ON THE TRANSMISSION OF LIGHT THROUGH A CLOUD OF RANDOMLY DISTRIBUTED PARTICLES

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1. *Introduction*

The study of the transmission of light through a cloud of particles is of practical importance, since it is connected with natural phenomena such as the opacity of clouds and smokes, and the visibility in fog and rain. So far, such a study has been restricted only to those cases where the particles are very small. Lord Rayleigh (1899), for instance, has used his theory of the scattering of light by small particles to determine the attenuation of the transmitted light. There is, however, no satisfactory simple theory, either for transparent or opaque particles of bigger size, distributed in a transparent medium. Mallock (1919) has attempted to give a theory of the transmission of light through a cloud of water droplets, but his theory is defective in many respects. He has assumed the droplets to be opaque, which is not true. Also, he has treated the problem on a geometrical basis, omitting the effect of diffraction, which is not justified even with opaque particles. Thirdly, his application of the theory of probability to the geometrical method is also not correct.*

In this paper, a general theory is developed of the propagation of light through a cloud of spherical particles, randomly distributed in a transparent medium. Both the cases when the particles are opaque, as well as when they are transparent are considered from the point of view of the wave-theory of optics. It is shown that in general, there is always an attenuation of the transmitted beam, which is of the exponential form. The numerical value of the attenuation coefficient has also been calculated in a number of cases. In particular, Mallock's formulæ are corrected during the course of the development of the theory.

2. *Geometrical Treatment of the Problem*

In this section, we shall develop a theory of the effect on the transmitted light of a number of opaque spherical particles distributed at random

* Richardson (1919) has also given a similar theory, but he too has taken the droplets to be opaque, and has not taken diffraction into account.

in a transparent medium, purely on the basis of geometric ideas. The case in study may be illustrated by the transmission of light through heavy smoke, where the particles composing the smoke can be considered to be opaque.

Suppose that a beam of cross-sectional area A traverses a column of the smoke of length l, and that N is the number of particles per unit volume of the medium. Assume further that the particles are all of the same size, and have a radius a. On a geometric basis, the effect of each particle would be to cut out a portion of the transmitted beam of area πa^2, and thus reduce the energy content by an amount corresponding to this area. In this way, each particle reduces the energy content of the beam by a certain fraction, and it would thus appear that if the number of particles n_0 be such that $n_0 \pi a^2 = \mathrm{A}$, then the beam would be completely cut out. However, it is not so, on account of the fact that some of the particles would screen those behind them, and thus increase the chance of a portion of the direct radiation coming through. It is therefore a question of probabilities to determine what fraction of the energy is transmitted by the particles in the medium. In our case, the number of particles, n, is evidently $= \mathrm{NA}l$.

The problem at hand is identical with one in which n disks, each of radius a, are thrown at random on an area A, and it is required to find the probable area covered by the n disks. Let P_m be the probable fraction of the area covered after m disks are thrown, and let p be the fraction of the area covered by a single disk when alone, *i.e.*, $p = \pi a^2/\mathrm{A}$. Now, on adding one more disk, it may fall on the area already covered, or on the area uncovered. The probability of its falling on the empty area is evidently $(1 - \mathrm{P}_m)$. Hence, the probable fraction of the area covered after $(m+1)$ disks are thrown is

$$\mathrm{P}_{m+1} = \mathrm{P}_m + p\,(1 - \mathrm{P}_m) = \mathrm{P}_m\,(1 - p) + p \tag{1}$$

Now, $\mathrm{P}_1 = p$, so that

$$\mathrm{P}_2 = p\,(1-p) + p,$$

$$\mathrm{P}_3 = p\,(1-p)^2 + p\,(1-p) + p, \text{ etc., giving}$$

$$\mathrm{P}_n = p\,[(1-p)^{n-1} + (1-p)^{n-2} + \cdots + 1]$$

$$= p\,[1-(1-p)^n]/[1-(1-p)] = [1-(1-p)^n]. \tag{2}$$

Thus, the probable fraction of the area not covered after n disks are thrown is

$$\mathrm{Q}_n = 1 - \mathrm{P}_n = (1-p)^n. \tag{3}$$

Coming back to the problem of the transmission of the beam of light, the probable fraction of its area A, not cut out after it has encountered

n particles is given by (3). Hence, the intensity of the transmitted beam is

$$I_t = I_i (1 - \pi a^2/A)^{ANl}, \tag{4}$$

where I_i = intensity of the incident beam. Since $\pi a^2/A$ is a small quantity, we get

$$I_t/I_i = e^{-\pi a^2 Nl}. \tag{5}$$

Thus, it is seen that the intensity of the transmitted beam is never zero, but that it diminishes exponentially with increase of the thickness of the medium. Also, from the relation $Q_n = (1-p)^n$, it is seen that the effects of the particles are *multiplicative*, each one diminishing the intensity in the ratio $(1-p):1$. This result is important, since we have started with the assumption that the effect of each particle is *subtractive*, *i.e.*, each subtracts a portion of the energy from the incident beam, corresponding to its area. However, on account of the random distribution of the particles, the effect becomes multiplicative.

