm{h}^1(\mathcal{L}) = 0$. By Definition 3.4.1 we have a canonical section

$$\mathfrak{s}_{\mathcal{L}} \in \mathrm{H}^0(X \times X, \mathcal{M}(\mathcal{L})),$$

the Szegö kernel of $\mathcal{L}$. On the other hand, by Definition 4.1.1 the prime form of $\mathcal{L}$ provides an isomorphism

$$E_{\mathcal{L}} : \mathcal{M}(\mathcal{L}) \otimes \Theta_X|_{\mathcal{L}} \longrightarrow \delta_{\mathcal{L}}^* \Theta_X,$$

so the theta function pulls back to a section of $\mathcal{M}(\mathcal{L})$. Normalizing the pullback to have value 1 on the diagonal, we recover the well–known formula for the Szegö kernel,

$$\mathfrak{s}_{\mathcal{L}}(x, y) = \frac{\theta(y - x + \mathcal{L}_0)}{\theta(\mathcal{L}_0) E(x, y)}. \tag{3}$$

Here we use Riemann's constant $\Omega_X^{\frac{1}{2}}$ (§ 4.1.2) to translate θ and $\mathcal{L}$ to Jac, whose group structure is written additively. In other words, up to the isomorphism given by the prime form and multiplication by a scalar (the value $\theta(\mathcal{L}_0)$), the Szegö kernel is the pullback of the theta function by the difference map of $\mathcal{L}$.

The Szegö kernel for the line bundle $\Omega^{\frac{1}{2}}_{\mathbb{CP}^1}$ over $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is

$$\frac{\sqrt{dz_1} \otimes \sqrt{dz_2}}{z_2 - z_1}$$

where (z_1, z_2) is the coordinate on $\mathbb{C}^2$.

4.3. **The Nonabelian Prime Form.** Fix a holomorphic vector bundle E of rank n over X. The projection of $X \times (X \times X)$ to X (respectively, $X \times X$) will be denoted by p_X (respectively, Π). Let $\underline{E} = p_X^* E$ and

$$\mathcal{E} := \underline{E} \otimes \mathcal{O}(\mathfrak{y} - \mathfrak{x}) \tag{4}$$

be the vector bundle over $X \times (X \times X)$. In other words, $\mathcal{E}$ is the family of vector bundles $E(y - x)$ over X parametrized by $X \times X$. Let $d(\mathcal{E})$ be the determinant line bundle over $X \times X$ for the family $\mathcal{E}$ of vector bundles over X. (Its dual is the pullback of the theta line bundle on the moduli space of vector bundles, by the map δ_E classifying the family $\mathcal{E}$.)

We recall that $d(\mathcal{E}) = \det R^0 \Pi_* \mathcal{E} \bigotimes (\det R^1 \Pi_* \mathcal{E})^*$, and hence the fiber of $d(\mathcal{E})$ over any $(x, y) \in X \times X$ is $\bigwedge^{\mathrm{top}} \mathrm{H}^0(X,\, E \otimes \mathcal{O}(y - x)) \bigotimes \left(\bigwedge^{\mathrm{top}} \mathrm{H}^1(X,\, E \otimes \mathcal{O}(y - x))\right)^*$. Let $\det E$ denote the line bundle $\bigwedge^n E$ on X.

4.3.1. **Proposition.** The dual determinant bundle $d(\mathcal{E})^*$ over $X \times X$ is isomorphic to the line bundle $\det E \boxtimes \det E^\vee(n\Delta)$.

4.3.2. *Introducing parameters.* We will prove a stronger version of Proposition 4.3.1, valid for arbitrary families of curves and vector bundles on them (i.e., over the moduli stack). Hence in the rest of § 4.3 we work over a fixed (complex) base scheme (or base analytic space) S, and all constructions are relative to S. Thus $\pi : \mathfrak{X} \to S$ will denote a smooth projective curve over S, of genus g (i.e., a smooth family of compact Riemann surfaces of genus g parametrized by S). For any $s \in S$, the curve $\pi^{-1}(s)$ is denoted by X_s. In this section all products are fiber products over S, so that the projection

$$\pi_2 : \mathfrak{X} \times \mathfrak{X} \longrightarrow S \tag{5}$$

will denote the family $\pi \times_S \pi : \mathfrak{X} \times_S \mathfrak{X}$ of algebraic surfaces $\{X_s \times X_s\}$ and $\Delta \subset \mathfrak{X} \times \mathfrak{X}$ will denote the relative diagonal. Let $\Pi : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{X} \times \mathfrak{X}$ be the projection to the second and third factors. Note that the family of Riemann surfaces over the fiber product $\mathfrak{X} \times \mathfrak{X}$ defined by the projection Π is the pullback of the family $\mathfrak{X}$ over S by the obvious projection of $\mathfrak{X} \times \mathfrak{X}$ to S.

The projection of $\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}$ to the first factor will be denoted by p_X. Let Δ_{01} (respectively, Δ_{02}) denote the diagonal divisor in $\mathfrak{X} \times (\mathfrak{X} \times \mathfrak{X})$ defined by all points whose first coordinate coincides with the second (respectively, third) coordinate.

Let E denote a vector bundle of rank n over $\mathfrak{X}$ (i.e., a holomorphic family of vector bundles E_s over X_s), and $d(E)$ its determinant line bundle over S. By $E^\vee$ we denote the relative Serre dual bundle $E^* \otimes \omega_\pi$, where ω_π is the relative dualizing sheaf (relative canonical line bundle) for the morphism $\pi : \mathfrak{X} \to S$.

Set $\underline{E} := p_X^* E$ and consider the following analogue of (4)

$$\mathcal{E} = \underline{E} \otimes \mathcal{O}(\Delta_{02} - \Delta_{01})$$

over $\mathfrak{X} \times (\mathfrak{X} \times \mathfrak{X})$. The determinant bundle $d(\mathcal{E})$ for the morphism Π is a line bundle over $\mathfrak{X} \times \mathfrak{X}$.

4.3.3. **Theorem.** The theta (or dual determinant) line bundle $(\Theta_X)_{\mathcal{E}} = d(\mathcal{E})^*$ is canonically isomorphic to

$$\det E \boxtimes \det E^\vee(n\Delta) \otimes \pi_2^* d(E)^*$$

(where $d(E)^* = (\Theta_X)_E$ is the determinant line bundle on S, and π_2, as in (5), is the natural projection of $\mathfrak{X} \times \mathfrak{X}$ to S).

4.3.4. *Remark: The universal situation.* The theorem is equivalent to the following universal statement on the moduli *stack* of curves and bundles. (A suitably modified statement also holds over the moduli *space* of stable bundles.) Let $\underline{\mathfrak{M}}_g(n)$ denote the moduli stack of pairs (X, E) of a smooth projective curve of genus g with a vector bundle over it of rank n, and $(\mathfrak{X}, \mathfrak{E})$ the universal curve and bundle. Let

$$\delta_{\mathfrak{E}} : \mathfrak{X} \times_{\underline{\mathfrak{M}}_g} \mathfrak{X} \times_{\underline{\mathfrak{M}}_g} \underline{\mathfrak{M}}_g(n) \to \underline{\mathfrak{M}}_g(n)$$

denote the universal difference map, classifying the vector bundle $\mathfrak{E} \boxtimes \mathfrak{E}^\vee$.

4.3.5. *Corollary.* There is an isomorphism

$$\delta_{\mathfrak{E}}^* \Theta_X \cong \det \mathfrak{E} \boxtimes \det \mathfrak{E}^\vee(n\Delta) \otimes \pi_2^* \Theta_X$$

of line bundles on the square of the universal curve over the moduli stack of curves and bundles $\underline{\mathfrak{M}}_g(n)$.

