VECTOR BUNDLES WITH HOLOMORPHIC CONNECTION
OVER A PROJECTIVE MANIFOLD WITH TANGENT BUNDLE
OF NONNEGATIVE DEGREE

INDRANIL BISWAS

(Communicated by Ron Donagi)

Abstract. For a projective manifold whose tangent bundle is of nonnegative
degree, a vector bundle on it with a holomorphic connection actually admits a
compatible flat holomorphic connection, if the manifold satisfies certain con-
ditions. The conditions in question are on the Harder-Narasimhan filtration
of the tangent bundle, and on the Neron-Severi group.

1. Introduction

Let $E$ be a holomorphic vector bundle over a connected complex projective
manifold $M$. Assume that $E$ admits a holomorphic connection. Then a natural
question to ask is whether $E$ admits a flat holomorphic connection. Since all the
rational Chern classes (of degree at least one) of a holomorphic vector bundle with a
holomorphic connection vanish, there is no topological obstruction for the existence
of a flat connection.

In this paper we consider this question for $M$ satisfying the condition that the
degree of the tangent bundle $T_M$ is nonnegative with respect to some polarization
on $M$.

Let

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_l \subset V_{l+1} = T_M$$

be the Harder-Narasimhan filtration of the tangent bundle $T_M$ with respect to a
polarization $L$ on $M$.

In Theorem 2.4 we prove the following (degree of a coherent sheaf on $M$ is
computed using $L$):

Theorem A. Assume that the degree of the tangent bundle $\deg T_M \geq 0$. Let $E$
be a holomorphic vector bundle on $M$ equipped with a holomorphic connection.

(1) If $\deg (T_M / V_i) \geq 0$ then the holomorphic vector bundle $E$ admits a compatible
flat connection. (This inequality condition is satisfied if, for example, $T_M$ is
semistable, since $\deg T_M \geq 0$.)
(2) Consider the case where $T_M$ is not semistable. Assume that the maximal semistable subsheaf of $T_M$, namely $V_1$, is locally free. If the rank of the Neron-Severi group, $NS(M)$, of $M$ is 1, i.e.,

$$H^{1,1}(M) \cap H^2(M, \mathbb{Q}) = \mathbb{Q},$$

then $E$ admits a compatible flat connection.

Under the assumptions either in part (1) or in part (2) of Theorem A, the vector bundle $E$ turns out to be semistable with respect to $L$ [Remark 2.12].

Generalizing the above question one may ask whether a holomorphic fiber bundle admitting a holomorphic connection actually admits a flat holomorphic connection. S. Murakami produced an example of a holomorphic fiber bundle over an abelian variety, with an abelian variety as fiber, such that the fiber bundle admits a holomorphic connection, but it does not admit any flat holomorphic connection [M1], [M2], [M3]. However part (1) of Theorem A implies that any holomorphic vector bundle over a projective manifold with trivial canonical line bundle, which admits a holomorphic connection, actually admits a flat holomorphic connection. Indeed, by a theorem of Yau [Ya] the tangent bundle of such a variety is semistable.

On the other hand, using a method of [Bi2], Theorem A can easily be generalized to principal $G$-bundles, where the structure group $G$ is a connected affine algebraic reductive group over $\mathbb{C}$. The example of Murakami shows that it is essential for $G$ to be noncompact.

2. Criteria for the existence of a flat connection

Let $M/\mathbb{C}$ be a connected smooth projective variety of complex dimension $d$. We will denote by $T_M$ (resp. $\Omega^1_M$) the holomorphic tangent bundle (resp. cotangent bundle) of $M$.

For a holomorphic vector bundle $V$, the corresponding coherent analytic sheaf given by its local holomorphic sections will also be denoted by $V$. The basic facts about holomorphic structures used here can be found in [Ko].

A holomorphic connection on a holomorphic vector bundle $E$ over $M$ is a first order differential operator

$$(2.1) \quad D : E \rightarrow \Omega^1_M \otimes E$$

satisfying the following Leibniz condition:

$$(2.2) \quad D(fs) = fD(s) + df \otimes s$$

where $f$ is a local holomorphic function on $M$ and $s$ is a local holomorphic section of $E$. Extend $D$ as a first order operator

$$D : \Omega^{p,q}_M \otimes E \rightarrow \Omega^{p+1,q}_M \otimes E$$

using the Leibniz rule. The curvature of $D$ is defined to be

$$D^2 := D \circ D$$

which is a holomorphic section of $\Omega^2_M \otimes \text{End } E$ ($\Omega^k_M := \bigwedge^k \Omega^1_M$). The notion of a holomorphic connection was introduced by M. Atiyah [At].

