A PROBLEM IN PROBABILITY RELATED TO THE PASSAGE OF LIGHT THROUGH A CLOUD OF PARTICLES

BY G. N. RAMACHANDRAN, F.A.SC.

(*Department of Physics, University of Madras, Guindy, Madras-25*)

Received July 23, 1960

1. INTRODUCTION

IF a beam of light passes through a cloud of opaque spherical particles, containing N particles per unit volume each of cross-sectional area α, then it is readily shown that the variation of intensity I_z with depth z is given by the differential equation

$$dI_z = -N\alpha I_z dz \tag{1}$$

whose solution is

$$I_z = I_0 e^{-N\alpha z}. \tag{2}$$

This corresponds to the so-called geometrical optics case, in which all diffraction effects have been neglected.

It is also possible to work out the above problem by considering the effect of each particle in turn on the energy content of the transmitted beam. This has been done by the author (Ramachandran, 1943. This paper also contains an account of the physical optical approach) and the theory leads to the formula

$$\frac{E_n}{E_0} = \left(1 - \frac{\alpha}{A}\right)^n \tag{3}$$

for the energy content of a beam of cross-sectional area A after it has encountered n particles, each presenting an area α to the beam. Since $n = NAz$ we have from Equation (3)

$$\frac{I_z}{I_0} = \frac{E_z}{E_0} = \left(1 - \frac{\alpha}{A}\right)^{NAz}. \tag{4}$$

It is clear that Equations (2) and (4) are not identical, but that the latter goes over into the former when $\alpha/A \ll 1$. Thus, the exponential formula (2) is really an approximation valid only when the continuous approximation in Equation (1) is made, although in practical examples, the difference between (3) and (4) cannot be noticed.

The problem of finding the energy content of the beam after it has encountered n particles of cross-sectional area α is identical with one in which n discs, each of area α, is thrown at random over an area A, and it is required to find the probable area covered by the n discs. An essential step in obtaining this is the following. Suppose A_m is the uncovered area after m discs have been thrown and $B_m = (A - A_m)$ is the covered area, then the probable additional area that is covered on throwing the $(m + 1)$th disc is

$$\Delta B_m = \left(\frac{A_m}{A}\right) \alpha. \tag{5}$$

This is obviously true if α, the area of the disc, is small compared with the area A_m, for the probability that its centre occurs in the uncovered area is exactly A_m/A and when it does so, it covers an additional area α, so that the probable additional area that is covered is $(A_m/A)\,\alpha$.

However, the result does not follow so simply if the area of the disc is not small, for the additional area that is obscured will be less than α if its centre lies close to the boundary of a region of the uncovered area A_m and further the $(m + 1)$th particle may cover a portion of the region A_m even if its centre lies outside the region A_m. The calculation of the mean additional area that is covered by the $(m + 1)$th disc would, therefore, appear to be very difficult. However, it is the purpose of this paper to show that Equation (5) is still true under all conditions for the probable (or mean) additional area that is covered and that it is *independent of the size or shape* of both the disc (α) and the aperture (A). This very surprising result follows quite generally from a simple theorem connected with the integral of the *Faltung* or convolution of two functions. In fact the general approach can be used to work out the mean transmission through a cloud of n particles, or discs, when each particle is not only of arbitrary cross-section but exhibits variations in transmission coefficient over its area. Even in this case, it follows that the probable (mean) reduction in energy of the beam due to the $(m + 1)$th particle is exactly equal to $(E_m/E_0)\,\eta$, where η is the energy absorbed by a single disc when a uniform beam of unit intensity falls on it. In order to bring out the physical ideas clearly, some simple examples are worked out below directly without reference to the general theorem.

2. Simple One-Dimensional Analogue—the Strip Problem

Suppose there is a straight slit PQ of length L having uniform width and suppose we throw opaque strips each of length l whose width is larger than that of the aperture, such that they fall on the aperture with their length parallel to the length of the aperture, but with their centres distributed completely at random within the length L_0 (Fig. 1).

