YANG–MILLS EQUATION FOR STABLE HIGGS SHEAVES

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ABSTRACT. We establish a Kobayashi-Hitchin correspondence for the stable Higgs sheaves on a compact Kähler manifold. Using it, we also obtain a Kobayashi-Hitchin correspondence for the stable Higgs G–sheaves, where G is any complex reductive linear algebraic group.

1. Introduction

The concept of Hermite–Einstein equations for stable sheaves was introduced by Bando and Siu in [BS]. It depends upon the notion of a certain class of hermitian metrics on reflexive sheaves, called *admissible*, for which the curvature is square integrable satisfying a pointwise boundedness condition. It also depends upon the extension of a solution of the corresponding Yang–Mills (= Hermite–Einstein) equation to the open subset where the sheaf is locally free. The approach is based on solving a heat equation.

The notion of a *Higgs bundle* is due to Hitchin and Simpson. They generalized the definition of stability to Higgs bundles, and also generalized the Yang–Mills equation to the Higgs bundles. In [Hit, Si] they established for Higgs bundles what is called the Kobayashi–Hitchin correspondence.

Our aim here is to combine both concepts to get a Kobayashi–Hitchin correspondence for Higgs sheaves. This is worked out in Theorem 3.1.

Also, stable principal Higgs G–sheaves are introduced, where G is any complex reductive linear algebraic group. It follows from our main theorem that tensor products of polystable Higgs sheaves are again polystable. Once shown this fact, a Kobayashi–Hitchin correspondence for stable Higgs G–sheaves follows.

2. Higgs sheaves and admissible metrics

Let X be a compact connected Kähler manifold equipped with a Kähler form ω. The adjoint of multiplication of differential forms by ω will be denoted by Λ_ω. We will use the summation convention throughout.

Definition 2.1. A *Higgs sheaf* on X consists of a torsionfree sheaf $\mathcal{E}$ on X together with a holomorphic section $\varphi = \varphi_\alpha dz^\alpha$ of $\Omega^1_X(End(\mathcal{E}))$ such that the form $\varphi \wedge \varphi = [\varphi_a, \varphi_\gamma] dz^\alpha \wedge dz^\gamma$, which is a holomorphic section of $\Omega^2_X(End(\mathcal{E}))$, vanishes identically.

Let $\mathcal{E}$ be a torsionfree sheaf on (X, ω). Let $S \subset X$ be the locus where $\mathcal{E}$ is not locally free. So S is a complex analytic subset with $\mathrm{codim}_X S \geq 2$. Following [BS] we call

2000 *Mathematics Subject Classification.* 53C07, 32L05.
Key words and phrases. Higgs sheaf, Kobayashi-Hitchin correspondence, polystability.

a hermitian metric h on the holomorphic vector bundle $\mathcal{E}|X\backslash S$ to be *admissible* if the following two hold:

$\mathbf{A_1}$: the curvature tensor F of h is square integrable, and
$\mathbf{A_2}$: $\Lambda_\omega F$ is bounded.

For a torsionfree sheaf $\mathcal{E}$, there is a natural embedding of $\mathcal{E}$ in its double dual $\mathcal{E}^{\vee\vee}$. It is easy to see that any admissible hermitian metric on $\mathcal{E}^{\vee\vee}$ restricts to an admissible hermitian metric on $\mathcal{E}$.

Let n be the complex dimension of X. The degree of a torsionfree sheaf $\mathcal{E}$, which is same as the degree of the determinant line bundle $\det \mathcal{E}$ [Kob, Ch. V, §6], is defined in terms of cohomology classes:

$$\deg \mathcal{E} = (c_1(\mathcal{E}) \cup [\omega]^{n-1}) \cap [X] \in \mathbb{R}\,,$$

where $[\omega] \in H^2(X, \mathbb{R})$ is the cohomology class represented by ω. Here, and below, we denote by ω^k the k–the exterior power of ω divided by $k!$.

Let $(\mathcal{E}, \varphi)$ be a Higgs sheaf. A coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ is called a *Higgs subsheaf*, if $\varphi(\mathcal{F}) \subset \Omega^1_X \otimes \mathcal{F}$.

Definition 2.2. A Higgs sheaf $(\mathcal{E}, \varphi)$ over X is called *stable* (respectively, *semistable*), if for any Higgs subsheaf $\mathcal{F}$, with $0 < \mathrm{rk}\mathcal{F} < \mathrm{rk}\mathcal{E}$, the inequality

$$\frac{\deg \mathcal{F}}{\mathrm{rk}\mathcal{F}} < \frac{\deg \mathcal{E}}{\mathrm{rk}\mathcal{E}}$$

(respectively, $\frac{\deg \mathcal{F}}{\mathrm{rk}\mathcal{F}} \leq \frac{\deg \mathcal{E}}{\mathrm{rk}\mathcal{E}}$) holds. A Higgs sheaf $(\mathcal{E}, \varphi)$ is called *polystable*, if it decomposes into a direct sum of stable Higgs subsheaves $(\mathcal{E}, \varphi) = \oplus_i (\mathcal{F}_i, \varphi_i)$ with

$$\frac{\deg \mathcal{F}_i}{\mathrm{rk}\mathcal{F}_i} = \frac{\deg \mathcal{E}}{\mathrm{rk}\mathcal{E}}$$

for all i.