Incidentally, it may be noted that if $n = 1/p$, *i.e.*, if the total area of all the particles is equal to the area of the aperture, then $Q_n = (1-1/n)^n = e^{-1}$. Hence, the probable fraction of the energy coming through is $1/e$. Mallock has wrongly assumed the value 1/2 for this quantity without any proof.

The actual value of the attenuation coefficient derived in this section is not correct, for, as is shown in Section 3, diffraction effects produce a doubling of its value. But, before proceeding to consider these, it will be interesting to discuss what the attenuation would be, according to geometric ideas, if the particles were not all of the same size. In this case, we must expect a distribution of size among the particles. As a general case, suppose that the number of particles per c.c., having a radius between a and $a + da$ is

$$dN = f(a)\, da, \tag{6}$$

where $f(a)$ is the function giving the law of distribution.

Now, since all the particles are not of the same size, the application of the theory of probability has to be modified. Let $p_1, p_2, p_3, \cdots$ be the fraction of the area covered by the first, second, third, etc., particles, when alone. These can be equal or different. Then, we have the general relation similar to (1), *viz.*,

$$P_{m+1} = P_m + p_{m+1}(1 - P_m) = P_m(1 - p_{m+1}) + p_{m+1} \tag{7}$$

Hence,

$$P_1 = p_1, \qquad Q_1 = (1 - p_1)$$

$$P_2 = p_1(1 - p_2) + p_2, \qquad Q_2 = (1 - p_1)(1 - p_2)$$

$$P_3 = p_1(1-p_2)(1-p_3) + p_2(1-p_3) + p_3, \quad Q_3 = (1-p_1)(1-p_2)(1-p_3),$$
etc.

$$\cdots\cdots \qquad \cdots\cdots$$

$$P_n = p_1(1-p_2)\cdots(1-p_n) + \cdots + p_n, \quad Q_n = (1-p_1)(1-p_2)\cdots(1-p_n) \quad (8)$$

Thus, every one of the quantities Q_n is symmetrical in $p_1, p_2, \cdots$ so that the order of the quantities $p_1, p_2, \cdots p_n$ is immaterial. Hence, we may regard each particle as producing an effect independent of the rest, and reducing the energy of the transmitted beam in the ratio $(1-p_n):1$. Hence, expression (4) can be generalised, and written as

$$I_l/I_i = (1-p_1)(1-p_2)\cdots(1-p_n) = e^{-\sum_1^n p_n}.$$

If we denote by a_n the radius of the nth particle, $p_n = \pi a_n^2/A$. Also, on account of the distribution (6), the summation $\sum_1^n p_n$ can be written as

$$\sum_1^n \pi a_n^2/A = \pi l \int a^2 dN. \quad (9)$$

Hence,
$$I_l/I_i = \exp\left[-\pi l \int a^2 dN\right]$$
$$= \exp\left[-\pi l \overline{a^2} N\right] \quad (10)$$

where $\overline{a^2}$ is the mean square of the radius.

As a particular case, we may take the Maxwellian law of distribution given by $f(a) = a^2 e^{-c^2a^2}$, or

$$dN = a^2 e^{-c^2a^2}\, da. \quad (11)$$

In this case, $\overline{a^2} = \frac{3\pi}{8}\bar{a}^2$, where $\bar{a}$ is the mean radius, so that

$$I_l/I_i = \exp(-3\pi^2 \bar{a}^2 N l/8). \quad (12)$$

This expression can also be put in terms of the most probable value of the radius, a_m, as

$$I_l/I_i = \exp\left[-3\pi a_m^2 N l/2\right]. \quad (13)$$

So far, we have been dealing with spherical particles. This assumption is, however, unnecessary, for the shape of the obstacle is unimportant. The really important quantity is the area of the beam that it cuts out. If we represent the area of a particle by α, then, $p = \alpha/A$, and (5) can be written in the form

$$I_l/I_i = \exp\left[-\alpha l N\right]. \quad (14)$$

In the case, where the size of the particles is not the same,

$$I_l/I_i = \exp\left[-l \int \alpha dN\right]. \quad (15)$$

3. *Transmission according to the Wave Theory*

It is well known that a small opaque circular obstacle placed in the path of a beam of light produces a diffraction pattern, whereby part of the energy is diffracted in directions away from the forward one. On account of this, it is evident that the reduction of energy in the forward direction will be greater than that due to the mere cutting off of a portion of the wave-front by the obstacle. In the following discussion, it will be shown that the reduction is exactly double of what would be expected on the geometrical theory.