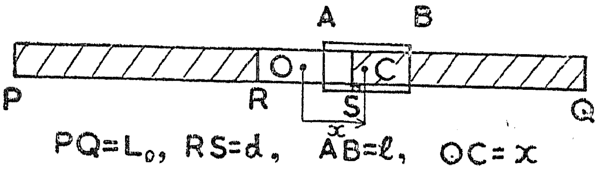

Fig. 1

Consider the situation after m strips are thrown. Some of them will overlap, and in general, there will be a number of gaps R_iS_i ($i = 1$ to k) of length d_i which are uncovered. Let $L_m = \sum_1^k d_i$ be the total uncovered length. Let the $(m+1)$th strip AB be now thrown and let the probability $p(x)\,dx$ of its falling with its centre C in an interval dx within PQ be given by $p(x)\,dx = k\,dx$, where k is a constant. (Note that we are now neglecting boundary effects.) In order to work out the mean additional length that is covered by this strip, consider one of the gaps R_iS_i which we will denote by RS for convenience, and let its length be d (Fig. 1). Suppose that it is the only gap in the slit, and denote the position of the new strip AB by the co-ordinate x of its centre C measured from the centre O of RS (Fig. 1). Then, denoting the additional length that is covered by this strip for different values of x by $f(x)$ the function $f(x)$ will vary as follows:

(*a*) $d > l$

$$f(x) = 0 \text{ for } x < -\left(\frac{d+l}{2}\right) \text{ and } x > +\left(\frac{d+l}{2}\right)$$

$$f(x) = l \text{ for } -\left(\frac{d-l}{2}\right) \leqslant x \leqslant \left(\frac{d-l}{2}\right)$$

and $f(x)$ varies linearly in the intermediate positions

(*b*) $d < l$

$$f(x) = 0 \text{ for } x < -\left(\frac{l+d}{2}\right) \text{ and } x > \left(\frac{l+d}{2}\right)$$

$$f(x) = d \text{ for } -\left(\frac{l-d}{2}\right) \leqslant x \leqslant \left(\frac{l-d}{2}\right)$$

and $f(x)$ varies linearly in between.

The variations are shown schematically in Figs. 2 (a) and (b) from which the integral of $f(x)$ from $-(d+l)/2$ to $+(d+l)\,2$ is seen to be ld in both cases. Thus, the mean length that is additionally obscured by AB is

$$\frac{\int_P^Q f(x)\,p(x)\,dx}{\int_P^Q p(x)\,dx} = \frac{k\int_{-(d+l)/2}^{+(d+l)/2} f(x)\,dx}{k\mathrm{L}} = \frac{ld}{\mathrm{L}} = \left(\frac{d}{\mathrm{L}}\right) l \tag{8}$$

i.e., it is a fraction d/L of l.

Clearly, if more than one gap $\mathrm{R}_i\mathrm{S}_i$ is present, the contribution from each gap of length d_i is equal to $(d_i/\mathrm{L})\,l$, so that the mean additional length cut off by the $(m+1)$th strip is

$$\frac{\Sigma\, d_i}{\mathrm{L}}\, l = \frac{\mathrm{L}_m}{\mathrm{L}}\, l. \tag{9}$$

Thus,

$$\mathrm{L}_{m+1} = \left(1 - \frac{l}{\mathrm{L}}\right) \mathrm{L}_m \tag{10}$$

or each additional strip cuts out, on the average, a fraction l/L of the uncovered area (neglecting boundary effects).

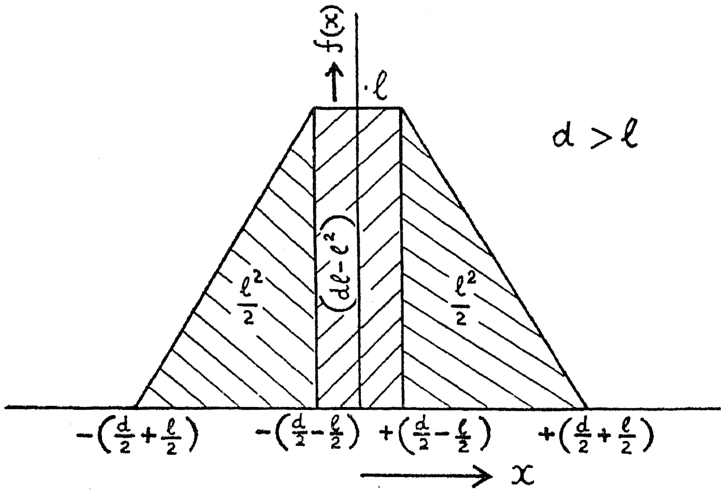

FIG. 2 (a)