4.3.6. *The Deligne pairing.* The proof of Theorem 4.3.3 relies on the Deligne pairing. We summarize here the properties of this construction that we will need, taken from [10, pp. 98, 147] and [4, pp. 367–368]. Given a family of curves $\pi : \mathfrak{X} \to S$ and two line bundles L and M on $\mathfrak{X}$, the Deligne pairing assigns a line bundle on the base S, denoted by $\langle L , M \rangle$. This pairing is a refined version of the pushforward of the product of Chern classes: the Chern class $c_1(\langle L\,,M\rangle)$ is the image, under the Gysin homomorphism $\mathrm{H}^4(\mathfrak{X},\mathbb{Q})\to\mathrm{H}^2(S,\mathbb{Q})$, of $c_1(L)c_1(M)$. There are canonical isomorphisms as follows:

a. $\langle L,M\rangle=\langle M,L\rangle$, $\langle L,M\otimes N\rangle=\langle L,M\rangle\otimes\langle L,N\rangle$, $\langle L,M^*\rangle=\langle L,M\rangle^*$, and $\langle L,\mathcal{O}_X\rangle=\mathcal{O}_S$.
b. $\langle L,M\rangle=d(L\otimes M)d(\mathcal{O}_X)d(L)^{-1}d(M)^{-1}$.
c. For pairs of vector bundles E_0,E_1 and F_0,F_1 of the same rank,
$$\langle\det(E_0-E_1),\det(F_0-F_1)\rangle=d((E_0-E_1)\otimes(F_0-F_1))$$
(where the differences denote virtual bundles).
d. For a relatively positive divisor $D\subset X$ with structure sheaf $\mathcal{O}_D$, there is an isomorphism $\langle L,\mathcal{O}_X(D)\rangle=d(L\otimes\mathcal{O}_D)\otimes d(\mathcal{O}_D)^{-1}$.

4.3.7. *Proof of Theorem 4.3.3.* There are short exact sequences
$$0\to\mathcal{O}(-\Delta_{01})\to\mathcal{O}\to\mathcal{O}_{\Delta_{01}}\to 0\quad\text{ and}$$
$$0\to\mathcal{O}\to\mathcal{O}(\Delta_{02})\to\omega^*_{\Delta_{02}}\to 0,$$
by the Poincaré adjunction formula. Since the determinant line is multiplicative in short exact sequences, we obtain
$$d(\mathcal{O}(-\Delta_{01}))=d(\mathcal{O}),\qquad d(\mathcal{O}(\Delta_{02}))=d(\mathcal{O})\otimes p_2^*\omega_\pi^* \tag{6}$$
where p_i is the projection of $\mathfrak{X}\times\mathfrak{X}$ to the ith factor. (Note that for a sheaf whose support has relative dimension zero, the determinant line is simply the top exterior power of the pushforward.)

We next calculate the Deligne pairing $\langle\mathcal{O}(\Delta_{02}),\mathcal{O}(-\Delta_{01})\rangle$ in two different ways. Using (d) (and symmetry) we obtain
$$\langle\mathcal{O}(\Delta_{02}),\mathcal{O}(-\Delta_{01})\rangle=\mathcal{O}(-\Delta),$$
as $d(\mathcal{O}_{\Delta_{02}})$ is the trivial line bundle, while using (b) we obtain
$$\langle\mathcal{O}(\Delta_{02}),\mathcal{O}(-\Delta_{01})\rangle=d(\mathcal{O}(\Delta_{02}-\Delta_{01}))\otimes d(\mathcal{O}(-\Delta_{02}))^*\otimes d(\mathcal{O}(\Delta_{01}))^*\otimes d(\mathcal{O}).$$
Comparing these two expressions for $\langle\mathcal{O}(\Delta_{02}),\mathcal{O}(-\Delta_{01})\rangle$ and using (6) we find
$$d(\mathcal{O}(\Delta_{02}-\Delta_{01}))^*=p_2^*\omega_\pi\otimes d(\mathcal{O})^*\otimes\mathcal{O}(\Delta). \tag{7}$$

We now calculate $\langle\det\underline{E},\mathcal{O}(\Delta_{02}-\Delta_{01})\rangle$ in two ways. First, using (a) and (d), we may write
$$\begin{aligned}\langle\det\underline{E},\mathcal{O}(\Delta_{02}-\Delta_{01})\rangle &= \langle\det\underline{E},\mathcal{O}(\Delta_{02})\rangle\otimes\langle\det\underline{E},\mathcal{O}(\Delta_{01})\rangle^*\\ &= d(\det\underline{E}\otimes\mathcal{O}_{\Delta_{02}})\otimes d(\mathcal{O}_{\Delta_{02}})^*\otimes(d(\det\underline{E}\otimes\mathcal{O}_{\Delta_{01}})\otimes d(\mathcal{O}_{\Delta_{01}})^*)^*\\ &= p_2^*\det E\otimes p_1^*\det E^* \end{aligned}\tag{8}$$

(since $d(\det \underline{E} \otimes \mathcal{O}_{\Delta_{0i}}) = p_i^* \det E$). On the other hand, using (a) and (c) (and the fact that det and determinant are homomorphisms from the Grothendieck K-group $K(\mathfrak{X})$ to the Picard group of line bundles over $\mathfrak{X}$ and S respectively), we write

$$\begin{aligned}\langle \det \underline{E}, \mathcal{O}(\Delta_{02} - \Delta_{01})\rangle &= \langle \det(\underline{E} - \mathcal{O}^{\oplus n}), \det(\mathcal{O}(\Delta_{02} - \Delta_{01}) - \mathcal{O})\rangle \\ &= d((\underline{E} - \mathcal{O}^{\oplus n}) \otimes (\mathcal{O}(\Delta_{02} - \Delta_{01}) - \mathcal{O})) \\ &= d(\mathcal{E}) \otimes d(\underline{E})^* \otimes d(\mathcal{O}(\Delta_{02} - \Delta_{01}))^{-n} \otimes d(\mathcal{O})^n.\end{aligned}$$

Comparing these and using (7) and (8) we obtain

$$\begin{aligned}d(\mathcal{E})^* &= p_1^* \det E \otimes p_2^* \det E^* \otimes (p_2^*\omega_\pi)^{\otimes n} \otimes \mathcal{O}(n\Delta) \otimes d(\underline{E})^* \\ &= \det E \boxtimes \det E^\vee(n\Delta) \otimes \pi_2^* d(E)^*\end{aligned}$$

where π_2 is as in (5). This completes the proof of the theorem.

4.4. **The determinant of the Szegö kernel.** In the rank one case, (3) expressed the Szegö kernel in terms of theta functions. In the higher rank case, however, $\mathfrak{s}_E$ is no longer a section of a line bundle, so cannot be expressed directly this way. The Szegö kernel $\mathfrak{s}_E$ for $E \in \mathfrak{M}_X(n) \setminus \Theta_X$ is a global section

$$\mathfrak{s}_E \in \mathrm{H}^0(X \times X, E \boxtimes E^\vee(\Delta))$$

(Definition 3.4.1). On the other hand, by Proposition 4.3.1 we know that the pullback of the theta function by the difference map

$$\delta_E : X \times X \to \mathfrak{M}_X(n), \qquad (x, y) \mapsto E \otimes \mathcal{O}(y - x)$$

is a section

$$\delta_E^* \theta \in \mathrm{H}^0(X \times X, \det E \boxtimes \det E^\vee(n\Delta)) \otimes \Theta_X|_E.$$

Thus it is natural to relate $\delta_E^*\theta$ to a "determinant" of the section $\mathfrak{s}_E$.

It is convenient to interpret the section $\mathfrak{s}_E$ of $p_1^*(E) \otimes p_2^*(E^\vee)(\Delta)$, by dualizing the second factor, as a homomorphism

$$\mathfrak{s}_E \in \mathrm{H}^0(X \times X, \mathrm{Hom}(p_2^*(E \otimes \mathcal{T}_X)(-\Delta), p_1^* E))$$

between two rank n vector bundles on $X \times X$. We may now take its determinant as a homomorphism

$$\det(\mathfrak{s}_E) \in \mathrm{H}^0(X \times X, \mathrm{Hom}(p_2^* \det(E \otimes \mathcal{T}_X)(-n\Delta), p_1^* \det(E)))\,,$$

equivalently obtaining a section $\det \mathfrak{s}_E \in \mathrm{H}^0(X \times X, \det E \boxtimes \det E^\vee(n\Delta))$. We will find a cohomological interpretation of this section, and of the nonabelian prime form, which will make the comparison with the theta function immediate.

4.4.1. **Theorem.** $\det \mathfrak{s}_E = \delta_E^*\theta/\theta(E)$ (under the identification given by the nonabelian prime form).