In this connection, it is important to consider what we mean by the intensity of the transmitted beam. The aperture of the incident beam is always finite, and will have a definite area A. Consequently, this will produce a diffraction pattern of its own. If we take the amplitude of the incident wave as unity, the intensity in the exact forward direction would be A^2/λ^2, and the pattern would extend over a solid angle of the order of λ^2/A. We may therefore speak of two quantities as representing the intensity of the transmitted beam: (*a*) the intensity in the exact forward direction, and (*b*) the total intensity in the diffraction pattern, which is in general comprised within a small solid angle. If the beam is brought to a focus on a photographic plate and we take a microphotometer record of the intensity, then the height of the peak would represent the quantity (*a*). If, on the other hand, we focus the beam on a photocell, the current produced would represent the quantity (*b*). The distinction between the two is necessary, since the reduction due to the existence of an obstacle is not the same for both, as is shown below.

We will now consider the general method of finding the reduction in the forward intensity. Let U be the amplitude of the diffraction pattern produced by the aperture A. Then $I = \int |U|^2 d\Omega$, where Ω is the solid angle, represents the total energy in the diffraction pattern, the integration being done over all the solid angles where $|U|^2$ is finite. On the other hand, $I_0 = |U_0|^2$ represents the intensity in the forward direction per unit solid angle, if U_0 is the amplitude in this direction.

Now, suppose that an obstacle of area a be introduced in the path of the beam. Then, this would produce a diffraction pattern of its own. Let u be the amplitude in the diffraction pattern due to an aperture complementary to the obstacle. Then, the amplitude in the pattern due to the obstacle is, by Babinet's principle equal to $-u$. The two patterns due respectively to the aperture A, and the obstacle a will be superposed, the second of which would obviously cover a larger solid angle. In this case, the total energy in the transmitted beam is

$$I' = \int |U - u|^2 \, d\Omega, \tag{16}$$

where the integration is to be carried out over the same range of solid angles, as was done for I. On the other hand, the intensity in the exact forward direction is

$$I_0' = |U_0 - u_0|^2. \tag{17}$$

From these formulæ, the reduction in intensity either of I or of I_0 can be calculated.

Coming now to the particular case in which we are interested, assume for simplicity that the obstacle α is spherical, and has a radius a. The diffraction would now consist of two parts, (1) the pattern due to the aperture A, covering a solid angle of the order of λ^2/A, and (2) the pattern due to the obstacle, covering a much larger solid angle of the order of λ^2/α.

Evidently, $I_0 = A^2/\lambda^2$, and $I_0' = (A - \alpha)^2/\lambda^2$, so that the ratio $I_0'/I_0 = (1 - 2\alpha/A)$, since α is small. Thus, the intensity in the exact forward direction is reduced in the ratio $(1 - 2\pi a^2/A): 1$, or double the reduction as given by the geometric theory.

In order to evaluate I′, assume that the pattern due to aperture extends over a solid angle Ω_0; Ω_0 is $\simeq \lambda^2/A$. Then,

$$I = \int_{\Omega_0} |U|^2 \, d\Omega = A. \tag{18}$$

Take now the whole pattern due to the aperture and the obstacle, and assume it to extend over the solid angle Ω. Then the total energy in this is

$$\int_{\Omega} |U - u|^2 \, d\Omega. \tag{19}$$

But, outside Ω_0, $U = 0$, so that the above expression can be written as

$$\int_{\Omega_0} |U - u|^2 \, d\Omega + \int_{\Omega - \Omega_0} |u|^2 \, d\Omega. \tag{20}$$

Hence,

$$I' = \int_{\Omega_0} |U - u|^2 d\Omega = \int_{\Omega} |U - u|^2 \, d\Omega - \int_{\Omega - \Omega_0} |u|^2 \, d\Omega.$$

Now, Ω_0 is $\ll \Omega$, so that the energy of the pattern due to the obstacle contained within Ω is very negligible. Hence, $\int_{\Omega - \Omega} |u|^2 \, d\Omega = \int_{\Omega} |u|^2 \, d\Omega$, which must be equal to α, by Babinet's principle. Also, $\int_{\Omega} |U - u|^2 \, d\Omega$ will be equal to $(A - \alpha)$, so that

$$I' = A - 2\alpha.$$

Thus,
$$I'/I = (1 - 2\alpha/A), \tag{21}$$
i.e., the total intensity is also reduced in the same ratio as the forward intensity, the reduction being double that given by the geometric theory.

A physical interpretation can also be given for this double loss in the total transmitted intensity. The cause has already been explained as due to the effect of diffraction. Now, by Babinet's principle, the energy in the diffraction pattern due to an opaque disk is the same as that in the pattern due to an aperture of the same size, and hence equal to α. In this way, the forward wave loses an amount of energy α due to diffraction, in addition to an equal amount lost due to the blocking by the opaque obstacle. This is exactly what is shown by the formulæ above.

However, it must be noted that we have assumed in this derivation that the energy of the diffraction pattern due to α, contained in the solid angle embraced by the pattern of the transmitted light is negligible. This will be true, only if $\alpha \ll A$. However, if this is not the case, then the additional loss of energy will not be α, but less. In general, therefore
$$I'/I = (1 - c\alpha/A), \tag{22}$$
where $c = 2$, only if α is small compared to A. If the obstacle is of dimensions comparable to those of A, then the pattern due to it will be quite small, and we have to take c nearly equal to 1, as is found from geometrical considerations. This is quite natural, since geometrical optics applies rigorously when large sizes are taken into consideration. However, in the exact forward direction, the reduction in intensity is always double that given by the geometric theory. The double loss of energy, and the conditions determining its occurrence do not seem to have been pointed by anybody so far.