Lemma 2.3. *The degree of a torsionfree sheaf $\mathcal{E}$ on X can be evaluated from the curvature of an admissible hermitian metric h on $\mathcal{E}$. Namely,*

$$\deg \mathcal{E} = \int_X c_1(\mathcal{E}, h) \wedge \omega^{n-1}\,. \tag{2.1}$$

Proof. Let η denote the (singular) hermitian metric on the determinant line bundle $\det \mathcal{E}$ which is induced by the given admissible hermitian metric h on $\mathcal{E}$. Now let η_0 be a background metric on $\det \mathcal{E}$ of class C^∞, and set $\eta = e^\chi \cdot \eta_0$, where χ is a real valued function on the open subset $X\backslash S$ where $\mathcal{E}$ is locally free. Since the hermitian metric on $\mathcal{E}$ is admissible, we know that χ is of class C^∞ on $X\backslash S$, and $\Box\chi$ is bounded. (Note that the condition that $\Lambda_\omega F$ is bounded implies that $\Box\chi$ is bounded.)

The degree of $\mathcal{E}$ with respect to ω can be computed from η_0. Therefore, to prove the lemma it suffices to show that

$$\int_X \Box\chi \cdot \omega^n = 0\,, \tag{2.2}$$

where χ is of class C^∞ on $X\backslash S$ with $\Box\chi$ bounded. Since $f := \Box\chi$ is in L_2, we can apply the global Green's operator to the function $f_0 = f - \alpha$, with $\alpha := (\int_X f\omega^n / \int_X \omega^n)$, and obtain a solution $\rho \in \mathbf{H}^2$ of $\Box\rho = f_0$ (cf. [Kod, Section 7]). By elliptic regularity, the function

$\rho|X\backslash S$ on $X\backslash S$ is of class C^∞. Let $\{U_i\}$ be an open covering of X together with a set of C^∞ functions μ_{U_i} on U_i satisfying $\Box\mu_{U_i} = \alpha$ on U_i. Then $(\chi - \rho - \mu_{U_i})|U_i\backslash S$ is harmonic. We note that the function $(\chi - \rho - \mu_{U_i})|U_i\backslash S$ can be extended as a harmonic function across the set $S \cap U_i$. Indeed, this follows from the fact that the complex codimension of the analytic subset $S \cap U_i \subset U_i$ is at least two. In particular, $\chi \in \mathbf{H}^2$, and (2.2) follows. $\Box$

Let $(\mathcal{E}, \varphi)$ be a Higgs sheaf. Then φ extends to a Higgs field on the double dual $\mathcal{E}^{\vee\vee}$. This Higgs field on $\mathcal{E}^{\vee\vee}$ will be denoted by $\varphi^{\vee\vee}$.

Lemma 2.4. *A torsionfree Higgs sheaf $(\mathcal{E}, \varphi)$ on (X, ω) is stable (respectively, semistable), if and only if $(\mathcal{E}^{\vee\vee}, \varphi^{\vee\vee})$ is stable (respectively, semistable). Similarly, $(\mathcal{E}, \varphi)$ is polystable if and only if $(\mathcal{E}^{\vee\vee}, \varphi^{\vee\vee})$ is so.*

Proof. We consider the embedding $\mathcal{E} \hookrightarrow \mathcal{E}^{\vee\vee}$, and note that the degrees of $\mathcal{E}$ and $\mathcal{E}^{\vee\vee}$ coincide. If $\mathcal{F}$ denotes a Higgs subsheaf of $(\mathcal{E}, \varphi)$, then its image in $\mathcal{E}^{\vee\vee}$ is a Higgs subsheaf of same rank and degree as those of $\mathcal{F}$. So the stability of $(\mathcal{E}^{\vee\vee}, \varphi^{\vee\vee})$ implies the stability of $(\mathcal{E}, \varphi)$. Conversely, for any Higgs subsheaf $\mathcal{G} \subset \mathcal{E}^{\vee\vee}$, the intersection $\mathcal{G} \cap \mathcal{E}$ is a Higgs subsheaf of $\mathcal{E}$ same rank and degree. This proves the converse. A similar proof works in the semistable and polystable cases. $\Box$

We will define a hermitian Yang–Mills–Higgs metric on a Higgs sheaf.

Definition 2.5. Let $(\mathcal{E}, \varphi)$ be a Higgs sheaf on (X, ω). An admissible hermitian metric h on $\mathcal{E}$ is called a *hermitian Yang–Mills–Higgs metric* if the hermitian Yang–Mills–Higgs equation

$$\Lambda_\omega(F + [\varphi, \varphi^*]) = \lambda \cdot \mathrm{Id}_{\mathcal{E}} \tag{2.3}$$

on $X\backslash S$ is satisfied for some constant $\lambda \in \mathbb{C}$; here S, as before, is the subset of X where $\mathcal{E}$ fails to be locally free.

We make the following observation: Let $U \subset \mathbb{C}^n$ be an open subset, and let $\mathcal{E}_U \subset \mathcal{O}_U^\ell$ be a coherent subsheaf. Then the restriction of any hermitian metric on $\mathcal{O}_U^\ell$ to $\mathcal{E}_U$ is an admissible metric over relatively compact subsets of U.