4.4.2. *Proof.* Recall that $\underline{E} = p_X^* E$ denotes the pullback of E from X to $X \times (X \times X)$, $\mathcal{E} = \underline{E}(\Delta_{02} - \Delta_{01})$, and $\Pi : X \times (X \times X) \to X \times X$ denotes the projection.

First we rewrite the sheaves $p_2^*(E \otimes \mathcal{T}_X)(-\Delta)$ and $p_1^* E$ in terms of Π and $\underline{E}$. We have

$$\begin{aligned} p_1^* E &= \Pi_*(\underline{E}|_{\Delta_{01}}) \\ p_2^*(E \otimes \mathcal{T}_X)(-\Delta) &= \Pi_*(\mathcal{E}/\underline{E}(-\Delta_{01})). \end{aligned}$$

The second statement follows by observing that $\underline{E}(\Delta_{02} - \Delta_{01})/\underline{E}(-\Delta_{01})$ is supported on Δ_{02}, and is given by

$$\underline{E}(\Delta_{02})|_{\Delta_{02}}(-\Delta_{01} \cap \Delta_{02}),$$

which under the identification $\Pi|_{\Delta_{02}} : \Delta_{02} \cong X \times X$ and adjunction formula becomes $p_2^*(E \otimes T_X)(-\Delta)$.

We next consider the long exact sequence of direct images, for the projection Π, of the sequence of sheaves

$$0 \to \underline{E}(-\Delta_{01}) \to \mathcal{E} \to \mathcal{E}/\underline{E}(-\Delta_{01}) \to 0\,. \tag{9}$$

So we have

$$\begin{aligned} R^0\Pi_*\underline{E}(-\Delta_{01}) \to \quad R^0\Pi_*\mathcal{E} \quad &\to R^0\Pi_*\mathcal{E}/\underline{E}(-\Delta_{01}) \to \\ R^1\Pi_*\underline{E}(-\Delta_{01}) \to \quad R^1\Pi_*\mathcal{E} \quad &\to R^1\Pi_*\mathcal{E}/\underline{E}(-\Delta_{01}) \end{aligned}$$

The last term vanishes for dimension reasons, while the first term vanishes since E (and in particular $E(-x)$ for any $x \in X$) has no global sections. Thus we obtain the following exact sequence:

$$0 \to R^0\Pi_*\mathcal{E} \to R^0\Pi_*\mathcal{E}/\underline{E}(-\Delta_{01}) \xrightarrow{\partial_{\mathcal{E}}} R^1\Pi_*\underline{E}(-\Delta_{01}) \to R^1\Pi_*\mathcal{E} \to 0 \tag{10}$$

A similar consideration applies to the Π pushforward of

$$0 \to \underline{E}(-\Delta_{01}) \to \underline{E} \to \underline{E}|_{\Delta_{01}} \to 0, \tag{11}$$

giving rise to

$$0 \to R^0\Pi_*\underline{E} \to R^0\Pi_*\underline{E}|_{\Delta_{01}} \xrightarrow{\partial_E} R^1\Pi_*\underline{E}(-\Delta_{01}) \to R^1\Pi_*\underline{E} \to 0 \tag{12}$$

In this case, however, both first and last terms vanish so that ∂_E is an isomorphism.

Thus we have a diagram

$$\begin{array}{ccc} p_2^*(E \otimes \mathcal{T}_X)(-\Delta) & \xrightarrow{\mathfrak{s}_E} & p_1^* E \\ \| & & \| \\ R^0\Pi_*\mathcal{E}/\underline{E}(-\Delta_{01}) & \xrightarrow{\partial_E^{-1} \circ \partial_{\mathcal{E}}} & R^0\Pi_*\underline{E}|_{\Delta_{01}} \end{array} \tag{13}$$

which we claim commutes. It suffices to check that $\partial_E^{-1} \circ \partial_{\mathcal{E}}$ restricted to the diagonal is the identity, since this property characterizes $\mathfrak{s}_E$ uniquely. However, over the diagonal the two sequences (9) and (11) coincide, so that $\partial_{\mathcal{E}}|_\Delta = \partial_E|_\Delta$.

The theorem now follows from the cohomological description of theta functions. Namely, the complexes

$$R^0\Pi_*\mathcal{E}/\underline{E}(-\Delta_{01}) \xrightarrow{\partial_{\mathcal{E}}} R^1\Pi_*\underline{E}(-\Delta_{01})$$

and

$$R^0\Pi_*\underline{E}|_{\Delta_{01}} \xrightarrow{\partial_E} R^1\Pi_*\underline{E}(-\Delta_{01})$$

(from (10), (12)) are easily seen to represent the determinant bundle for the coherent sheaves $\mathcal{E}$ and $\underline{E}$ on $X \times (X \times X)$, in the sense of § 2.2. For example, for the first complex $\mathcal{K}^{\cdot}$ we must check that for any coherent sheaf $\mathcal{G}$ on $X \times X$ we have

$$R^i\Pi_*(\mathcal{E} \otimes \Pi^*\mathcal{G}) \cong \mathrm{H}^i(\mathcal{K}^{\cdot} \otimes \mathcal{G}).$$

Thus we consider as before (when $\mathcal{G}$ was $\mathcal{O}_{X\times X}$) the long exact sequence associated to (9) tensored with $\Pi^*\mathcal{G}$. But by the projection formula the first and last terms of the resulting six–term sequence will again vanish, and the two middle terms will give our $\mathcal{K}^0 \otimes \mathcal{G}$ and $\mathcal{K}^1 \otimes \mathcal{G}$, as desired.

It follows that we have a canonical isomorphism of line bundles

$$\underline{\mathrm{Hom}}_{X\times X}(p_2^* \det(E \otimes \mathcal{T}_X)(-n\Delta), p_1^* \det(E)) \cong d(\mathcal{E})^* \otimes d(\underline{E}),$$

(where $d(\underline{E})$ is the trivial line bundle $d(E) \otimes_{\mathbb{C}} \mathcal{O}_{X\times X}$) and that $\det \partial_{\mathcal{E}} = \delta_E^*\theta$ and $\det \partial_E$ is the constant $\theta(E)$. Such an isomorphism was constructed in Theorem 4.3.3, the nonabelian prime form. Since $X \times X$ is projective, the isomorphisms must be proportional. In fact, the isomorphisms are equal, since along the diagonal both restrict to the identity map $\mathcal{O} \to d(E)^* \otimes d(E)$. This gives the cohomological description of the prime form. Tracing through these identifications we obtain

$$\det \mathfrak{s}_E = \delta_E^*\theta/\theta(E),$$

as desired.

5. Connections and the Szegö kernel

5.1. **Nonabelian Szegö Kernels.** For our purpose it is convenient to introduce the subsheaf $\mathcal{M}^{\mathrm{Id}}(E) \subset \mathcal{M}(E)$ consisting of all sections whose restriction to the diagonal Δ is a scalar multiple of the identity,

$$\mathcal{M}^{\mathrm{Id}}(E) \;=\; \{s \in \mathcal{M}(E) \big|\; s|_\Delta \in \mathcal{O}_\Delta \cdot \mathrm{Id}\}.$$

In other words, $\mathcal{M}^{\mathrm{Id}}(E)$ is the inverse image of the line-subbundle of $\operatorname{End} E = \mathcal{M}(E)|_\Delta$ defined by scalar operators. For E of rank 1 we have $\mathcal{M}^{\mathrm{Id}}(E) = \mathcal{M}(E)$.

5.1.1. **Proposition.**

(1) If $\mathrm{h}^0(E) = \mathrm{h}^1(E) = 0$, then $\mathrm{H}^0(X \times X, \mathcal{M}^{\mathrm{Id}}(E)) = \mathbb{C} \cdot \mathfrak{s}_E$, where $\mathfrak{s}_E$, the Szegö kernel of E, is the unique section with $\mathfrak{s}_E|_\Delta = \mathrm{Id}$.

(2) Otherwise, the inclusion $\mathrm{H}^0(X \times X, E \boxtimes E^\vee) \hookrightarrow \mathrm{H}^0(X \times X, \mathcal{M}^{\mathrm{Id}}(E))$ is an isomorphism, i.e., all global sections of $\mathcal{M}^{\mathrm{Id}}(E)$ vanish on Δ.