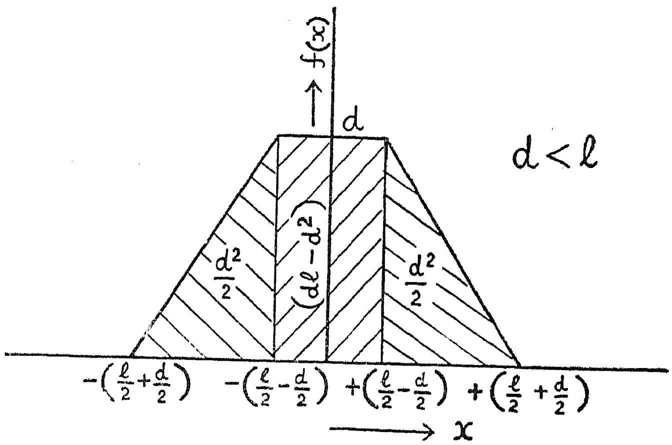

FIG. 2 (*b*)

The most interesting aspect of the above derivation is the fact that it does not depend on the magnitude of the length l of the strip and *the final result is true irrespective of whether it is larger or smaller than the individual gaps*. This comes because what is gained in the swings is lost in the roundabouts.

It might appear that such an exact balance will be possible only if there is a *linear* variation of $f(x)$ when it is varying, as is the case here. However, such a condition is not all necessary, as will be seen from the more general problem to be considered in the next section.

3. General One-Dimensional Case—Strip with Variable Absorption

Suppose in the example considered in Section 2, the strips are not opaque, but exhibit variations in transmission. This may be defined by a function $g(x')$ which gives the variation of the absorption coefficient, (*i.e.*, the fraction of the incident intensity that is absorbed) with distance x' from a chosen point (say the centre C of the strip). Thus, if light of intensity unity (per unit length of the slit PQ) is incident on the strip of length l, then the energy absorbed by it is

$$\eta = \int_{-l/2}^{+l/2} g(x')\, dx'. \tag{11}$$

Thus each strip absorbs an energy ηI where I is the incident (linear) intensity. We shall now consider what happens when the $(m+1)$th strip is added. Suppose light of (linear) intensity I is incident on the slit PQ. Then its total energy is $E_0 = LI$. Let the distribution of intensity along the length of the slit after m strips have been thrown be given by the function $h_m(x)$, x being now measured from P [Fig. 3 in which the function $g(x')$ is also schematically represented]. Clearly,

$$E_m = \int_0^L h_m(x)\, dx. \tag{12}$$

Now, let the probability of the centre C of the $(m+1)$th disc falling in an interval dx at x be $p(x)\,dx = k\,dx$ as before. When it does fall there, the amount of energy removed from the transmitted beam is given by

$$\int_0^L g(x''-x)\, h_m(x'')\, dx'' = \int_{-\infty}^{+\infty} g(x''-x)\, h_m(x'')\, dx''$$
$$= gCh_m(x). \tag{13}$$

since $h_m(x'')$ is zero for values of x'' outside the range 0 to L.

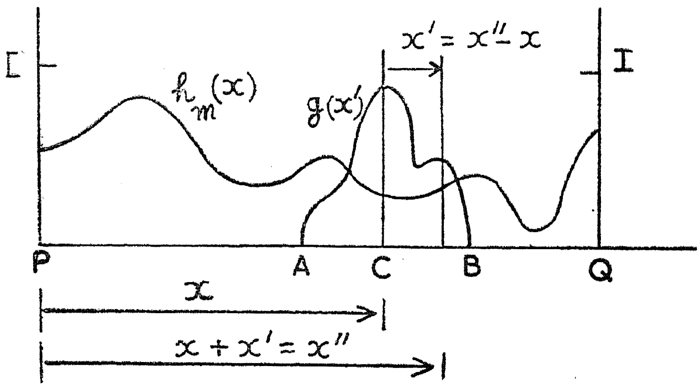

FIG. 3

The symbol gCh_m stands for the "correlation function" of the function $g(x)$ with $h_m(x)$ (see Appendix I for the definition of 'correlation function'). Since $g(x')$ becomes zero outside the length l of the strip, the limits of integration in the first expression in Equation (13) are justified if boundary effects are neglected.