Let $\mathcal{E} \in Coh(X)$ be torsionfree. There exists an open covering $\{U_\alpha\}$ of X together with presentations

$$\mathcal{O}_{U_\alpha}^{k_\alpha} \xrightarrow{\varphi_\alpha^\vee} \mathcal{O}_{U_\alpha}^{\ell_\alpha} \longrightarrow \mathcal{E}^\vee|U_\alpha \longrightarrow 0\,. \tag{2.4}$$

According to Hironaka's flattening theorem [Hir], there exists a finite sequence of blow–ups with smooth centers $\pi_i : X_i \to X_{i-1}$, where $i = 1, \ldots, \ell$ and $X_0 = X$, such that the pull–back of $\mathcal{E}^\vee$ to X_ℓ modulo torsion is locally free. Set $\widetilde{X} := X_\ell$, and let

$$\pi := \pi_1 \circ \ldots \circ \pi_\ell : \widetilde{X} \to X$$

be the projection. So $(\pi^*(\mathcal{E}^\vee))/\text{torsion}$ is a holomorphic vector bundle over $\widetilde{X}$.

Set $\widetilde{U}_\alpha := \pi^{-1}(U_\alpha)$. We note that for a resolution of type (2.4), the equality

$$\mathrm{coker}(\pi^*\varphi_\alpha^\vee) = (\pi^*(\mathcal{E}^\vee)/\text{torsion})|\widetilde{U}_\alpha$$

holds independent of the choice of a local resolution $\pi|\widetilde{U}_\alpha : \widetilde{U}_\alpha \to U_\alpha$ (cf. [RG, Section 5] in particular [RG, 5.4]).

Assume that $\mathcal{E}$ is reflexive, that is, $\mathcal{E} = \mathcal{E}^{\vee\vee}$. We have exact sequences

$$0 \longrightarrow \mathcal{E}|\widetilde{U}_\alpha \longrightarrow \mathcal{O}^{\ell_\alpha}_{\widetilde{U}_\alpha} \xrightarrow{\varphi_\alpha} \mathcal{O}^{k_\alpha}_{\widetilde{U}_\alpha}\, .$$

Setting $\widetilde{\mathcal{E}} := (\pi^*(\mathcal{E}^\vee)/\text{torsion})^\vee$ we get that

$$\widetilde{\mathcal{E}}|\widetilde{U}_\alpha = \ker(\pi^*\varphi_\alpha) = (\pi^*\mathcal{E}/\text{torsion})|\widetilde{U}_\alpha$$

is actually a *sub–vector bundle* of $\mathcal{O}^{\ell_\alpha}_{\widetilde{U}_\alpha}$, where $\widetilde{U}_\alpha = \pi^{-1}(U_\alpha)$. Observe that $\mathcal{E}|U_\alpha \hookrightarrow \mathcal{O}^{\ell_\alpha}_{U_\alpha}$ is a subbundle wherever it is locally free.

Let h_α be a hermitian metric on $\mathcal{O}^{\ell_\alpha}_{U_\alpha}$. Let $\widetilde{h}_\alpha$ denote the restriction of the hermitian metric π^*h_α to the subbundle $\widetilde{\mathcal{E}}|U_\alpha$. Now $\widetilde{h}_\alpha$ clearly is a hermitian metric (with no degeneracies) which descends to a hermitian metric on $\mathcal{E}|U_\alpha$ over U_α wherever $\mathcal{E}$ is locally free. Taking a partition of unity subordinate to $\{U_\alpha\}$, we get a hermitian metric $\widetilde{h}$ on $\widetilde{\mathcal{E}}$, which descends to a hermitian metric on $\mathcal{E}$ over $X \setminus S$ (the subset where $\mathcal{E}$ is locally free). Let h denote the hermitian metric on $\mathcal{E}|X \setminus S$ obtained this way from $\widetilde{h}$.

The complex manifold $\widetilde{X}$ is Kähler. We denote by η a Kähler form on $\widetilde{X}$, and we set

$$\omega_\epsilon := \pi^*\omega + \epsilon\eta$$

for $0 \le \epsilon \le 1$. We will later need the following argument which shows that h is admissible with respect to ω on X:

Let F be the curvature tensor of h (over $X\backslash S$). The pull–back of F to $\widetilde{X} \setminus \pi^{-1}(S)$ extends to $\widetilde{X}$ as the curvature tensor of the hermitian metric $\widetilde{h}$ on the vector bundle $\widetilde{\mathcal{E}}$. We use the same notation F for the pull–back.

Now

$$\Lambda_\epsilon F \omega_\epsilon^n = F \wedge \omega_\epsilon^{n-1}\, . \tag{2.5}$$

The right–hand side is a bounded $End(\widetilde{\mathcal{E}})$–valued form on $\widetilde{X}$ with estimates uniform with respect to ϵ. With $\epsilon \to 0$, since $\omega_0 = \pi^*\omega$, we see the boundedness of $\Lambda_\omega F$ on $X\backslash S$. In a similar way we see that

$$\mathrm{tr}(F \wedge F) \wedge \omega^{n-2} = \lim_{\epsilon\to 0} \mathrm{tr}(F \wedge F) \wedge \omega_\epsilon^{n-2}$$

is integrable on X. In view of (2.5), it therefore follows that F is square integrable. Consequently, h is admissible with respect to ω.