5.1.2. *Proof.* Consider the exact sequence of cohomology

$$0 \to \mathrm{H}^0(X \times X, E \boxtimes E^\vee) \to \mathrm{H}^0(X \times X, \mathcal{M}^{\mathrm{Id}}(E)) \to \mathrm{H}^0(\Delta, \mathcal{O}_\Delta) \xrightarrow{H} \mathrm{H}^1(X \times X, E \boxtimes E^\vee)$$

obtained from the short exact sequence

$$0 \longrightarrow E \boxtimes E^\vee \longrightarrow \mathcal{M}^{\mathrm{Id}}(E) \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 \tag{14}$$

In the projection $\mathcal{M}^{\mathrm{Id}}(E)|_\Delta \longrightarrow \mathcal{O}_\Delta$, the section of $\mathcal{O}_\Delta$ defined by the constant function 1 corresponds to the identity automorphism of E.

By the Künneth formula, the cohomology

$$\mathrm{H}^1(X \times X, E \boxtimes E^\vee) = \left(\mathrm{H}^0(X, E) \otimes \mathrm{H}^1(X, E^\vee)\right) \oplus \left(\mathrm{H}^1(X, E) \otimes \mathrm{H}^0(X, E^\vee)\right).$$

This space has a canonical element coming from the Serre duality pairing between the cohomologies of E and $E^\vee$. It is straight-forward to check that the image $H(1)$ of the function $1 \in \mathrm{H}^0(\mathcal{O}_\Delta) = \mathbb{C}$ (in the above exact sequence of cohomology) is this Serre duality element. It follows that the homomorphism H is an embedding whenever either $\mathrm{H}^0(X, E)$ or $\mathrm{H}^1(X, E)$ is nonzero, while if both vanish then $H = 0$. The proposition follows immediately.

5.1.3. *Remark.* This construction is a special case of the description of Serre duality via residues at the diagonal for any smooth projective variety X of dimension d. Consider the map

$$\mathrm{H}^d_\Delta(X \times X, E \boxtimes E^\vee) \longrightarrow \mathrm{H}^d(X \times X, E \boxtimes E^\vee) = \bigoplus_{0 \le i \le d} \mathrm{H}^i(X, E) \otimes \mathrm{H}^{d-i}(X, E^\vee)$$

from top local cohomology at the diagonal to middle cohomology on $X \times X$. The local cohomology is identified by the Grothendieck–Sato formula with the differential operators $\mathrm{H}^0(X, \mathcal{D}(E, E))$ from E to E. The image of the identity operator is the Serre duality pairing.

5.1.4. *Remark: Even nonsingular theta characteristics.* If the theta characteristic $\Omega_X^{\frac{1}{2}}$ lies in the complement $\mathrm{Pic}^{g-1}(X) \setminus \Theta_X$, in other words $\Omega_X^{\frac{1}{2}}$ is even and nonsingular, then the Szegö kernel $\mathfrak{s}$ associated to $\Omega_X^{\frac{1}{2}}$ is the classical Szegö kernel introduced in [19]. Since the transpose of the Szegö kernel for E clearly agrees with the Szegö kernel for $E^\vee$, it follows that $\mathfrak{s}$ is symmetric. Hence its restriction to 2Δ necessarily agrees with the canonical trivialization μ_1 of $\mathcal{M}_1|_{2\Delta} = \Omega_X^{\frac{1}{2}} \boxtimes \Omega_X^{\frac{1}{2}}(\Delta)|_{2\Delta}$.

5.2. **An identity for the Szegö kernel.** Let E be a vector bundle over X with $h^0(E) = h^1(E) = 0$, and Szegö kernel $\mathfrak{s}_E$. Let $\mathfrak{s}_E^{ij}$ denote the pullback of $\mathfrak{s}_E$ to $X\times X\times X$ along the ith and jth factors. Regarding $\mathfrak{s}_E$ as a homomorphism from p_2^*E to $E\boxtimes\Omega_X(\Delta)$ over $X \times X$, the composition $\mathfrak{s}_E^{01} \circ \mathfrak{s}_E^{12}$ gives a vector bundle homomorphism from the pullback p_3^*E along the third factor to $E\boxtimes\Omega_X\boxtimes\Omega_X(\Delta_{01}+\Delta_{12})$; here $p_3 : X\times X\times X \longrightarrow X$ be the projection to the third factor. We will consider $\mathfrak{s}_E^{01} \circ \mathfrak{s}_E^{12}$ as a meromorphic homomorphism from p_3^*E to $E \boxtimes \Omega_X \boxtimes \Omega_X$, with pole of order one along the (connected reduced) divisor $\Delta_{01} + \Delta_{12}$.

Let $f : X \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$ be a nonconstant meromorphic function. Take a point $c \in \mathbb{C}$ such that $f^{-1}(c)$ is reduced. In other words, if the degree of f is l, then $f^{-1}(c)$ consists of l distinct points. Without loss of generality (translating f if necessary) we assume $c = 0$. The following proposition describes an identity satisfied by the Szegö kernel.

5.2.1. **Proposition.** For any $(x, y) \in X \times X \setminus \Delta$, the identity

$$\sum_{\alpha\in f^{-1}(0)} \frac{1}{df(\alpha)}\mathfrak{s}_E(x,\alpha) \circ \mathfrak{s}_E(\alpha, y) = \frac{f(y) - f(x)}{f(y)f(x)}\mathfrak{s}_E(x, y)$$

is satisfied.

5.2.2. *Proof.* Fix a point $y \in X$ and fix a vector $e \in E_y$ in the fiber. Also, fix a nonzero holomorphic tangent vector $v \in T_yX \setminus \{0\}$. Now, the map defined by

$$x \longmapsto \langle \sum_{\alpha\in f^{-1}(0)} \frac{1}{df(\alpha)}\mathfrak{s}_E(x,\alpha) \circ \mathfrak{s}_E(\alpha, y)(e)\, , v\rangle$$

is a meromorphic section of E. Note that the condition on 0 ensures that $df(\alpha) \neq 0$. Therefore, $df(\alpha)$ and v trivialize the fibers $(\Omega_X)_\alpha$ and $(\Omega_X)_y$ respectively; here $\langle - , v\rangle$ denotes the contraction of $(\Omega_X)_y$ by v. We will denote this meromorphic section of E by $A(e, v)$.

Similarly,

$$x \longmapsto \langle\frac{f(y) - f(x)}{f(y)f(x)}\mathfrak{s}_E(x, y)\, , v\rangle$$

is a meromorphic section of E, which we will denote by $B(e, v)$.

The proposition is equivalent to the assertion $A(e, v) = B(e, v)$ for all e, v. Since E does not admit any nonzero holomorphic section, it suffices to show that the poles of $A(e, v)$ and $B(e, v)$ coincide. Indeed, in that case $A(e, v) - B(e, v)$ is a holomorphic section and thus must be identically zero.

First consider the case $y \notin f^{-1}(0)$. Then the poles of the section $A(e, v)$ are at $f^{-1}(0)$, each pole is of order one and the residue at $\alpha \in f^{-1}(0)$ is

$$\frac{\langle \mathfrak{s}_E(\alpha, y)\, , v\rangle}{df(\alpha)} \in E_x \otimes T_\alpha X\, .$$

Since $\mathcal{O}_X(\alpha)$ is identified (by the adjunction formula) with $T_\alpha X$, the residue is an element of $E_x \otimes T_\alpha X$; here $1/df(\alpha)$ denotes the element of $T_\alpha X$ dual to the nonzero element $df(\alpha)$ of $(\Omega_X)_\alpha$. That the residue of $A(e, v)$ coincides with $\langle \mathfrak{s}_E(\alpha, y)\, , v\rangle/df(\alpha)$ is an immediate consequence of the property of $\mathfrak{s}_E$ that its restriction to Δ is the identity automorphism of E.

The poles of $B(e, v)$ are contained in $f^{-1}(0) \cup \{y\}$. Now, y is a removable singularity since the pole of $\mathfrak{s}_E$ at y is canceled by the vanishing of $f(y) - f(x)$. Clearly, the pole of $B(e, v)$ at any $\alpha \in f^{-1}(0)$ is of order one, and the residue is $\langle \mathfrak{s}_E(\alpha, y)\, , v\rangle/df(\alpha)$. (First note that the numerator $f(y) - f(x)$ cancels with $f(y) - 0$ and then the expression of residue follows from the fact that the residue of $1/f(x)$ is $1/df(\alpha)$.)