If $\overline{\partial}_E : E \rightarrow \Omega^1_M \otimes E$ denotes the first order differential operator defining the holomorphic structure on $E$, then the operator

$$D + \overline{\partial}_E$$

...
is a connection on $E$ in the usual sense. Moreover, the curvature of this connection is $D^2$; in particular, it is a holomorphic section of $\Omega^2_M \otimes \text{End} \ E$. Conversely, the $(1,0)$ part of a connection on $E$, such that the $(0,1)$ part of it is $\overline{\partial}_E$ and its curvature is a holomorphic section of $\Omega^2_M \otimes \text{End} \ E$, is actually a holomorphic connection.

In particular, if $\nabla$ is a flat connection on a $C^\infty$ complex vector bundle $M$, then the $(0,1)$ part of the connection operator defines a holomorphic structure on $E$ and the $(1,0)$ part defines a holomorphic connection.

Let $L$ be a polarization on $M$, or equivalently, $L$ is an ample line bundle on $M$. For a coherent sheaf $F$ on $M$, the degree of $F$, denoted by $\deg F$, is defined as follows ($d = \dim_C M$):

$$\deg F := \int_M c_1(F) \cup c_1(L)^{d-1}.$$ 

A torsion-free coherent sheaf $F$ is called semistable if for every (nonzero) coherent subsheaf $V \subset F$, the following inequality holds:

$$\frac{\text{rank} V}{\deg V} \leq \frac{\text{rank} F}{\deg F}.$$ 

Moreover, if the strict inequality holds for every proper coherent subsheaf $V$ with $F/V$ torsion-free, then $F$ is called stable.

The quotient $\text{rank} F/\deg F$ is called the slope of $F$ and is usually denoted by $\mu(F)$.

Any torsion-free coherent sheaf $F$ admits a unique filtration by coherent subsheaves, known as the Harder-Narasimhan filtration, of the following type ([Ko], page 174, Theorem 7.15):

$$0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_k \subset F_{k+1} = F$$

where $F_1$ is the maximal semistable subsheaf of $F$. The Harder-Narasimhan filtration is determined by the property that $F_{i+1}/F_i$ is the maximal semistable subsheaf of $F/F_i$. This implies that $\mu(F_{i+1}/F_i) < \mu(F_i/F_{i-1})$.

Let

(2.3) $$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_l \subset V_{l+1} = T_M$$

be the Harder-Narasimhan filtration of the tangent bundle $T_M$.

A flat connection on a holomorphic vector bundle $E$ on $M$ is said to be compatible if the $(0,1)$ part of the connection is $\overline{\partial}_E$ (equivalently, (local) flat sections are holomorphic sections). A compatible flat connection is same as a flat holomorphic connection.

**Theorem 2.4.** Assume that the degree of the tangent bundle $\deg T_M \geq 0$. Let $E$ be a holomorphic vector bundle on $M$ equipped with a holomorphic connection.

1. If $\deg(T_M/V_i) \geq 0$ then the holomorphic vector bundle $E$ admits a compatible flat connection. (This inequality condition is satisfied if, for example, $T_M$ is semistable since $\deg T_M \geq 0$.)

2. Consider the case where $T_M$ is not semistable. Assume that the maximal semistable subsheaf of $T_M$, namely $V_1$, is locally free. If the rank of the Neron-Severi group, $NS(M)$, of $M$ is 1, i.e.,

$$H^{1,1}(M) \cap H^2(M, \mathbb{Q}) = \mathbb{Q},$$

then $E$ admits a compatible flat connection.
Proof. Assume that \( \deg (T_M/V_l) \geq 0 \). Then from Lemma 2.1 of [Bi2] (also Remark 3.7(ii) of [Bi1]) we know that the vector bundle \( E \) is semistable. To be self-contained as much as possible we will quickly recall the proof of the semistability of \( E \). Since \( E \) admits a holomorphic connection, Theorem 4 (page 192) of [At] says that all the (rational) Chern classes, \( c_k(E) \), where \( k \geq 1 \), of \( E \) vanish. In particular \( \deg E = 0 \).