Lemma 2.6. *Let $(\mathcal{E}, \varphi)$ be a Higgs sheaf equipped with an admissible hermitian metric h on the compact Kähler manifold (X, ω). Then φ is bounded on X. In particular, φ is square integrable.*

Proof. We consider $\mathrm{tr}(\varphi \wedge \varphi^*) \wedge \omega^{n-1}$ pulled back to $\widetilde{X}$, where it is easily seen to be bounded in terms of ω^n. □

3. Heat equation for Higgs sheaves

The following is the main result proved here.

Theorem 3.1. *Let (X, ω) be a compact Kähler manifold and $(\mathcal{E}, \varphi)$ a stable (torsionfree) Higgs sheaf on (X, ω). Then there exists an admissible hermitian Yang–Mills–Higgs metric on $(\mathcal{E}, \varphi)$. Furthermore, the admissible hermitian Yang–Mills–Higgs connection is unique.*

This theorem will be proved later.

Lemma 3.2. *It suffices to prove the theorem under the assumption that $\mathcal{E}$ is reflexive.*

Proof. By Lemma 2.4, the Higgs sheaf $(\mathcal{E}^{\vee\vee}, \varphi^{\vee\vee})$ is stable. A Hermitian Yang–Mills–Higgs metric on $\mathcal{E}^{\vee\vee}$ can be restricted to $\mathcal{E}$ as such, since the equation (2.3) in Definition 2.5 is required only on the locally free locus of $\mathcal{E}$.

We mention that for the existence proof it would be sufficient to relax the condition of admissibility of the initial hermitian metric to require the conditions $\mathbf{A_1}$ and $\mathbf{A_2}$ to hold on the complement of some complex analytic subset in X, of codimension at least two, that contains the set S where $\mathcal{E}$ fails to be locally free. In this sense an admissible metric for $\mathcal{E}$ would also yield an admissible metric for $\mathcal{E}^{\vee\vee}$. It would follow from [BS, Theorem 2c] that (2.3) holds on the locally free locus of the given sheaf.

For the uniqueness of the connection, we observe that any admissible hermitian metric on $\mathcal{E}$ can be interpreted as an admissible metric on the double dual $\mathcal{E}^{\vee\vee}$ in the above slightly more general sense. A posteriori it satisfies (2.3) wherever $\mathcal{E}^{\vee\vee}$ is locally free. $\square$

We first assume that $\mathcal{E}$ is already a vector bundle on X, equipped with an hermitian metric h. We consider the "augmented curvature"

$$\widetilde{F} = \varphi \wedge \varphi^* + \varphi^* \wedge \varphi + F \,, \tag{3.1}$$

where F is the curvature of h. For a differentiable family h_t of hermitian metrics h_t, $t \geq 0$, we denote the curvature by F_t and set

$$\widetilde{F}_t = \varphi \wedge \varphi^* + \varphi^* \wedge \varphi + F_t \,,$$

where the adjoint forms φ^* are taken with respect to h_t. The heat equation in the sense of Higgs bundles is

$$\frac{dh_t}{dt} \cdot h_t^{-1} = -(\sqrt{-1}\Lambda_\omega \widetilde{F}_t - \lambda \cdot \mathrm{Id}_{\mathcal{E}}) \,, \tag{3.2}$$

with initial metric $h_0 = h$; the constant λ is determined by

$$\int_X \mathrm{tr}(\sqrt{-1}\Lambda_\omega \widetilde{F}_t - \lambda \cdot \mathrm{Id}_{\mathcal{E}}) \wedge \omega^n = 0 \,.$$

In the latter equation $\widetilde{F}_t$ can be replaced by F_t, as $\mathrm{tr}(\varphi \wedge \varphi^* + \varphi^* \wedge \varphi) = 0$.

The quantity $\widetilde{F}$ in (3.1) and the corresponding heat equation (3.2) can be interpreted in terms of a certain connection, induced by both the hermitian metric and the Higgs field. However, we will not take this standpoint in our arguments.

The standard equations and estimates for Higgs bundles are formally the same as in the classical case.

Lemma 3.3. *Let $\Box$ denote the Laplacian for differentiable sections of $End(\mathcal{E})$ on (X, ω). Then*

$$\frac{d}{dt}\Lambda_\omega \widetilde{F}_t = \Box \Lambda_\omega \widetilde{F}_t \,. \tag{3.3}$$

Proof. It follows from (3.2) that

$$\begin{aligned}
\frac{d}{dt}\Lambda_\omega \widetilde{F}_t &= \frac{d}{dt}\Lambda_\omega(\overline{\partial}(\partial h_t \cdot h_t^{-1})) \\
&= \Lambda_\omega \overline{\partial}\left(\partial(\frac{d}{dt}h_t \cdot h_t^{-1}) + [\frac{d}{dt}h_t \cdot h_t^{-1}, \partial h_t \cdot h_t^{-1}]\right) \\
&= \Lambda_\omega \overline{\partial}\partial_{h_t}(\frac{d}{dt}h_t \cdot h_t^{-1}) \\
&= \Box \Lambda_\omega \widetilde{F}_t\,,
\end{aligned}$$

where the connection ∂_{h_t} is taken with respect to h_t. $\Box$

Next, equation (3.3) implies immediately that

$$\frac{d}{dt}(|\Lambda_\omega \widetilde{F}_t|^2) = \Box(|\Lambda_\omega \widetilde{F}_t|^2) - |\nabla \Lambda \widetilde{F}_t|^2\,. \tag{3.4}$$