Since the residues of $A(e, v)$ and $B(e, v)$ coincide, the proposition is proved under the assumption that $y \notin f^{-1}(0)$. If $y \in f^{-1}(0)$, then it follows by continuity as y is in the closure of the complement of $f^{-1}(0)$. (This case can also be proved by exactly the same way as done above for $y \notin f^{-1}(0)$. The only point to take into account is that using the tangent vector v the line $(\mathcal{O}_{X\times X}(\Delta))_{(y,y)}$ has to be trivialized.)

5.2.3. *Remark: The degenerate case of the Szegö kernel identity.* Proposition 5.2.1 can be extended to the case $y = x$, that is, across the diagonal. Taking the limit as x approaches y, the identity in Proposition 5.2.1 becomes

$$\sum_{\alpha\in f^{-1}(0)} \frac{1}{df(\alpha)} \mathfrak{s}_E(y, \alpha) \circ \mathfrak{s}_E(\alpha, y) = \frac{df(y)}{f(y)^2} \mathrm{Id}_{E_y}\, .$$

Indeed, this is an immediate consequence of the fact that the evaluation $\mathfrak{s}_E(y, y)$ is the identity automorphism of E_y (see Definition 3.4.1).

5.3. **The sheaf of Szegö kernels.** We would like to describe a version of Proposition 5.1.1 with parameters, in other words to let the bundle E (and possibly the curve X) vary. For this purpose we require a universal bundle to substitute for $E \boxtimes E^\vee$. Such a universal bundle is available (tautologically) over the moduli *stack* of bundles, so all of our arguments will work in the stack setting. Alternatively, for concreteness we restrict to the stable locus $\mathfrak{M}^s_X(n) \subset \mathfrak{M}_X(n)$, recalling that the codimension of the strictly semistable locus is greater than one, so that line bundles and their sections are determined by the restriction to the stable locus. Recall (§ 2.2.1) that there is a universal bundle $\mathfrak{E} \boxtimes \mathfrak{E}^\vee$ over $X \times X \times \mathfrak{M}^s_X(n)$ whose restriction to $X \times X \times \{E\}$ is $E \boxtimes E^*$. Using this bundle, we define a canonical vector bundle $\mathcal{M}^{\mathrm{Id}} \subset \mathfrak{E} \boxtimes \mathfrak{E}^\vee(\Delta)$ over $X \times X \times \mathfrak{M}^s_X(n)$ whose restriction to any $X \times X \times \{E\}$ is $\mathcal{M}^{\mathrm{Id}}(E)$.

Let $\pi_2 : X \times X \times \mathfrak{M}^s_X(n) \longrightarrow \mathfrak{M}^s_X(n)$ be the natural projection (as in (5)). We now describe the direct image sheaf

$$\mathfrak{S} = \pi_{2*}\mathcal{M}^{\mathrm{Id}}$$

on $\mathfrak{M}_X(n)$, where the Szegö kernel naturally lives, for the above projection π_2.

5.3.1. **Proposition.** $\mathfrak{S}$ is naturally a torsion–free subsheaf of the determinant line bundle Θ^*_X. The inclusion is an isomorphism on $\mathfrak{M}^s_X(n) \setminus \Theta^{\mathrm{sing}}_X$, where its inverse is given by the *normalized Szegö kernel* $\overline{\mathfrak{s}} := \mathfrak{s} \cdot \theta$, a nowhere vanishing section of $\mathfrak{S} \otimes \Theta_X$ off Θ^{sing}_X.

5.3.2. *Proof.* Consider the relative version

$$0 \to \mathfrak{E} \boxtimes \mathfrak{E}^\vee \to \mathcal{M}^{\mathrm{Id}} \to \mathcal{O}_\Delta \to 0$$

of (14), and push it forward along π_2, obtaining the exact sequence

$$0 \longrightarrow \pi_{2*}(\mathfrak{E} \boxtimes \mathfrak{E}^\vee) \longrightarrow \mathfrak{S} \xrightarrow{\mathrm{Res}} \mathcal{O} \longrightarrow \cdots$$

The sheaf $\pi_{2*}(\mathfrak{E} \boxtimes \mathfrak{E}^\vee)$ vanishes off of Θ_X, and is a line bundle on Θ°_X. In fact, for $E \in \Theta^\circ_X$, the vector spaces $\mathrm{H}^0(X, E)$ and $\mathrm{H}^1(X, E)^* \cong \mathrm{H}^0(X, E^\vee)$ are one–dimensional, so that we have isomorphisms

$$\mathrm{H}^0(X \times X, E \boxtimes E^\vee) \cong \mathrm{H}^0(X, E) \otimes \mathrm{H}^0(X, E^\vee) \cong \Theta^*_X|_E.$$

As E varies over Θ°_X we obtain an isomorphism $\Theta^*_X|_{\Theta^\circ_X} \cong \pi_{2*}(\mathfrak{E} \boxtimes \mathfrak{E}^\vee)$.

Since $\mathcal{M}^{\mathrm{Id}}$ is a locally free sheaf, $\mathfrak{S} = \pi_{2*}\mathcal{M}^{\mathrm{Id}}$ is torsion–free. On $\mathfrak{M}^s_X(n) \setminus \Theta_X$, the morphism Res above is an isomorphism. Moreover, $\mathfrak{S}$ remains locally free on $\mathfrak{M}_X(n) \setminus \Theta^{\mathrm{sing}}_X$. To see this observe (by Proposition 5.1.1 and the above discussion) that on Θ°_X, $\mathfrak{S}$ is identified with the line bundle Θ^*_X, so (by Nakayama's lemma) it is locally free of rank one near any $E \in \Theta^\circ_X$.

The morphism Res is a regular section of the line bundle $\underline{\mathrm{Hom}}(\mathfrak{S}, \mathcal{O})$. It is nonvanishing off Θ_X, and vanishes to first order along Θ_X. This identifies $\underline{\mathrm{Hom}}(\mathfrak{S}, \mathcal{O})$ with Θ_X and provides the desired embedding $\mathfrak{S} \subset \Theta^*_X$, an isomorphism off of Θ^{sing}_X. Moreover, since by definition the residue of $\mathfrak{s}_E$ is the identity for every E, tracing through the isomorphism we see that the section $\overline{\mathfrak{s}} = \mathfrak{s} \otimes \theta$ of $\mathfrak{S} \otimes \Theta_X = \underline{\mathrm{Hom}}(\Theta^*_X, \mathfrak{S})$ on $\mathfrak{M}^s_X(n) \setminus \Theta_X$ is identified with the inverse of the identification above. In particular this section extends across Θ°_X in a nonzero fashion.

5.3.3. *Remark.* Note that the statement (and proof) above generalize immediately when the curve X and bundle are allowed to vary over the moduli space $\mathfrak{M}_g(n)$.

5.4. **Connections on Θ_X.** Consider the affine space $\mathcal{C}onn_X(E_0)$ of twisted connections on E. This is a torsor for the space of Higgs fields, which is the cotangent space to $\mathfrak{M}_X(n)$ at E (at smooth points – in particular for E stable). Thus as E varies, the sheaf on $\mathfrak{M}_X(n)$ defined by $\mathcal{C}onn_X(E_0)$ forms an Ω–torsor. This Ω–torsor will be denoted by $\mathcal{C}onn_X(n)$. On the complement $\mathfrak{M}_X(n) \setminus \Theta_X$, the restriction of the Szegö kernel $\mathfrak{s}_E|_{2\Delta}$ to 2Δ gives a canonical section of $\mathcal{C}onn_X(n)$, denoted $\mathfrak{s}|_{2\Delta}$.

5.4.1. **Theorem.** There is a unique isomorphism of Ω–torsors $\mathcal{C}onn_X(n) \cong \mathcal{C}onn_X(\Theta)$ on $\mathfrak{M}_X(n)$ sending $\mathfrak{s}|_{2\Delta}$ to $\mathrm{d}\log\theta$.