Let \( W \) be the maximal semistable subsheaf of \( E \). The key observation is that \( W \) is left invariant by the holomorphic connection operator \( D \) on \( E \). Indeed, the homomorphism \( W \rightarrow \Omega^1_M \otimes \frac{E}{W} \) induced by \( D \) is \( \mathcal{O}_M \)-linear (a simple consequence of the Leibniz identity (2.2)). The Harder-Narasimhan filtration of a tensor product is simply the tensor product of the corresponding Harder-Narasimhan filtrations. Applying this to \( \Omega^1_M \otimes (E/W) \), since the degree of any subsheaf of \( \Omega^1_M \) is nonpositive (this is equivalent to the assertion that the degree of a quotient sheaf of \( T_M \) is nonnegative, which, in turn, is warranted by the assumption that \( \deg (T_M/V_l) \geq 0 \)), the slope of the maximal semistable subsheaf of \( \Omega^1_M \otimes (E/W) \) is strictly less than \( \mu(W) \). Thus the homomorphism in (2.5) must be the zero homomorphism. In other words, \( W \) has an induced holomorphic connection. This implies that \( W \) is locally free of degree zero. So \( W \) cannot be a proper subsheaf of \( E \). In other words, \( E \) must be semistable.

Since \( E \) is semistable with vanishing first and second Chern classes, the Corollary 3.10 (page 40) of [Si] implies that \( E \) admits a flat connection compatible with its holomorphic structure.

To prove part (2) of Theorem 2.4 we assume that \( T_M \) is not semistable. The maximal semistable subsheaf of \( T_M \), namely \( V_1 \) (in (2.3)), is assumed to be locally free.

Our first step will be to prove that \( V_1 \) is closed under the Lie bracket operation on \( T_M \). Towards this goal consider the homomorphism

\[
\Gamma : V_1 \otimes V_1 \rightarrow \frac{T}{V_1}
\]

defined by composing the Lie bracket operation with the natural projection of \( T_M \) onto \( T_M/V_1 \). Since the Lie bracket satisfies the Leibniz identity, namely

\[
[fv, w] = f[v, w] - (dw, v),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the obvious contraction, the map \( \Gamma \) is actually \( \mathcal{O}_M \)-linear, i.e., \( \Gamma \) is a homomorphism of vector bundles.

Now we are given that \( \mu(V_1) > \mu(T_M) \geq 0 \). So

\[
\mu(V_1 \otimes V_1) = 2\mu(V_1) > \mu(V_1) > \mu(V_2/V_1),
\]

the last inequality being a general property of Harder-Narasimhan filtrations. The image of the homomorphism \( \Gamma \) is simultaneously a quotient of \( V_1 \otimes V_1 \) as well as a
subsheaf of $T_M/V_1$. But $V_2/V_1$, by definition, is the maximal semistable subsheaf of $T_M/V_1$. So if $\Gamma \neq 0$ then
\[ \mu(V_1 \otimes V_1) \leq \mu(\text{image } \Gamma) \leq \mu(V_2/V_1). \]
The first inequality is a consequence of the fact that $V_1 \otimes V_1$ is semistable. (A tensor product of semistable vector bundles is again semistable [MR], Remark 6.6 (iii).) This contradicts the inequality (2.7) unless image $\Gamma = 0$. But $\Gamma = 0$ is equivalent to $V_1$ being closed under the Lie bracket operation. In other words, $V_1$ is a nonsingular holomorphic foliation on $M$.

If $E$ is semistable we may complete the proof of Theorem 2.4 by repeating the use of the Corollary 3.10 of [Si] as done in the proof of part (1) of Theorem 2.4. So we may, and we will, assume that $E$ is not semistable. Let
\[ 0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_m \subset W_{m+1} = E \]
be the Harder-Narasimhan filtration of $E$.

Our next step will be to show that the sheaf $W_1$ has an induced holomorphic partial connection along the foliation $V_1$. In other words, we want to show that the operator $D$ in (2.1) induces an operator
\[ (2.8) \quad D' : W_1 \rightarrow V_1^* \otimes W_1 \]
which satisfies the Leibniz condition (2.2); $df$ in (2.2) is realized as a section of $V_1^*$ in (2.8) by using the natural projection of $\Omega^1_M$ onto $V_1^*$. The notion of a partial connection was introduced by R. Bott.