Also, equation (3.3) yields an estimate of differentiable functions

$$\frac{d}{dt}|\Lambda_\omega \widetilde{F}_t| \le \Box |\Lambda_\omega \widetilde{F}_t|\,. \tag{3.5}$$

From there

$$\frac{d}{dt}\int_X |\Lambda_\omega \widetilde{F}_t|^2 \omega^n = -\int_X |\nabla \widetilde{F}_t|^2 \omega^n \tag{3.6}$$

and

$$\int_X |\Lambda_\omega \widetilde{F}_t| \omega^n \le \int_X |\Lambda_\omega \widetilde{F}_0| \omega^n\,. \tag{3.7}$$

Now the estimate (3.5) implies

$$|\Lambda_\omega \widetilde{F}_t|(x) \le \int_X H(t,x,y) |\Lambda_\omega \widetilde{F}_0(y)| \omega(y)^n, \tag{3.8}$$

where $H(t,x,y)$ is the heat kernel for differentiable functions on (X,ω).

The finite time solutions of the heat equation (3.2) are guaranteed by a result of Simpson [Si], as well as the convergence of a subsequence of the hermitian metrics to a solution of the hermitian Yang–Mills–Higgs equation (2.3) after applying suitable gauge transformations.

Now following [BS] we consider the heat equation (3.2) for $(\widetilde{\mathcal{E}}, \pi^*\varphi, \widetilde{h})$ on (X, ω_ϵ) with $0 < \epsilon \le 1$. No assumption of stability is needed in order to get solutions for all finite t according to [Si].

For $\widetilde{\mathcal{E}}$ on $(\widetilde{X}, \omega_\epsilon)$ we consider the heat equation (3.2) and denote the solutions by $\widetilde{h}_{t,\epsilon}$, with augmented curvatures $\widetilde{F}_{t,\epsilon}$, and as before we set Λ_ϵ to be the adjoint of exterior multiplication with the form ω_ϵ. Like in [BS] the equalities (3.3) — (3.5) on $(\widetilde{X}, \omega_\epsilon)$ for $(\widetilde{\mathcal{E}}, \widetilde{h})$ together with [BS, Proposition 2] imply that

$$\Lambda_\epsilon \widetilde{F}_{t,\epsilon} \in C^\infty(\widetilde{X}, End(\widetilde{\mathcal{E}}))$$

are uniformly bounded with respect to $0 < \epsilon \le 1$ and $t \ge 0$. Next, [BS, Lemma 6] together with the boundedness of $\Lambda_\epsilon \widetilde{F}_{t,\epsilon}$ implies that $\widetilde{F}_{t,\epsilon}$ is square integrable with uniform bound on the norm. As an application we note that the solutions $h_{t,\epsilon}$ of (3.2) are "uniformly admissible". The limits

$$h_t = \lim_{\epsilon\to 0} h_{t,\epsilon}$$

solve the heat equation on $X\backslash S$ for $\mathcal{E}$ with admissible h_t (with curvatures F_t) for all $t \geq 0$ and uniform bounds for $|\Lambda F_t|$ and $\|F_t\|_{L^2}$.

Now with $\epsilon \to 0$ the equation (3.6) holds on (X, ω) for $(\mathcal{E}, h_t)$ implying that

$$\int_0^\infty \int_X |\nabla \Lambda_\omega \widetilde{F}_t|^2 \omega^n \leq \int_X |\Lambda_\omega \widetilde{F}_0|^2 \omega^n \,.$$

In particular, there exists a sequence of real numbers $t_i \to \infty$ such that

$$\int_X |\nabla \Lambda_\omega \widetilde{F}_{t_i}|^2 \omega^n \to 0\,.$$

Under the assumption of stability of $(\mathcal{E}, \varphi, \omega)$, on the complement (in $X\backslash S$) of a subset $S' \subset X\backslash S$ of finite Hausdorff measure in real codimension four there exists a subsequence $h_{t_{i(j)}}$ which converges to a limit h_∞ (after applying suitable gauge transformations). The limit is a hermitian Yang–Mills–Higgs metric on this part [Si, p. 895]. Let $\widetilde{F}$ be the augmented curvature of the limit metric. Now $\Lambda_\omega \widetilde{F}$ is bounded, in particular $\Lambda_\omega F$ is bounded. By [BS, Theorem 2] the hermitian metric h_∞ is in $L^p_{2loc}(X\backslash S)$, also h_∞ is locally bounded on $X\backslash S$. Finally, the ellipticity of the hermitian Yang–Mills–Higgs equation implies the regularity of h_∞ on all of $X\backslash S$.

This proves the existence part of Theorem 3.1.