The result (13) is for the case when the centre C of the strip falls at x. Consequently, the *mean* energy abstracted from the transmitted beam is obtained by averaging overall values of x from 0 to L. This gives a value

$$\frac{\int_0^L p(x)\, gCh_m(x)\, dx}{\int_0^L p(x)\, dx} = \frac{1}{L}\int_0^L gCh_m(x)\, dx. \tag{14}$$

By the correlation integral theorem discussed in Appendix I (Equation A 10) the last integral is equal to

$$\int_0^L h_m(x)\, dx \cdot \int_{-l/2}^{+l/2} g(x)\, dx = E_m \eta. \tag{15}$$

Thus, the mean reduction in energy due to the $(m+1)$th strip is $\eta E_m/L$ or

$$E_{m+1} = \left(1 - \frac{\eta}{L}\right) E_m \tag{16}$$

which, by iteration, leads to the result

$$E_n = \left(\frac{1-\eta}{L}\right)^n E_0. \tag{17}$$

It will be noticed again that Equation (16) is true irrespective of the nature of the variations of the functions $h_m(x)$ and $g(x)$. This is analogous to the result found for the opaque strip, namely, that the relative dimensions of l and the widths d_i of the various gaps is immaterial to the problem. Thus, the intensity may fluctuate to any extent over the length of the slit and the transmission of the strip AB may also widely vary over its length, but these are immaterial to the problem so long as we are interested only in the probable (or mean) reduction in energy due to the $(m+1)$th strip. The mean reduction $(E_m - E_{m+1})$ is then simply a fraction η/L of the total energy content E_m.

The interesting point is that the evaluation of the function (13), giving the reduction in intensity when the centre of the strip falls at x, may be extremely difficult in any practical case. However, the theorem of Appendix I reduces the mean of this to an extremely simple form, namely, to the evaluation of the transmission of a single strip when kept in a uniform beam. Thus, we are able to show that Equation (16) is exactly true, analogous to Equation (10).

Incidentally, the centre C of the strip AB may be replaced by any fixed point on the strip in the above derivation and all the results would still be true. This is of interest in connection with the generalisation to two dimensions.

4. Extension to Two Dimensions—the Disc Problem

We shall now return to the disc problem, but will no longer restrict ourselves to circular discs. On the other hand, suppose that the discs are of arbitrary shape, but that they are all identical in shape and size (each of area a) and that they fall in exactly the same orientation on the area A, but at random positions. The randomness may be specified by stating that the probability that a chosen point on the disc falls in an area dA at $\boldsymbol{r}$ is equal to $K\,dA$, where K is a constant.

Let us denote the uncovered regions after m disc are thrown by A_m, and use A_m to denote also their area. Analogously to the strip problem, we shall prove that the *mean* additional area that is covered by the $(m+1)$th disc is $(A_m/A)\,\alpha$. A reference to Fig. 4 will show how surprising this result is at first sight, all the more so, since an attempt at proving this directly even for a circular disc turns out to be very difficult, because the shape of the individual apertures in the region A_m cannot be specified analytically.

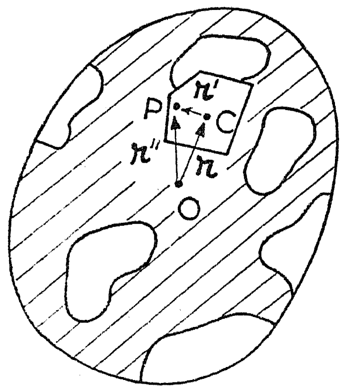

FIG. 4

We shall, therefore, prove a very general result by resorting to the theorem on the integral of convoluted functions and then deduce the above as a special case. Consequently, let the function $g(\boldsymbol{r}')$ denote the variation of absorption on the disc, the vector $\boldsymbol{r}'$ being measured from a specified origin C on the disc. Similarly let $h_m(\boldsymbol{r})$ be the distribution of intensity over the plane of the area A, after m discs are thrown, $\boldsymbol{r}$ being measured from an origin O on it. Clearly

$$E_m = \int_A h_m(\boldsymbol{r})\, d\boldsymbol{r}. \tag{18}$$

(Here $d\boldsymbol{r}$ denotes an element of area dA as in Appendix I.) Now, let the $(m+1)$th disc be thrown. Then for a particular position of this disc with $\vec{OC} = \boldsymbol{r}$, the energy absorbed by the disc is analogous to Equation (13),

$$\int_A h_m(\boldsymbol{r}'')\, g(\boldsymbol{r}'' - \boldsymbol{r})\, d\boldsymbol{r}'' = gCh_m(\boldsymbol{r}). \tag{19}$$