We will now consider what the effect of a cloud of particles would be according to wave theory. Assuming that the incident wave is represented by $u = \exp(2\pi ict/\lambda)$ at $x = 0$, which corresponds to the beginning of the cloud, the wave at a distance x within the medium can be represented by
$$u = e^{-kx} e^{\frac{2\pi i}{\lambda}(ct - \mu x)}$$
where μ is the refractive index, and k, the absorption coefficient. Taking now a thin layer of the medium of thickness dl, the number of particles in this thickness per unit area of cross-section will be Ndl. As already seen, the effect of each particle would be to reduce the amplitude by α. Hence, the wave, after passing the thickness dl would be represented by
$$(1 - N\alpha dl)\, e^{\frac{2\pi i}{\lambda}(ct - dl)}.$$

Comparing this with the above equation, we get $\mu = 1$, and $\exp(-kdl) = (1 - N\alpha dl) = \exp(-N\alpha dl)$. Hence, the absorption coefficient for the amplitude is $k = N\alpha$, and that for intensity will be double this, namely $2N\alpha$. Hence, the intensity of the beam after passing through a length l of the medium is

$$I_l = I_i \, e^{-2N\alpha l} \tag{23}$$

where I_i is the incident intensity. For spherical particles of radius a, this gives

$$I_l / I_i = e^{-2\pi a^2 N l}. \tag{24}$$

This expression differs from that given by the geometric theory only in that there is an additional factor 2 in the exponent, the reason for which has already been explained.

4. *Passage of Light through Transparent Spheres*

This case is of practical importance in connection with the study of the transmission of light through a cloud of water droplets, as in fog, cloud or rain. Mallock has wrongly supposed that one can take the water particles as opaque. This is, *prima facie*, not true, since the droplets are by their very nature transparent, and not opaque. Consequently, portions of the wave-front transmitted through the droplets must also be taken into consideration. The portions passing through the droplet will, however, be retarded with respect to the rest of the wave-front, the extent of the retardation depending on the distance each portion has passed through the drop. Changes in amplitude will also occur in the wave-front on account of the convergence of the rays passing through the drop. These features must be taken into account in a proper theory of the phenomena.

The actually observed facts also show that the drops cannot be regarded as opaque. The work of Barus (1907, 1908), and of others has shown that the transmitted light is often highly coloured and that this colour is complementary to the colour of the light scattered by the particles themselves in the forward direction. Also, these colours are found to undergo periodic changes with increase in particle size, reappearing for values of the radii in the ratio of natural numbers. These phenomena indicate that the light transmitted through the drop must be taken into account.

The most rigorous method of attacking the problem would be to use the electromagnetic theory of light, and to determine the effect of a dielectric sphere on the field. Such a theory has been given by Mie (1908), Debye (1909), Rayleigh (1910) and Bromwich (1920). However, this method does not give any simple expression for the intensity of the diffracted and the transmitted light, if the size of the droplets is not small compared with the wave-length of light. The nature of the phenomena could only be studied by numerical computation, which again is prohibitively tedious. It is therefore desirable to give a theory based on optical principles, which would yield expressions that can readily be applied.

We shall now attempt to give such a theory for the diffraction of light by a transparent sphere. The theory will be developed to include also the diffraction of light in directions inclined to the forward one. As already said, one must take into account both the amplitude and phase changes that occur in the transmitted wave-front. However, if we assume that the difference in refractive index between the sphere and the surrounding medium is small, then we can as a good approximation, take that the effects of refraction are negligible. In this case, we can neglect the changes in amplitude, and take only the phase-changes into consideration, the path retardation of any portion of the wave-front being calculated on the assumption that it has passed straight through the drop.

T. A. S. Balakrishnan (1941) has developed a theory for the diffraction by a single droplet based on similar ideas. In this paper, the theory is worked out more fully, and it is extended to the consideration of the transmission and forward scattering by a cloud of transparent spherical droplets.