To construct $D'$ first note that, by projecting $\Omega^1_M$ onto $V_1^*$, the operator $D$ in (2.1) induces an operator
\[ (2.9) \quad D_1 : W_1 \rightarrow V_1^* \otimes E. \]
Now projecting $E$ onto $E/W_1$, the operator $D_1$ in (2.9) induces an operator
\[ D_2 : W_1 \rightarrow V_1^* \otimes \frac{E}{W_1}. \]
The Leibniz identity (2.2) implies that $D_2$ is $\mathcal{O}_M$-linear; i.e., the order of the differential operator $D_2$ is zero. In other words, $D_2$ is a homomorphism of vector bundles.

We will show that $D_2 = 0$ by following the steps of the argument for $\Gamma = 0$ (in (2.6)).

If $D_2 \neq 0$ then $\mu(\text{image } (D_2)) \geq \mu(W_1)$, since image $(D_2)$ is a quotient of the semistable sheaf $W_1$. On the other hand, since
\[ \text{image } (D_2) \subseteq V_1^* \otimes \frac{E}{W_1}, \]
we conclude that the slope of image $(D_2)$ is at most the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$.

Thus if $D_2 \neq 0$, then $\mu(W_1)$ is less than or equal to the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$.

On the other hand, since $V_1^*$ is semistable with strictly negative slope, the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$ is strictly less than the slope of the maximal semistable subsheaf of $E/W_1$ – which in turn is strictly less than the slope of $W_1$. Thus the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$ is strictly less than $\mu(W_1)$. This contradicts the inequality obtained in the previous paragraph. So we have $D_2 = 0$. 


Since $D_2 = 0$, the differential operator $D_1$ in (2.9) induces a first order differential operator $D'$ as in (2.8). Clearly $D'$ satisfies the Leibniz identity, as $D$ satisfies it.

The operator $D'$ maps (local) holomorphic sections of $W_1$ to holomorphic sections of $V^*_1 \otimes W_1$. So $D'$ is a partial connection on $W_1$ along $V_1 \oplus T_{M,1}$ in the sense of [BB] (Sections 2 and 3); $T_{M,1}$ is the anti-holomorphic tangent bundle.

However, unfortunately, $W_1$ is not necessarily locally free. (A coherent sheaf equipped with a holomorphic connection must be locally free, but $D'$ is only a partial connection.) To circumvent the problems caused by such a possibility of not being locally free, we will consider the determinant line bundle $d(W_1) := \det W_1 = \bigwedge^r W_1$ where $r$ is the rank of $W_1$. The details of the construction of the determinant bundle of a torsion-free coherent sheaf can be found in Chapter 5, §6 of [Ko]. We note that the determinant bundle of a torsion-free sheaf is locally free of rank one, i.e., it is a line bundle.

The partial connection $D'$ induces a partial connection on $d(W_1)$, which we will also denote by $D'$. More precisely, for a local section of $d(W_1)$

$$s := s_1 \wedge s_2 \wedge \ldots \wedge s_r \in \Gamma(U, d(W_1))$$

the action of $D'$ on it is defined as follows:

$$D'(s) := \sum_{j=1}^r s_1 \wedge \ldots \wedge D'(s_j) \wedge \ldots \wedge s_r.$$ 

It is straight-forward to check that the operator $D'$ defined above satisfies the Leibniz identity. Thus $D'$ is a partial holomorphic connection on $d(W_1)$ along $V_1$.

We may extend the partial connection $D'$ to an actual connection on $d(W_1)$ following [BB]. Fix a Kähler metric, say $H$, on $M$. Let $\nabla'$ be a hermitian connection on $d(W_1)$; the $(0, 1)$ part of $\nabla'$ is assumed to be $\overline{\partial}_{d(W_1)}$. For any $v \in T_{M,1}^0$ let $v = v_1 \oplus v_2$ be the decomposition as $T_{M,1}^0 = V_1 \oplus V_1^\perp$ using the metric $H$. For $v' \in T_{M,1}^{0,1}$ and a smooth section $\phi$ of $d(W_1)$ define:

$$\nabla_{v \oplus v'} \phi := \langle D' \phi, v_1 \rangle + \nabla_{v_2} \phi + \langle \overline{\partial}_{d(W_1)} \phi, v' \rangle.$$ 

Clearly $\nabla$ is a connection in the usual sense whose $(0, 1)$ part coincides with $\overline{\partial}_{d(W_1)}$, and it is an extension of the partial connection $D'$.