The uniqueness part needs some further preparation.

Considering the adjoint action of the Higgs field φ on $\mathcal{F} := End(\mathcal{E}) = \mathcal{E}\otimes\mathcal{E}^\vee$, we obtain a Higgs field $\widetilde{\varphi}$ on $End(\mathcal{E})$. Let h and h' be two admissible hermitian Yang–Mills–Higgs metrics on $(\mathcal{E}, \varphi)$ with connections θ and θ'. Then it follows immediately that h'^{-1} is admissible on $(\mathcal{E}^\vee, \varphi^\vee)$ and it is a hermitian Yang–Mills–Higgs metric. Now

$$\theta_{\mathcal{F}} := \theta \otimes id_{\mathcal{E}^\vee} - id_{\mathcal{E}} \otimes \theta'^\vee$$

is a hermitian Yang–Mills–Higgs connection on the sheaf of endomorphisms $(\mathcal{F}, \widetilde{\varphi})$ induced by $h_{\mathcal{F}} = h \otimes h'^{-1}$.

Lemma 3.4. *Let $\sigma \in H^0(X, \mathcal{F})$ be any holomorphic section which commutes with the Higgs field, i.e., $[\sigma, \varphi] = 0$. Then σ is parallel with respect to $\theta_{\mathcal{F}}$ over the locally free locus of $\mathcal{F}$.*

Proof. On the complement $X\backslash S$ of the singular locus of the sheaf $\mathcal{F}$, the pointwise norm of any given section σ satisfies

$$\square(|\sigma|^2)(x) = |\partial_{\theta_{\mathcal{F}}}\sigma|^2(x) - \langle[\Lambda_\omega F, \sigma], \sigma\rangle(x)\,.$$

As $\deg \mathcal{F} = 0$, and $[\sigma, \varphi] = 0$, we have (pointwise)

$$-\langle[\Lambda_\omega F, \sigma], \sigma\rangle = \langle\Lambda_\omega[\widetilde{\varphi}, \widetilde{\varphi}^*](\sigma), \sigma^*\rangle = \langle\Lambda_\omega[\varphi, [\varphi^*, \sigma]], \sigma\rangle = \langle[\varphi^*, \sigma], [\varphi^*, \sigma]\rangle \geq 0\,.$$

Hence

$$\square(|\sigma|^2) \geq |\partial_{\theta_{\mathcal{F}}}\sigma|^2 \geq 0 \quad \text{over} \quad X\backslash S\,. \tag{3.9}$$

Now by [BS, Theorem 2] the pointwise norm $|\sigma|$ is bounded, and $|\sigma|^2$ can be extended to X as a subharmonic function. The maximum principle shows that $|\sigma|$ is constant so that (3.9) implies $\partial_{\theta_{\mathcal{F}}}\sigma = 0$. Hence σ is a flat section. □

Now we prove the uniqueness part of Theorem 3.1.

The assumptions of Lemma 3.4 are satisfied for $\sigma = id_{\mathcal{E}}$. So the identity map of $\mathcal{E}$ is a flat section of $\mathcal{F}$ with respect to $\theta_{\mathcal{F}}$, i.e.,

$$\theta \circ id_{\mathcal{E}} = id_{\mathcal{E}} \circ \theta' ,$$

and the two connections agree. □

Corollary 3.5. *Let (X,ω) be a compact Kähler manifold and $(\mathcal{E},\varphi)$ a torsionfree Higgs sheaf on (X,ω). There exists an admissible hermitian Yang–Mills–Higgs metric on $(\mathcal{E},\varphi)$ if and only if $(\mathcal{E},\varphi)$ is polystable. Furthermore, a polystable Higgs sheaf admits a unique admissible hermitian Yang–Mills–Higgs connection.*

Proof. Since a polystable Higgs sheaf is a direct sum of stable Higgs sheaves of same slope $(= \frac{\text{degree}}{\text{rank}})$, it follows from Theorem 3.1 that a polystable Higgs sheaf admits a unique admissible hermitian Yang–Mills–Higgs connection. The decomposition of a polystable Higgs sheaf into a direct sum of stable Higgs sheaves is an orthogonal decomposition.

To prove the converse, assume that $(\mathcal{E},\varphi)$ has an admissible hermitian Yang–Mills–Higgs metric h. So h is a nonsingular hermitian metric on the vector bundle $\mathcal{E}|X\setminus S$ satisfying the hermitian Yang–Mills–Higgs equation, where $X\setminus S$ is the subset where $\mathcal{E}$ is locally free. Since the complex codimension of S is at least two, from the hermitian Yang–Mills–Higgs equation it follows that $(\mathcal{E},\varphi)$ is polystable (see [Si, p. 878, Proposition 3.3]). □

4. Hermitian Yang–Mills–Higgs connection on a Higgs G–sheaf

As before, let X be a compact connected Kähler manifold equipped with a Kähler form ω.

Definition 4.1. By a *large open subset of* X we will mean a dense open subset U of X such that the complement $X\setminus U$ is a complex analytic subspace of X of complex codimension at least two.