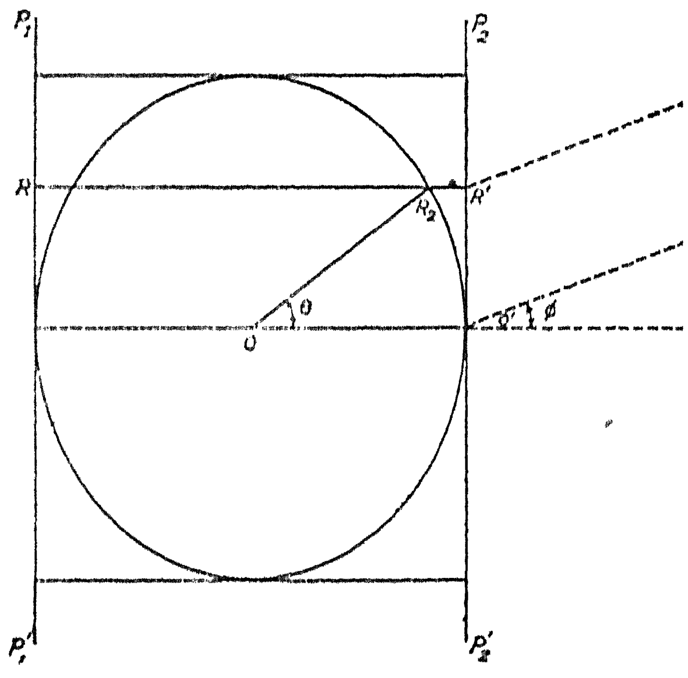

FIG. 1. Passage of Light through a Sphere

5. *Derivation of the Formulæ*

In this section, the formulæ for the diffraction of light by a spherical droplet is derived, for the case when the amplitude changes can be neglected, and the wave-front can be supposed to have passed straight through the drop. Let P_1RP_1' be a plane wave-front which, after passing through a droplet of centre O, and radius a, emerges out as $P_2R'P_2'$ (Fig. 1). The phase of the emergent wave-front will evidently not be uniform throughout, but will lag behind in the portions which have passed through the drop. Let RR_2R' be any ray, and let the angle between OR_2 and OO' be θ. Then, this ray suffers a retardation of $2a(\mu-1)\cos\theta$ with respect to the portions of the wave-front passing outside the drop. Thus, the plane wave-front, on the whole, would be distorted into one having a concave part in the centre.

Our idea is to find the Fraunhofer diffraction pattern of this emergent wave-front, *i.e.*, we wish to find the intensity in a direction making an angle ϕ with the incident direction. For this, we proceed in a manner analogous to the usual theory of diffraction by a circular aperture (*vide* Max Born, *Optik*, pp. 155 *et seq.*). Taking polar co-ordinates, if (r, α) be the co-ordinates of an element at R′ with respect to O′ as origin, and the horizontal radius as the initial position of the radius vector, then the diffracted amplitude in the direction ϕ due to this element is

$$\frac{1}{\lambda}\sin\frac{2\pi}{\lambda}\{ct-2(\mu-1)a\cos\theta+r\cos\alpha\sin\phi\}\,r\,dr\,d\alpha,$$

where the amplitude of the incident wave-front is assumed to be unity, and λ is the wave-length of the light. Substituting $a\sin\theta$ for r, and integrating between the proper limits, we get that the amplitude in the direction ϕ due to the portion of the wave-front which has passed through the drop is

$$X_1=\frac{a^2}{\lambda}\int_0^{\pi/2}\int_0^{\pi}\sin\frac{2\pi}{\lambda}\{ct-2(\mu-1)a\cos\theta+a\sin\theta\cos\alpha\sin\phi\}\sin\theta\cos\theta\,d\theta\,d\alpha.$$

Putting $\chi=2\pi ct/\lambda$, $\xi=4\pi(\mu-1)a/\lambda$, and $\eta=2\pi\sin\phi\,a/\lambda$,

$$X_1=\frac{a^2}{\lambda}\int_0^{\pi/2}\int_0^{2\pi}\sin(\chi-\xi\cos\theta+\eta\sin\theta\cos\alpha)\sin\theta\cos\theta\,d\theta\,d\alpha.$$

Integrating with respect to α, this can be written as

$$X_1=\frac{2\pi a^2}{\lambda}\int_0^{\pi/2}J_0(\eta\sin\theta)\sin(\chi-\xi\cos\theta)\sin\theta\cos\theta\,d\theta. \tag{25}$$

The effect of the portion of the original undisturbed wave-front covered by the drop is obtained by substituting $\mu=1$ in the above expression, which makes $\xi = 0$, and gives the value

$$X_2 = \frac{2\pi a^2}{\lambda} \sin \chi \int_0^{\pi/2} J_0 (\eta \sin \theta) \sin \theta \cos \theta \, d\theta$$

$$= \frac{2\pi a^2}{\lambda} \cdot \frac{J_1 (\eta)}{\eta} \cdot \sin \chi. \tag{26}$$

The amplitude X of the diffracted wave due to the complete wave-front which has traversed the drop is the sum of the amplitude due to the portion which has passed through the drop, and that due to the portion which has passed outside it. The former is given by X_1, and the latter by subtracting X_2 from the effect due to the undisturbed wave-front, which we may call X_0. In the region outside that covered by the source, X_0 is evidently zero, so that

$$X = X_1 - X_2. \tag{27}$$

In the forward direction, *i.e.*, for $\phi = 0$, X_0 is finite, so that the resultant amplitude, X', in the forward direction, is

$$X' = X_0 + X_1' - X_2', \tag{28}$$

where X_1' and X_2' are the values of X_1 and X_2, when $\phi = 0$. For this particular case, since $\phi = 0$, η vanishes, and the integral in the expression for X_1 can be easily integrated, giving

$$X_1' = \frac{2\pi a^2}{\lambda \xi^2} \int_0^{\pi/2} \sin (\chi - \xi \cos \theta) (\xi \cos \theta) \, d (\xi \cos \theta)$$

$$= \frac{2\pi a^2}{\lambda \xi^2} \sin \chi \, [\xi \sin \xi + \cos \xi - 1] + \frac{2\pi a^2}{\lambda \xi^2} \cos \chi \, [\xi \cos \xi - \sin \xi].$$