Consequently, the mean energy abstracted from the beam by the $(m+1)$th disc is similar to Equation (14) and is equal to

$$\frac{1}{A}\int_A gCh_m(\boldsymbol{r})\, d\boldsymbol{r}, \tag{20}$$

which by the correlation-integral theorem (A 20) of Appendix I, is equal to

$$\frac{1}{A}\cdot\int_A h_m(\boldsymbol{r})\, d\boldsymbol{r}\cdot\int_a g(\boldsymbol{r})\, d\boldsymbol{r} = \frac{E_m\eta}{A} \tag{21}$$

where

$$\eta = \int_a g(\boldsymbol{r})\, d\boldsymbol{r} \tag{22}$$

is the energy absorbed by one disc when unit intensity is incident on it.

Hence,

$$E_{m+1} = E_m \left(1 - \frac{\eta}{A}\right) \tag{23}$$

and by iteration

$$E_n = E_0 \left(\frac{1 - \eta}{A}\right)^n . \tag{24}$$

Clearly, when the discs are opaque, η is just equal to a the area of the disc, and Equation (22) reduces to the form of Equation (3). The result is however true for a disc of any shape.

As stated in Appendix I, the correlation-integral theorem is true even if the two functions which are convoluted are functions of any number of variables. However, there is no analogue of the light transmission problem in three dimensions of real space.

5. Discussion

We had neglected the boundary effects in all the previous sections. The variations occurring near the boundary would have negligible effects only if the area a is small compared with the total area A (similarly for the one-dimensional case). However, the main point of the results proved above is that the variations *inside* the area A all get averaged out exactly, irrespective of the shape or nature of the discs and of the individual apertures within A, *provided we assume that the probability of a chosen point on the disc occurring anywhere within the area A is a constant.* If this last proviso is satisfied, all our results are mathematically correct. Thus, this condition is satisfied in the case of a light beam passing through a random cloud of particles of uniform density when the cross-section of the beam is completely inside the cloud.

It is interesting to note that although the phenomenon in its elementary form is subtractive, *i.e.*, each disc removes a certain amount of energy from the beam, the effect of each additional disc is found to be multiplicative owing to probability effects, when the average is taken. It would be of interest to study the *distribution* of the energy E_n after encountering n discs, quite apart from its mean value, which we have shown is equal to $E_0 (1 - a/A)^n$, but this is not attempted here.

The case when the particles of arbitrary but identical shape and size take up not only a random position within the area A, but also a random orientation, can also be readily worked out. Since for a definite orientation the mean follows the iterative formula (23), it undergoes no further change when averaged over all possible orientations. Consequently, Equation (24) still holds for E_n and the only relevant factor is η, the absorption factor for each disc.

The case when the particles vary in size has been treated in the earlier paper by the author when the size is small. A study of the proof shows that the results are equally valid when the size is not small, and by following arguments similar to those in the last section, it is possible to show that the following equations hold for the mean energy:

$$E_m = E_{m-1} \left(\frac{1 - \eta_m}{A} \right) \quad (25)$$

and

$$E_n = E_0 \left(1 - \frac{\eta_1}{A}\right) \left(1 - \frac{\eta_2}{A}\right) \cdots \left(1 - \frac{\eta_n}{A}\right) \quad (26)$$

where η_i is the energy that would be absorbed by the ith disc if unit intensity is incident upon it.

ACKNOWLEDGEMENT

I wish to thank Dr. P. M. Mathews for the discussion I had with him which led to the development of these ideas.

REFERENCES

Ramachandran, G. N. .. *Proc. Ind. Acad. Sci.*, 1943, **17 A**, 171.

Sneddon, I. N. .. *Fourier Transforms*, McGraw-Hill, New York, 1951.

APPENDIX

Faltung and Correlation Integral Theorems

1. Theorems on Faltung of Functions

The *Faltung* (convolution) of two functions $g(x)$ and $h(x)$ of a single variable x, denoted by $gFh(x)$ [$= f(x)$ say] is defined by*

$$gFh(x) = f(x) = \int_{-\infty}^{+\infty} g(x')\, h(x - x')\, dx', \tag{A 1}$$

which is also equivalent to the form

$$f(x) = \int_{-\infty}^{+\infty} g(x - x'')\, h(x'')\, dx''. \tag{A 2}$$

It is obvious that the functions gFh and hFg are identical.