Let

$$\mathcal{I} \subseteq \Omega_{M,1}^{1,1} \oplus \Omega_{M,1}^{0,2}$$

be the degree 2 component of the ideal, in the exterior algebra $\bigwedge (\Omega_{M,1}^{1,0} \oplus \Omega_{M,1}^{0,1})$, generated by the subspace of $\Omega_{M,1}^{0,1}$ that annihilates $\nabla_1$.

The following simple lemma will be useful:

**Lemma 2.10.** The curvature $\nabla^2$, which is a smooth 2-form on $M$, is actually a section of $\mathcal{I} \oplus \Omega_{M,1}^{2,0}$.

The proof of Lemma 2.10 is a simple computation. It is actually a straightforward extension of (3.33), page 295 of [BB] to partial holomorphic connections (extension from partial flat connections). All we need to observe is that the cur-
vature of $\nabla'$ is of type (1, 1) (since $\nabla'$ is assumed to be hermitian) and that the curvature of the partial connection $D'$ is a holomorphic section of $\bigwedge^2 V_1^*$. Since the restriction of $\nabla$ to a leaf of the foliation $V_1$ coincides with $D' + \partial d(W_1)$, the restriction of $\nabla^2$ to a leaf is a section of $\bigwedge^2 V_1^*$. It is easy to see that this implies Lemma 2.10.

Continuing with the proof of Theorem 2.4, our next step will be to establish a lemma on vanishing of characteristic classes of $d(W_1)$, analogous to the Proposition (3.27), page 295, of [BB].

**Lemma 2.11.** Let $q$ be an integer with $q > \dim M - \dim V_1$. Then $c_1(d(W_1))^q = 0$.

**Proof of Lemma 2.11.** The characteristic class $c_1(d(W_1))^q \in H^{q,q}(M)$, and it is represented by the differential form $(\nabla^2/2\pi\sqrt{-1})^q$. Since the space of forms on $M$ admits Hodge decomposition, to prove Lemma 2.11 it is enough to show that the differential form $(\nabla^2)^q$ is a section of the vector bundle $$\bigoplus_{j>q} \Omega^{2q-j} M.$$ But Lemma 2.10 implies that $(\nabla^2)^q$ is indeed of the above type. To see this first note that by Lemma 2.10, both the (1, 1) and the (0, 2) part of $\nabla^2$ is contained in the ideal generated by the subspace of $\Omega^{0,1}_M$ that annihilates $V_1$. But the dimension of this annihilator is $\dim M - \dim V_1$. So the component of $(\nabla^2)^q$ in $$\bigoplus_{j\leq q} \Omega^{2q-j} M$$ vanishes identically. This completes the proof of the lemma.

To complete the proof of Theorem 2.4 we first note that the given condition that the rank of the Neron-Severi group, $NS(M)$, is 1 implies that if $(\omega)^{j} = 0$, where $\omega \in NS(M) \otimes \mathbb{Q}$ ($= H^2(M, \mathbb{Q}) \cap H^{1,1}(M)$) and $1 \leq j \leq \dim_{\mathbb{C}} M$, then $\omega = 0$. This is simply because $\omega$ is a (possibly zero) rational multiple of the hyperplane class, and the $j$-th power of the hyperplane class is nonzero. Substituting $c_1(d(W_1))$ for $\omega$ and using Lemma 2.11 we get that $c_1(d(W_1)) = 0$. Thus we have $$\deg W_1 = \deg d(W_1) = 0.$$ But $W_1$ is the maximal semistable subsheaf of $E$ and $\deg E = 0$. This contradicts the assumptions that $E$ is not semistable and that $W_1$ is the maximal semistable subsheaf of $E$. We already noted that if $E$ is semistable then the Corollary 3.10 (page 40) of [Si] completes the proof of the theorem. This completes the proof of Theorem 2.4.

**Remark 2.12.** The proof of Theorem 2.4 shows that under the assumptions in either part 1 or part 2 of the statement of Theorem 2.4, the vector bundle $E$ is actually semistable.

**Acknowledgments**

The author is thankful to C. Simpson and T. Pantev for explaining some results of [Si] used here.
REFERENCES


SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, BOMBAY 400005, INDIA

E-mail address: indranil@math.tifr.res.in