Also, $X_2' = \frac{\pi a^2}{\lambda} \sin \chi$, since $\underset{\eta \to 0}{\mathrm{Lt}} \frac{J_1 (\eta)}{\eta} = \frac{1}{2}$, so that

$$X' = X_0 + 2\pi a^2/\lambda \cdot \sin \chi \cdot (\cos \xi/\xi^2 + \sin \xi/\xi - 1/2 - 1/\xi^2)$$
$$+ 2\pi a^2/\lambda \cdot \cos \chi \, (\cos \xi/\xi - \sin \xi/\xi^2).$$

Now, if A be the aperture of the beam, then X_0 is clearly equal to A/λ. $\sin \chi$, so that

$$X' = \frac{A \sin \chi}{\lambda} \left[1 + \frac{2\pi a^2}{A} \left\{\frac{\cos \xi}{\xi^2} + \frac{\sin \xi}{\xi} - \frac{1}{2} - \frac{1}{\xi^2}\right\}\right]$$

$$+ \frac{A \cos \chi}{\lambda} \left[\frac{2\pi a^2}{A} \left\{\frac{\cos \xi}{\xi} - \frac{\sin \xi}{\xi^2}\right\}\right]. \tag{29}$$

Now, the quantity $2\pi a^2/A$ is small, (since A can be made as large as we please), so that its square can be neglected. Hence, the intensity in the forward direction due to the wave-front that has passed through a droplet is

$$I_0' = \frac{A^2}{\lambda^2}\left[1 - \frac{4\pi a^2}{A}\left\{\frac{1}{2} + \frac{1}{\xi^2} - \frac{\cos\xi}{\xi^2} - \frac{\sin\xi}{\xi}\right\}\right] = \frac{A^2}{\lambda^2}\left[1 - \frac{4\pi a^2}{A} f(\xi)\right] \text{ (say).}$$

This expression gives the reduction in intensity due to a single droplet. In order to find the effect of a cloud of droplets, we proceed in a manner similar to what we adopted for a cloud of opaque particles. Let the incident wave be represented by $u = \sin\left[\frac{2\pi}{\lambda}(ct - \mu x)\right]$, the origin being at the boundary of the cloud. Then, from (29), the amplitude of the wave after passing through a layer of the cloud of thickness dl is

$$u' = [1 - \mathrm{N}dl\, 2\pi a^2 f(\xi)] \sin\frac{2\pi}{\lambda}(ct - dl) - \mathrm{N}dl\, 2\pi a^2 g(\xi) \cos\frac{2\pi}{\lambda}(ct - dl), \quad (30)$$

where $g(\xi)$ is the expression $(\sin\xi/\xi^2 - \cos\xi/\xi)$. From the wave-theory, we know that u' must be of the form

$$u' = e^{-kdl} \sin\frac{2\pi}{\lambda}(ct - \mu dl) \quad (31)$$

Putting the expression (30) above in this form

$$u' = [\{1 - 2\pi \mathrm{N} a^2 f(\xi)\, dl\}^2 + \{2\pi \mathrm{N} a^2 g(\xi)\, dl\}^2]^{1/2} \sin\left\{\frac{2\pi}{\lambda}(ct - dl) - \epsilon\right\},$$

where $\tan\epsilon = 2\pi \mathrm{N} a^2 g(\xi)\, dl / \{1 - 2\pi \mathrm{N} a^2 f(\xi)\, dl\}$.

Since both the quantities $2\pi \mathrm{N} a^2 f(\xi)\, dl$ and $2\pi \mathrm{N} a^2 g(\xi)\, dl$ are small compared to unity, this may be written as

$$u' = [1 - 2\pi \mathrm{N} a^2 f(\xi)\, dl] \sin\{2\pi (ct - dl) - \epsilon\}$$

$$= e^{-2\pi \mathrm{N} a^2 f(\xi) dl} \sin\left\{\frac{2\pi}{\lambda}(ct - dl) - \epsilon\right\},$$

where $\tan\epsilon = 2\pi \mathrm{N} a^2 g(\xi) dl = \epsilon$.

Comparing this with (31), we see that $k = 2\pi \mathrm{N} a^2 f(\xi)$ and $\mu = 1 + \mathrm{N}\lambda a^2 g(\xi)$. Since the absorption coefficient for intensity is double that for amplitude, we get that the intensity after passing through a length l of the cloud is

$$I_l = I_0 \exp[-4\pi a^2 \mathrm{N} l f(\xi)], \quad (32)$$

where I_0 is the incident intensity.