5.4.2. *Proof.* The Szegö kernel $\mathfrak{s}|_{2\Delta}$ is a meromorphic section of the Ω–torsor $\mathcal{C}onn_X(n)$, trivializing it on $\mathfrak{M}_X(n) \setminus \Theta_X$. Moreover it has only a first order pole along Θ_X, in fact as we will see a *logarithmic* pole. It follows therefore that the class of $\mathcal{C}onn_X(n)$ in $\mathrm{H}^1(\mathfrak{M}_X(n), \Omega^1)$ is a complex multiple of the divisor class $[\Theta_X]$, with the multiplicity given by the residue of $\mathfrak{s}|_{2\Delta}$ along Θ_X. (Note that it makes sense to speak of the residue of a section of any Ω–torsor with first–order poles, since the expressions of this section in any two local trivializations differ by a regular one–form.) Hence we need to check $\mathfrak{s}|_{2\Delta}$ is logarithmic, calculate its residue and compare it to that of $\mathrm{d}\log\theta$.

We first describe the conormal bundle $N^*_{\Theta_X^\circ}$ to Θ_X°. By the adjunction formula, the conormal bundle is isomorphic to the restriction of the dual line bundle Θ_X^*. This isomorphism is given by $\mathrm{d}\log\theta|_{\Theta_X^\circ}$, a nonvanishing section of $N^*_{\Theta_X^\circ} \otimes \mathcal{O}(\Theta_X)|_{\Theta_X^\circ} \subset \Omega^1_{\mathfrak{M}_X(n)}(\Theta_X)|_{\Theta_X^\circ}$. (This is also identified with the one–jet $J^1\theta$, which along the divisor of θ is a section of $\Omega^1_{\mathfrak{M}_X(n)} \otimes \mathcal{O}(\Theta_X)$.)

On the other hand, identifying Θ_X° with the Brill–Noether moduli space of bundles with a single holomorphic section, standard deformation theory identifies its tangent space at $E \in \Theta_X^\circ$ with the kernel of the cup product map

$$\mathrm{H}^1(X, \mathrm{End}\, E) \longrightarrow \mathrm{Hom}(\mathrm{H}^0(X,E), \mathrm{H}^1(X,E)).$$

(Given a section of E and a first order deformation of the bundle, we obtain a class in $\mathrm{H}^1(X,E)$, while a compatible deformation of the section makes this class into a coboundary.) Dually, the conormal bundle is identified with the image of the Petri map

$$\mathrm{H}^0(X,E) \otimes \mathrm{H}^0(X,E^\vee) \longrightarrow \mathrm{H}^0(X, E \otimes E^\vee)\,,$$

or equivalently the restriction to the diagonal

$$\mathrm{H}^0(X \times X, E \boxtimes E^\vee) \to \mathrm{H}^0(X, \mathrm{End}\, E \otimes \Omega_X)\,,$$

which are injective when $\mathrm{h}^0(E) = \mathrm{h}^0(E^\vee) = 1$. These two identifications of the conormal bundle are compatible with the natural isomorphism $\Theta_X^*|_E = \mathrm{H}^0(X,E) \otimes \mathrm{H}^0(X,E^\vee)$ for $E \in \Theta_X^\circ$.

For $E \in \Theta_X^\circ$ the restriction of the normalized Szegö kernel $\bar{\mathfrak{s}}|_{2\Delta}$ is a section of $E \boxtimes E^\vee(\Delta)|_{2\Delta} \otimes \Theta_X|_{\{E\}}$ vanishing on the diagonal. In other words, it is a $\Theta_X|_{\{E\}}$–twisted section of $E \boxtimes E^\vee|_\Delta = \operatorname{End} E \otimes \Omega$. Moreover, this twisted Higgs field is the image of

$$\bar{\mathfrak{s}}_E \in \mathrm{H}^0(X \times X, E \boxtimes E^\vee) \otimes \Theta_X|_{\{E\}}\,,$$

hence (by the above identifications) it is a nonvanishing function $\bar{\mathfrak{s}}|_{2\Delta} \in N^*_{\Theta_X^\circ} \otimes \Theta_X \cong \mathcal{O}$ along Θ_X°.

It follows that $\mathfrak{s}|_{2\Delta}$ has log–poles along Θ_X°, since its singular part is contained in the conormal direction. Therefore its residue can be read of as the function on Θ_X° given by the restriction $\bar{\mathfrak{s}}|_{2\Delta}$. But by Proposition 5.3.1, the normalized Szegö kernel (along Θ_X°) is simply the expression of the canonical isomorphism $\Theta_X^*|_E = \mathrm{H}^0(X,E) \otimes \mathrm{H}^0(X,E^\vee)$. In other words, the residue of $\mathfrak{s}|_{2\Delta}$ along Θ_X is 1, and the class of the Ω–torsor $\mathcal{C}onn_X(n)$ is the (Chern) class of Θ_X.

Consider the meromorphic identification of Ω–torsors $\mathcal{C}onn_X(n) \to \mathcal{C}onn_X(\Theta)$ defined by $\mathfrak{s}|_{2\Delta} \mapsto \mathrm{d}\log\theta$. This isomorphism is clearly regular outside Θ_X, and the above identification of the singular parts shows that it remains regular along Θ_X. Thus we have constructed a global and uniquely characterized isomorphism of Ω–torsors, as desired.

5.4.3. *Uniqueness and symplectic structure.* As explained in § 2.4, Ω–torsors on moduli spaces enjoy a remarkable rigidity. Since the isomorphism in Theorem 5.4.1 is valid over arbitrary families of curves, and there are no one–forms on $\mathfrak{M}_g(n)$ relative to $\mathfrak{M}_g$, it follows that this isomorphism agrees with the isomorphisms described in [12] and [5]. The latter isomorphisms are in fact isomorphisms of *twisted cotangent bundles*, i.e., they carry the symplectic structure on flat connections (given by the cup product on de Rham cohomology of a connection) to that on $\mathcal{C}onn_X(\Theta)$ (see § 2.3.1). While we have not addressed the relation of Szegö kernels to the natural symplectic structure on $\mathcal{C}onn_X(n)$ above, this relation follows immediately by comparison with these other constructions:

5.4.4. **Corollary.** The section of $\mathcal{C}onn_X(n)$ over $\mathfrak{M}_X(n)\backslash\Theta_X$ given by the Szegö kernel is Lagrangian with respect to the natural twisted cotangent bundle symplectic structure on the moduli space of connections.

6. Extended Connections

6.1. **Extended Higgs Fields.** Let $\mathfrak{M}_g(n)$ denote the moduli space of semistable bundles of rank n and Euler characteristic zero over the moduli space of curves $\mathfrak{M}_g$. This is an orbifold (Deligne–Mumford stack) which fibers over the moduli space of curves, with fiber at X the moduli space $\mathfrak{M}_X(n)$. It carries a universal theta divisor, also denoted by Θ_X, which specializes to the theta divisor on each $\mathfrak{M}_X(n)$. (See [9] for a discussion of this moduli space.)

The tangent space to $\mathfrak{M}_g(n)$ at a smooth point (X, E) is calculated, by standard deformation theory, to be $\mathrm{H}^1(X, \mathrm{At}(E))$, where $\mathrm{At}(E)$ is the Atiyah bundle of E. The cotangent space to $\mathfrak{M}_g(n)$ may thus be described concretely as an H^0 of the dual sheaf of kernels. Using the residue pairing along the diagonal and the trace on $\operatorname{End} E$, the result is (after transposition of factors) the following (see [5]):

6.1.1. *Definition.* The space $\mathcal{E}x\mathcal{H}iggs_X(E)$ of extended Higgs fields on E is the sheaf

$$\mathcal{E}x\mathcal{H}iggs_X(E) = \{\phi \in E \boxtimes E^*(-\Delta)|_{2\Delta}\}/(\mathrm{Ker}(\mathrm{tr}) \subset E \boxtimes E^*(-2\Delta)|_\Delta)$$

supported over 2Δ, where

$$\mathrm{tr} : E \boxtimes E^*(-2\Delta)|_\Delta = \operatorname{End} E \otimes \Omega_X^{\otimes 2} \longrightarrow \Omega_X^{\otimes 2}$$

is the trace map. (Note that $E \boxtimes E^*(-2\Delta)|\Delta \subset E \boxtimes E^*(-\Delta)|_{2\Delta}$ is the subsheaf vanishing on $\Delta \subset 2\Delta$.)