Let G be a connected reductive linear algebraic group defined over the field of complex numbers. The Lie algebra of G will be denoted by $\mathfrak{g}$. Let

$$\mathfrak{g}' := [\mathfrak{g}\,,\mathfrak{g}] \tag{4.1}$$

be the semisimple part of $\mathfrak{g}$. Let $\mathfrak{z}(\mathfrak{g})\subset\mathfrak{g}$ be the center of $\mathfrak{g}$. The projection

$$\mathfrak{g}\longrightarrow\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \tag{4.2}$$

identifies $\mathfrak{g}'$ with the quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$.

We recall from [GS] the definition of a principal G–sheaf.

A *principal G–sheaf* over X is a triple of the form $(E_G\,,E\,,\psi)$, where

(1) E_G is a rational principal G–bundle over X, which means that E_G is a holomorphic principal G–bundle over some large open subset U of X,

(2) for any character χ of G, the holomorphic line bundle $E_G \times^{\chi} \mathbb{C}$ over U associated to E_G for χ extends to a holomorphic line bundle over X,
(3) E is a torsionfree coherent analytic sheaf on X, and
(4)

$$\psi\,:\, E_G(\mathfrak{g}') \longrightarrow E|_U \tag{4.3}$$

is a holomorphic isomorphism of vector bundles over a large open subset U over which E_G is a holomorphic principal G–bundle and E is locally free, where $E_G(\mathfrak{g}')$ is the vector bundle over U associated to E_G for the G–module $\mathfrak{g}'$ defined in (4.1).

We note that the second condition that $E_G \times^{\chi} \mathbb{C}$ extends to a holomorphic line bundle over X is automatically satisfied when X is a complex projective manifold. Since the adjoint bundle $\mathrm{ad}(E_G)$ is the direct sum of $E_G(\mathfrak{g}')$ with the trivial vector bundle with fiber $\mathfrak{z}(\mathfrak{g})$, the fourth condition ensures that the adjoint bundle $\mathrm{ad}(E_G)$ over U extends to X as a torsionfree coherent analytic sheaf.

Remark 4.2. The above mentioned large open subset U is not a part of the definition of a principal G–sheaf. In other words, we do not distinguish between the two G–sheaves given by $(E\,,U\,,E_G\,,\psi)$ and $(E\,,U'\,,E'_G\,,\psi')$ respectively where $E_G|_{U\cap U'} = E'_G|_{U\cap U'}$ and $\psi|_{U\cap U'} = \psi'|_{U\cap U'}$. However, we may take U to be the open subset of X over which the torsionfree coherent analytic sheaf E is a vector bundle. In this sense, there is a natural choice of the large open subset U. We note that the open subset over which E is a vector bundle is also the largest open subset over which E_G is a holomorphic principal G–bundle. (See [GS] for the details.)

We will now define a principal Higgs G–sheaf.

Definition 4.3. A *principal Higgs G–sheaf* on X consists of data of the following type:

- a principal G–sheaf $(E_G\,,E\,,\psi)$ on X,
- a holomorphic section

$$\varphi \,\in\, H^0(U,\, \Omega^1_X \otimes \mathrm{ad}(E_G))$$

of the adjoint bundle $\mathrm{ad}(E_G) := E_G(\mathfrak{g})$ defined on the large open subset $U \subset X$ over which E is locally free (see Remark 4.2), and
- a holomorphic homomorphism of coherent analytic sheaves

$$\widehat{\varphi}\,:\, E \longrightarrow \Omega^1_X \otimes E$$

satisfying the following two conditions:

(1) the composition

$$E \xrightarrow{\widehat{\varphi}} \Omega^1_X \otimes E \xrightarrow{\widehat{\varphi}} \Omega^2_X \otimes E$$

vanishes identically, and
(2) the restriction of $\widehat{\varphi}$ to the large open subset U coincides, using ψ (in (4.3)), with the homomorphism $E_G(\mathfrak{g}') \longrightarrow \Omega^1_X \otimes E_G(\mathfrak{g}')$ defined by $\alpha \longmapsto [\varphi, \alpha]$.

We note that for any principal Higgs G–sheaf $(E_G\,,E\,,\psi\,,\varphi\,,\widehat{\varphi})$, the pair $(E\,,\widehat{\varphi})$ is a Higgs sheaf. Similarly, the pair $(E_G(\mathfrak{g}')\,,\varphi)$ is so, and furthermore, these two Higgs sheaves are identified using the isomorphism ψ.

A principal Higgs G–sheaf $(E_G\, , E\, , \psi\, , \varphi\, , \widehat{\varphi})$ is called *stable* (respectively, *semistable*) if for every triple of the form $(U'\, , Q\, , E_Q)$, where

- $U' \subset M$ is a large open subset contained in the open subset of M over which E_G is a holomorphic principal G–bundle,
- $Q \subset G$ is a proper maximal parabolic subgroup, and
- $$E_Q \subset E_G|_{U'} \tag{4.4}$$
  is a holomorphic reduction of structure group of $E_G|_{U'}$ to Q over U' such that $\varphi|_{U'}$ is a section of $\Omega^1_X \otimes \mathrm{ad}(E_Q)$,

the following inequality

$$\text{degree}(\mathrm{ad}(E_G|_{U'})/\mathrm{ad}(E_Q)) > 0 \tag{4.5}$$

(respectively, $\text{degree}(\mathrm{ad}(E_G|_{U'})/\mathrm{ad}(E_Q)) \geq 0$) holds.