6. *Colours of the Transmitted Beam*

Equation (32) above gives the formula for the attenuation of the directly transmitted beam. Ordinarily, for particles whose radii are larger than the wavelength of light, ξ is large, so that terms in $1/\xi^2$ can be neglected. (32) then goes over into the form

$$I_l/I_0 = \exp[-4\pi a^2 \mathrm{N} l (1/2 - \sin\xi/\xi)]. \quad (33)$$

This expression clearly shows that the attenuation of the transmitted light is different for different wavelengths, the attenuation coefficient depending on ξ, and hence on λ. Taking particles of a fixed size, the value of ξ will evidently depend on the wavelength λ of the light, and from (33) it is clear that the intensity will be a maximum for those for which $\sin \xi$ is $+1$, and a minimum for which $\sin \xi$ is -1. Now, the visible spectrum comprises the range of values $0{\cdot}4\,\mu$ to $0{\cdot}7\,\mu$ of λ, so that if the drops are small enough, there will be only one maximum, and the colour of the source will be due to the predominant effect of the range of wave-lengths in the neighbourhood of this maximum. This explains the reason why the transmitted light is vividly coloured. In Fig. 2, $f(\xi)$ is plotted against ξ, and it shows the manner in which $f(\xi)$ undergoes maxima and minima.

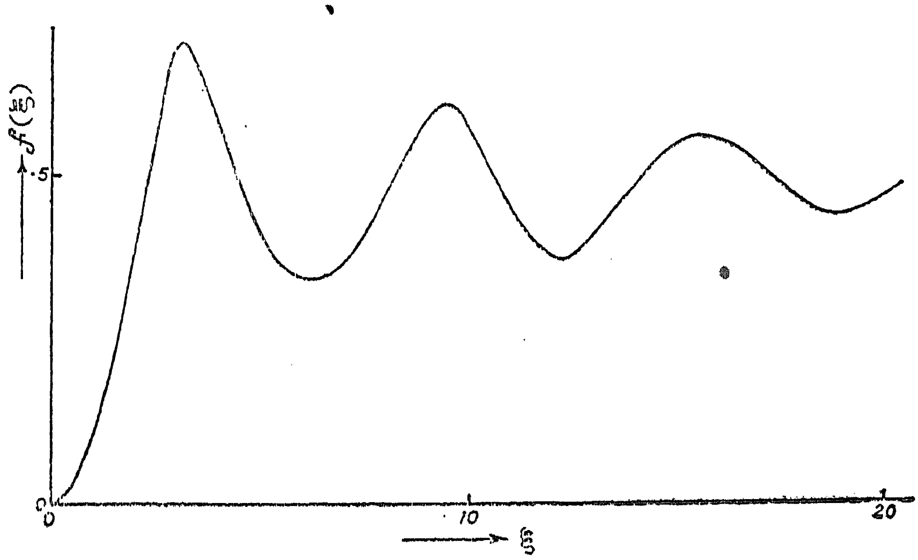

FIG. 2. Graph of $f(\xi)$ against ξ

As the size of the droplet is changed, the value of λ for which the maximum occurs will vary, and consequently, the colour of the transmitted beam will also vary. Since the varying function is $\sin \xi$, we should expect a cyclic change of colours as ξ increases; the colours will repeat themselves for values of ξ which are in the ratio of natural numbers, *i.e.*, for values of the radius a which are in this ratio. This is in accordance with Barus' observations (1907).

If the size of the droplets is further increased, the range of values of ξ may comprise two or more maxima, and also the difference between the maxima and minima of intensity will not be pronounced, since $(\sin \xi)/\xi$ becomes intrinsically small. Hence the colours will become less and less vivid. In the limiting case of very large values of ξ, $\sin \xi/\xi$ becomes small compared to $\frac{1}{2}$, and (33) reduces to

$$I_l/I_0 = \exp\left[-2\pi a^2 N l\right]. \qquad (34)$$

There is no term depending on λ, so that no colours will be exhibited. It is also interesting to notice that this is identical with the intensity in the forward direction given by a large opaque sphere of the same size. The following simple physical argument shows that this must be the case. When the particles are large, the path retardation between the central ray, and the rays passing outside the drops will comprise a number of wavelengths. We can therefore divide the portion of the wave-front which has passed through the droplets into half-period zones, of which the outer ones will practically cancel out. The resultant effect will only be due to the innermost rings, the area of which relative to the total area of the wave-front that has passed through the drop is smaller, the larger the drop. Thus, the forward intensity must approach the value for an opaque disc. In fact, the vanishing of the term $(\sin \xi)/\xi$ mathematically represents this fact.