Although the limits $-\infty$ and $+\infty$ is used for the integral, in practice both functions might be zero beyond finite limits, in which case the faltung $f(x)$ would also have a non-zero value only within finite limits. In such a case, it would be sufficient to evaluate the integral (A 1) or (A 2) within those limits.

It is possible to show that the *integral of the faltung of two functions is the product of the integrals of the individual functions.* We shall call this the Faltung Integral Theorem. If we restrict ourselves to functions which occur in physical problems, the only conditions required for its validity appear to be that the integrals of $g(x)$ and $h(x)$ should be finite.

Let

$$G = \int_{-\infty}^{+\infty} g(x)\, dx \quad \text{and} \quad H = \int_{-\infty}^{+\infty} h(x)\, dx.$$

Then,

$$F = \int_{-\infty}^{+\infty} f(x)\, dx = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} g(x')\, h(x - x')\, dx'$$

$$= \int_{-\infty}^{+\infty} g(x')\, dx' \int_{-\infty}^{+\infty} h(x - x')\, d(x - x') = GH. \tag{A 3}$$

* To avoid confusion with the use of the star for the complex conjugate and also so as to be in accordance with the notation of the correlation function below, the symbol gFh is used in preference to $g * h$ for the faltung of the functions g and h.

The condition for this to be valid is that the change in the order of integration in (A 7) should be justified which is ordinarily valid for physical functions.

Although the result (A 3) is a consequence of a well-known theorem on the Fourier transform of the convolution of two functions [*see e.g.*, Sneddon (1951)], it does not seem to have been specifically noted and applied so far. It appears to have applications in various physical problems.

If g and h are functions of a number of variables x_j (which may be collectively represented by a vector $\boldsymbol{r}$), then the following results hold. Define the faltung by

$$f(\boldsymbol{r}) = g\mathrm{F}h(\boldsymbol{r}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g(\boldsymbol{r}')\, h(\boldsymbol{r} - \boldsymbol{r}')\, d\boldsymbol{r}'. \tag{A 4}$$

Then, we have the general faltung integral theorem

$$\mathrm{F} = \mathrm{GH} \tag{A 5}$$

where

$$\mathrm{F} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g(\boldsymbol{r})\, d\boldsymbol{r} \tag{A 6}$$

and similar integrals define G and H.

2. THEOREMS ON CORRELATION FUNCTIONS

It will be noticed that the integrals (13) and (19) in the body of the paper are very similar to that in the definition of the faltung function, but are different from the latter. In fact, integrals of the type met with in this paper also occur in other physical problems and it would be worthwhile to define a function, which may be called the 'correlation function'.

The "correlation function of $g(x)$ with $h(x)$" denoted by $g\mathrm{C}h(x)$ is defined by

$$g\mathrm{C}h(x) = \int_{-\infty}^{+\infty} g(x')\, h(x + x')\, dx' \tag{A 7}$$

Clearly Equation (A 7) is also equivalent to the form

$$g\mathrm{C}h(x) = \int_{-\infty}^{+\infty} g(x'' - x)\, h(x'')\, dx'' \tag{A 8}$$

However, $gCh(x)$ is not necessarily equal to $hCg(x)$, but

$$gCh(x) = hCg(-x) \tag{A 9}$$

The term " correlation " for the function $gCh(x)$ is justified by the fact that the value of this function at x is obtained by taking the product of g at x' and of h at $x + x'$, *i.e.*, obtained by an additional shift x, and integrating over x'.

The correlation function of $g(x)$ with $h(x)$ can be shown to be equal to the faltung of $g(-x)$ and $h(x)$. Hence, from the faltung integral theorem

$$\int_{-\infty}^{+\infty} gCh(x)\, dx = \int_{-\infty}^{+\infty} g(-x)\, dx \cdot \int_{-\infty}^{+\infty} h(x)\, dx = \text{GH}. \tag{A 10}$$

Thus, the integral of the correlation function of g with h is also equal to the product of the two integrals of the functions g and h separately. We shall call this the *Correlation Integral Theorem*. It will be noticed that the integrals which occur in the body of the paper are all of the type defining a ' correlation function '. Since these functions vanish outside a finite range of the argument, the limits in (A 10) can be set finite in these cases. The extension of theorem to a number of independent variables x_i is obvious.

Only the results required for the present paper have been derived in this Appendix. It is proposed to discuss the properties of the faltung function and the correlation function in a separate paper with examples of their applications in physical problems.