The direct image $p_{1*}\mathcal{E}x\mathcal{H}iggs_X(E)$ is a vector bundle of rank $n^2 + 1$ over X. This vector bundle will also be called the extended Higgs fields on E (as we have a natural identification defined by taking this direct image).

6.1.2. It is easy to check that the bundle $\mathcal{E}x\mathcal{H}iggs_X(E)$ is an extension

$$0 \to \Omega_X^{\otimes 2} \to \mathcal{E}x\mathcal{H}iggs_X(E) \to \operatorname{End} E \otimes \Omega_X \to 0 \tag{15}$$

of the sheaf of Higgs fields $\operatorname{End} E \otimes \Omega_X$ by the sheaf of quadratic differentials $\Omega_X^{\otimes 2}$.

6.1.3. **Proposition.** The cotangent space to $\mathfrak{M}_g(n)$ at any point (X, E) is given by the space of global extended Higgs fields on E, namely $\mathrm{H}^0(X, \mathcal{E}x\mathcal{H}iggs_X(E))$.

6.1.4. *Proof.* The tangent space to $\mathfrak{M}_g(n)$ at any point (X, E) is given by $\mathrm{H}^1(X, \mathrm{At}(E))$, where $\mathrm{At}(E)$ is the Atiyah bundle defined in § 3.2.2 (see [5]). The Serre duality gives $\mathrm{H}^1(X, \mathrm{At}(E))^* \cong \mathrm{H}^0(X, \mathrm{At}(E)^* \otimes \Omega_X)$. The proof of the proposition will be completed by giving a canonical isomorphism of $\mathrm{At}(E)^* \otimes \Omega_X$ with the vector bundle $\mathcal{E}x\mathcal{H}iggs_X(E)$.

To construct the isomorphism, first note that the pulled back line bundle $(p_1^*\Omega_X)|_{2\Delta}$ over 2Δ is naturally identified with the line bundle $\mathcal{O}_{X\times X}(-\Delta)|_{2\Delta}$. To construct this isomorphism, observe that a holomorphic function f defined on a connected symmetric analytic open subset $U \subset X\times X$ (by symmetric we mean invariant under the involution σ of $X\times X$ that interchanges factors) satisfying the identity $f\circ\sigma = -f$ has the following property: if f vanishes on $\Delta \cap U$ of order at least two, then f must vanish on $\Delta \cap U$ of order at least three. This observation gives an isomorphism of $\mathcal{O}_{X\times X}(-\Delta)|_{2\Delta}$ with $(p_1^*\Omega_X)|_{2\Delta}$. Indeed, taking any function f on such a symmetric open subset U that vanishes on $\Delta \cap U$ exactly of order one, project the section $\mathrm{d}f$ of Ω^1_U to a section of $(p_1^*\Omega_X)|_U$, where the projection is defined using decomposition $\Omega^1_{X\times X} \cong p_1^*\Omega_X \oplus p_2^*\Omega_X$. The resulting isomorphism of $\mathcal{O}_{X\times X}(-\Delta)|_{2\Delta}$ with $(p_1^*\Omega_X)|_{2\Delta}$ does not depend on the choice of f. This is an immediate consequence of the above observation.

From the definition of $\mathrm{At}(E)$ in § 3.2.2 it follows immediately that $\mathrm{At}(E)^*$ is a quotient of $J^1(E)\otimes E^*$. Recall that $J^1(E)$ is, by definition, the direct image $p_{2*}((p_1^*E)|_{2\Delta})$. Consider the quotient sheaf

$$\mathcal{V} := \{\phi \in E \boxtimes E^* \otimes p_1^*\Omega_X|_{2\Delta}\}/(\mathrm{Ker}(\mathrm{tr}) \subset E \boxtimes E^* \otimes p_1^*\Omega_X(-\Delta)|_{\Delta})$$

on 2Δ, where

$$\mathrm{tr} : E \boxtimes E^* \otimes p_1^*\Omega_X(-\Delta)|_{\Delta} = \mathrm{End}\, E \otimes \Omega_X^{\otimes 2} \longrightarrow \Omega_X^{\otimes 2}$$

is the trace map. Using the above remarks it follows that $p_{1*}\mathcal{V} \cong \mathrm{At}(E)^* \otimes \Omega_X$.

Finally, using the earlier obtained isomorphism of $\mathcal{O}_{X\times X}(-\Delta)|_{2\Delta}$ with $(p_1^*\Omega_X)|_{2\Delta}$, the vector bundle $\mathrm{At}(E)^*\otimes\Omega_X$ over X is identified with $\mathcal{E}x\mathcal{H}iggs_X(E)$. This completes the proof of the proposition.

6.2. **Extended connections.** We introduce the space of extended connections, a torsor over the space of extended Higgs fields, using the language of kernels.

Recall that a connection on a vector bundle E may be realized, for any ν, by a section of $\mathcal{M}_\nu(E)$ on 2Δ, whose restriction to Δ is the identity endomorphism of E. It is convenient to describe the sheaf of sections on 2Δ as the quotient of $\mathcal{M}_\nu(E)$ by sections that vanish on 2Δ: we have an identification

$$\mathcal{C}onn_X(E) \cong \{\varphi \in \mathcal{M}_\nu(E)/\mathcal{M}_\nu(E)(-2\Delta) \mid \varphi|_\Delta = \mathrm{Id}_E\}.$$

In order to define extended connections, we would like to extend the connection kernels on 2Δ by *scalar* operators on 3Δ. Thus instead of quotienting out $\mathcal{M}_\nu(E)$ by *all* sections that vanish on 2Δ, we quotient out by those whose leading term is traceless. Let $\mathrm{End}^0(E) \subset \mathrm{End}\, E$ denote the subbundle of traceless endomorphisms of E and define

$$\mathcal{M}_\nu(E)(-2\Delta)^0 := \Omega_X^{\otimes 2} \otimes \mathrm{End}^0(E) \subset \mathcal{M}_\nu(E)(-2\Delta)|_\Delta = \Omega_X^{\otimes 2} \otimes \mathrm{End}\, E\,.$$

So $\mathcal{M}_\nu(E)(-2\Delta)^0$ coincides with the kernel of the homomorphism

$$\mathcal{M}_\nu(E)(-2\Delta)|_\Delta = \Omega_X^{\otimes 2} \otimes \mathrm{End}\, E \xrightarrow{\mathrm{tr}_E} \Omega_X^{\otimes 2}.$$

(Note that $\mathcal{M}_\nu(E)(-2\Delta)|_\Delta$ is identified with the sections of $\mathcal{M}_\nu(E)|_{3\Delta}$ vanishing on 2Δ.)

6.2.1. *Definition.* The sheaf of extended connections is

$$\mathcal{E}x\mathcal{C}onn_X^{\nu}(E) = \{\varphi \in \mathcal{M}_{\nu}(E)/\mathcal{M}_{\nu}(E)(-2\Delta)^0 \,\big|\, \varphi|_{\Delta} = \mathrm{Id}_E\},$$

the sections of the quotient of $\mathcal{M}_{\nu}(E)$ over 3Δ by traceless sections $\mathcal{M}_{\nu}(E)(-2\Delta)^0$, whose restriction to Δ is the identity automorphism of E.

6.2.2. It follows that restriction to 2Δ defines a map $\Pi_{\nu} : \mathcal{E}x\mathcal{C}onn_X^{\nu}(E) \to \mathcal{C}onn_X(E)$ to the space of connections on E. If we take $\varphi_1, \varphi_2 \in \mathcal{E}x\mathcal{C}onn_X^{\nu}(E)$ defining the same connection on E, that is $\varphi_1|_{2\Delta} = \varphi_2|_{2\Delta}$, then their difference $\varphi_1 - \varphi_2 = q\,\mathrm{Id}_E$ with q a quadratic differential on X. Thus restriction to 2Δ makes $\mathcal{E}x\mathcal{C}onn_X^{\nu}(E)$ an affine bundle for $\mathrm{H}^0(X, \Omega_X^{\otimes 2})$ over $\mathcal{C}onn_X(E)$.