7. *The Intensity of Forward Scattering*

As already mentioned, we can employ our theory to consider the nature of the light scattered in the forward direction by the droplets. This would comprise the region just outside the region covered by the source. Here, ϕ is small, but the effect of the incident wave is zero. The amplitude is therefore $X = X_1' - X_2'$. Substituting for X'_1 and X'_2,

$$X = \frac{2\pi a^2}{\lambda} \sin\chi \left[\frac{\cos\xi}{\xi^2} + \frac{\sin\xi}{\xi} - \frac{1}{2} - \frac{1}{\xi^2}\right] + \frac{2\pi a^2}{\lambda} \cos\chi \left[\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}\right]. \quad (35)$$

As before, neglecting terms in $(1/\xi^2)$ as an approximation, we get the intensity as equal to

$$(2\pi a^2/\lambda)^2 \,(1/4 - \sin\xi/\xi). \quad (36)$$

This expression refers to the intensity diffracted by a single droplet. But the light reaching it, and the light diffracted have both to travel through the fog, and the total path will not differ from l, since ϕ is small. Hence, the expression (36) has to be multiplied by the expression

$$\exp\,[-4\pi a^2 N l\,(1/2 - \sin\xi/\xi)].$$

Also, since we have assumed a random distribution of the droplets, there will be no phase correlation between the light scattered by different particles. Hence, the light diffracted out of the cloud in the forward direction, per c.c., is $I_f = N\,(2\pi a^2/\lambda)^2 \cdot (1/4 - \sin\xi/\xi)$.

$$\exp\,[-4\pi a^2 N l\,(1/2 - \sin\xi/\xi)]. \quad (37)$$

This is of the form Bxe^{-Cx}, where $x = N\,(1/4 - \sin\xi/\xi)$. The corresponding expression for the transmitted light is $I_l = I_0 e^{-Cx}$. Now, the former expression is zero when $x = 0$, and reaches a maximum for $x = 1/C$, after which it diminishes. The latter is a maximum at $x = 0$, and steadily diminishes as x increases. Hence, if the maximum value of x, upon which the colours depend, is less then $1/C$, then corresponding to it I_f will be a maximum, while I_t is a minimum. Similarly, for the minimum value of x, the reverse will be the case. Thus, the colour in the region immediately surrounding the source will be complementary to that of the source itself.

However, it is to be noted that for this phenomenon to be prominent, the maximum value of x must be less than $1/C$, or Nl must be less than a quantity which depends on the size of the droplets. In other words, if the density of the fog is fixed, there is a limit to the depth of the fog beyond which the colours are not complementary, but will tend to be more and more identical. The phenomenon of complementary colours can best be observed with fairly thin columns of cloud, as for instance when the sun is seen through the puffs of steam emitted by a locomotive.

In the limiting case of very large drops, the expression (36) for the forward intensity reduces to $(\pi a^2/\lambda)^2$. This is again identical with that for an opaque obstacle of the same size.

8. *Concluding Remarks*

The theory developed in this paper for the transmission of light through a medium containing a cloud of particles cannot be expected to be strictly valid for very small sizes of the particles, since Huygens' principle, and the idea of complete wave-fronts, on which it is based cannot be applied when very small dimensions are concerned. The correct procedure in these cases is to apply the theory of scattering of light, as for example, the theory developed by Mie and others (*loc. cit.*). These however do not yield any simple expressions; but for very small droplets, the theory of Rayleigh can be used. In this case, the attenuation has actually been worked out by Lord Rayleigh (*loc. cit.*) and can be represented by

$$I_t/I_0 = \exp\left[-Nl\,\frac{128\pi^5}{3}\cdot\frac{(\mu^2-1)^2}{(\mu^2+2)^2}\cdot\frac{a^6}{\lambda^4}\right]. \tag{38}$$

In the case when $(\mu - 1)$ is small, $(\mu^2 + 2)^2$ can be put as equal to 9, so that

$$I_t/I_0 = \exp\left[-128\pi^5(\mu^2-1)^2 a^6 Nl/27\lambda^4\right]. \tag{39}$$

This theory is certainly not true for larger particles. It is not possible to say exactly at what stage the line of argument advanced in this paper cases to be valid, although we can take it to be applicable, if the radius is larger than the wave-length of light. This point will be discussed more in detail in a later publication.

I wish to express my sincere thanks to Prof. Sir C. V. Raman for suggesting the problem, and for the kind and encouraging guidance he gave during the course of the investigation.

Summary

The phenomenon of the transmission of light through a cloud of particles distributed at random in a transparent medium is theoretically investigated on the basis of wave-optics. The cases of both transparent and of opaque particles are considered, and it is found that the transmitted beam is progressively attenuated, the intensity diminishing exponentially with the increase in the thickness of the medium. The actual values of the attenuation coefficient are calculated in both cases. It is found that for an opaque particle, the diminution of intensity in the forward direction is actually double what would be expected from simple geometric considerations. For transparent particles, the transmitted intensity shows spectral variations, and this explains some of the phenomena hitherto not well understood, such as the colours shown by the transmitted light, and the complementary nature of this colour and the colour of the light scattered by the cloud in the forward direction.

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