6.2.3. **Proposition.** For every $\nu \in \mathbb{Z}$, the space $\mathcal{E}x\mathcal{C}onn_X^{\nu}(E)$ is naturally an affine space for $\mathcal{E}x\mathcal{H}iggs_X(E)$.

6.2.4. *Proof.* We claim that for every ν, the vector bundle

$$\mathcal{E}x\mathcal{H}iggs_X^{\nu}(E) = \{\varphi \in \mathcal{M}_{\nu}(E)/\mathcal{M}_{\nu}(E)(-2\Delta)^0 \,\big|\, \varphi|_{\Delta} = 0\}$$

is isomorphic to $\mathcal{E}x\mathcal{H}iggs_X(E)$. The isomorphism is given by tensoring with the section $\mu_{-\nu}$, i.e., by the identification

$$\Gamma(\mathcal{M}_{\nu}(-\Delta)|_{3\Delta}) = \Gamma(\Omega_X^{\nu/2} \boxtimes \Omega_X^{\nu/2}((i-1)\Delta)|_{2\Delta}) \xrightarrow{\otimes \mu_{-\nu}} \Gamma(\mathcal{O}(-\Delta)|_{2\Delta}).$$

It is clear that $\mathcal{E}x\mathcal{C}onn_X^{\nu}(E)$ is an affine bundle for the vector bundle $\mathcal{E}x\mathcal{H}iggs_X^{\nu}(E)$, and hence for $\mathcal{E}x\mathcal{H}iggs_X(E)$.

6.3. **Varying the curve.** We would like to use the Szegö kernel $\mathfrak{s}_E$ to define a section of a twisted cotangent bundle over the moduli space $\mathfrak{M}_g(n)$, extending the construction $\mathfrak{s}|_{2\Delta}$ along $\mathfrak{M}_X(n)$. Thus we introduce the extended analogue of $\mathcal{C}onn_X(E_0)$: the space $\mathcal{E}x\mathcal{C}onn_X(E_0)$ of twisted extended connections on E is

$$\mathcal{E}x\mathcal{C}onn_X(E_0) = \{s \in \mathrm{H}^0(3\Delta, \mathcal{M}(E)|_{3\Delta}) \,|\, s|_{\Delta} = \mathrm{Id}_E\}/(\mathrm{Ker}(\mathrm{tr}) \subset E \boxtimes E^{\vee}(-\Delta)|_{3\Delta}).$$

Again this space depends only on E and not on the choice of theta characteristic $\Omega_X^{\frac{1}{2}}$.

Now consider the projection of $\mathfrak{s}_E|_{3\Delta}$ to $\mathcal{E}x\mathcal{C}onn_X(E_0)$, which we also denote by $\mathfrak{s}_E|_{3\Delta}$. This defines a section of the Ω–torsor $\mathcal{E}x\mathcal{C}onn_g(n)$ over $(X, E) \in \mathfrak{M}_g(n) \setminus \Theta_g$.

6.3.1. **Theorem.** There is a unique isomorphism of Ω–torsors

$$\mathcal{E}x\mathcal{C}onn_g(n) \cong \mathcal{C}onn_g(\Theta_g)$$

over $\mathfrak{M}_g(n)$ sending $\mathfrak{s}|_{3\Delta}$ to $\mathrm{d}\log\theta$.

6.3.2. *Proof.* The proof is a direct generalization of that of Theorem 5.4.1. Namely, we claim that $\mathfrak{s}|_{3\Delta}$ is a section of $\mathcal{E}x\mathcal{C}onn_g(n)$ with logarithmic poles along the universal theta divisor Θ_g, and with residue 1. As was the case for fixed curve, the logarithmic derivative of the theta function identifies the conormal bundle to Θ_g° with the restriction of Θ_g^*. Deformation theory now identifies the tangent space to Θ_g° with the kernel of a natural map

$$\mathrm{H}^1(X, \mathrm{At}(E)) \longrightarrow \mathrm{Hom}(\mathrm{H}^0(X,E), \mathrm{H}^1(X,E)).$$

Dually, the Petri map factors through extended Higgs bundles,

$$\mathrm{H}^0(X,E) \otimes \mathrm{H}^0(X,E^\vee) \longrightarrow \mathrm{H}^0(X, \mathcal{E}x\mathcal{H}iggs_X(E)) \longrightarrow \mathrm{H}^0(X, E \otimes E^\vee)\,,$$

where its image describes the conormal line to Θ_g°.

For $(X,E) \in \Theta_g^\circ$, the normalized Szegö kernel gives rise to a Θ_g–twisted extended Higgs field $\bar{\mathfrak{s}}|_{3\Delta}$: $\bar{\mathfrak{s}}_E$ is a section of $\mathcal{M}(E) \otimes \mathcal{O}(\Theta_g)|_E$, whose value on the diagonal $\theta(E)\,\mathrm{Id}_E$ vanishes for $(X,E) \in \Theta_g$. Since it comes as the restriction of a global kernel

$$\bar{\mathfrak{s}}_E \in \mathrm{H}^0(X \times X, E \boxtimes E^\vee) \otimes \Theta_X|_{\{E\}}\,,$$

it lies in the conormal bundle. Thus $\mathfrak{s}|_{3\Delta}$ indeed has logarithmic singularities along Θ_g°. Again by Proposition 5.3.1, this gives the natural trivialization of the twisted conormal bundle, so that the residue is 1. We finally conclude that the meromorphic identification of Ω–torsors $\mathcal{E}x\mathcal{C}onn_g(n) \to \mathcal{C}onn_g(\Theta)$ defined by $\mathfrak{s}|_{3\Delta} \mapsto \mathrm{d}\log\theta$ remains regular along Θ_g, as desired.

6.4. **Final Remarks.** In [5], Beilinson and Schechtman give a canonical local description of the Atiyah bundle of the determinant line bundle, with its Lie algebroid structure, for an arbitrary family of curves and vector bundles. They formulate the result in the language of kernel functions. This was the model for our definition of extended connections. In this language, their description of the Atiyah algebra is equivalent to a canonical, local identification of the twisted cotangent bundle $\mathcal{C}onn_g(\Theta)$ with the twisted cotangent bundle formed by the space of extended connections (equipped with a natural symplectic form). This follows since the Atiyah algebra of a bundle $\mathcal{L}$ is the Poisson algebra of affine–linear functions on the affine bundle of connections on $\mathcal{L}$. The approach presented above is global, and ignores the symplectic structure. The compatibility of our identification with that of [5] follows as in § 5.4.3 from the rigidity (§ 2.4) of Ω–torsors on the moduli space of curves and bundles. In particular the Lagrangian property of the Szegö extended connections follows, as in Corollary 5.4.4.

6.4.1. *Relations with conformal field theory.* The point of view of [5] is inspired by conformal field theory, in particular the Virasoro–Kac–Moody uniformization of moduli spaces (reviewed in [16, 7]). This point of view can be expanded to describe the role of the Szegö kernel. In the abelian case, it is well known (see [25, 20]) that the Szegö kernel is given by the two–point functions of fermions twisted by the line bundle $\mathcal{L} \in \mathrm{Pic}^{g-1}(X) \setminus \Theta_X$. The conformal field theoretic explanation for why $\mathrm{d}\log\theta$ comes from a kernel function on all of $X \times X$ is that it is given by a current one–point function $\langle J(z)\rangle$ on X, which by operator product expansion comes as the singular part at the diagonal of a fermion two–point function $\langle\psi(z)\psi^*(w)\rangle$ on $X \times X$ (see [16] for an algebraic development of the necessary conformal field theory). This argument should generalize to the higher rank case, with the free fermion replaced by n free fermions. We hope to return to this in future work.

The relation with logarithmic derivatives of theta functions only "sees" the restriction of the Szegö kernel to 2Δ or 3Δ. To find similar interpretations of $\mathfrak{s}|_{(n+1)\Delta}$ would require an extension of the moduli of curves to "$\mathcal{W}$–moduli", putatively associated to the vertex algebra $\mathcal{W}(\mathfrak{s}l_n)$ ([16]), where the cotangent directions include not only quadratic differentials but cubic, quartic, and up to n–ary differentials on the curve.

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