Since $\mathrm{ad}(E_Q)$ is a subbundle of $\mathrm{ad}(E_G)$ over a large open subset, and $\mathrm{ad}(E_G)$ extends to X as a coherent analytic sheaf, it follows that $\mathrm{ad}(E_Q)$ also extends to X as a coherent analytic sheaf.

By a *Levi subgroup* of a parabolic subgroup $P \subset G$ we will mean a connected reductive subgroup of P whose projection to the quotient $P/R_u(P)$ is an isomorphism, where $R_u(P)$ is the unipotent radical of P.

A principal Higgs G–sheaf

$$(E_G\, , E\, , \psi\, , \varphi\, , \widehat{\varphi})$$

is called *polystable* if either $(E_G\, , E\, , \psi\, , \varphi\, , \widehat{\varphi})$ is stable, or there is a pair $(L(P)\, , E_{L(P)})$, where

- $L(P) \subset P \subset G$ is a Levi subgroup of some parabolic subgroup P of G, and
- $E_{L(P)} \subset E_G|_U$ is a holomorphic reduction of structure group to $L(P) \subset G$, over the large open subset U over which E is locally free, with the property that the section $\varphi|_U$ lies in the image of the natural inclusion
  $$H^0(U,\, \Omega^1_X \otimes \mathrm{ad}(E_{L(P)})) \hookrightarrow H^0(U,\, \Omega^1_X \otimes \mathrm{ad}(E_G|_U))$$

such that the following two hold:

1. the principal Higgs $L(P)$–bundle $(E_{L(P)}\, , E'\, , \psi\, , \varphi\, , \widehat{\varphi})$ is stable, where E' is the coherent analytic subsheaf of E generated by $\mathrm{ad}(E_{L(P)})$ using ψ, and
2. for each character χ of $L(P)$ which is trivial on the center of G, the line bundle $E_{L(P)}(\chi)$ over U associated to $E_{L(P)}$ for the character χ is of degree zero.

Note that there is a natural inclusion of $\mathrm{ad}(E_{L(P)})$ in $\mathrm{ad}(E_G)$; hence there is a natural homomorphism from $\mathrm{ad}(E_{L(P)})$ to $E_G(\mathfrak{g}')$. (See [RS], [AB] for the definition of polystable principal bundles.)

We will now define hermitian Yang–Mills–Higgs connections on principal G–sheaves.

Fix a maximal compact subgroup

$$K(G) \subset G\, . \tag{4.6}$$

If E'_G is a holomorphic principal G–bundle over a complex manifold, and $E'_{K(G)} \subset E'_G$ is a C^∞ reduction of structure group of E'_G to $K(G)$, then the G–bundle E'_G has a unique complex connection which is induced by a connection on $E'_{K(G)}$. This unique connection will be called the *Chern connection*.

Set

$$Z := G/[G\,,G] \tag{4.7}$$

to be the quotient group, which is a product of copies of $\mathbb{G}_m = \mathbb{C}^*$. Note that Z is a finite quotient of the connected component, containing the identity element, of the center of G.

Let E_G be a holomorphic principal G–bundle over a large open subset U of X. Let φ be a Higgs field on E_G over U. Let

$$E_Z := E_G(Z) \tag{4.8}$$

be the principal Z–bundle over U obtained by extending the structure group of E_G using the quotient map $G \longrightarrow Z$ in (4.7). The Higgs field on E_Z over U induced by φ will be denoted by φ^z.

The above defined principal Higgs Z–bundle $(E_Z\,,\varphi^z)$ extends to a holomorphic principal Higgs Z–bundle over X. Indeed, this follows from the facts that Z is a product of copies of $\mathbb{C}^*$, and any holomorphic line bundle over U extends to a holomorphic line bundle over X. To see that any holomorphic line bundle L over U extends to X, consider the determinant line bundle $\det(\iota_* L)$ over X, where ι is the inclusion map of U in X. See [Kob, Ch. V, §6] for the construction of the determinant line bundle of a torsionfree coherent analytic sheaf on X; note that from the condition that the codimension of the complex analytic set $X \setminus U \subset X$ is at least two it follows that the direct image $\iota_* L$ is a coherent analytic sheaf on X. The holomorphic extension of E_Z to X is clearly unique. Since φ^z is a holomorphic section of $\Omega^1_X \otimes_{\mathbb{C}} \mathfrak{z}$ over U, where $\mathfrak{z}$ is the Lie algebra of Z, and the codimension of the complex analytic set $X\setminus \subset X$ is at least two, the section φ^z extends to a holomorphic section of $\Omega^1_X \otimes_{\mathbb{C}} \mathfrak{z}$ over X.

Since any Higgs line bundle over X has a unique hermitian Yang–Mills–Higgs connection, any holomorphic principal Higgs Z–bundle over X also has a unique hermitian Yang–Mills–Higgs connection.

Let $(E_G\,, E\,, \psi\,, \varphi\,, \widehat{\varphi})$ be a principal Higgs G–sheaf on (X, ω). Let $U \subset X$ be the large open subset over which E_G is a holomorphic principal G–bundle (see Remark 4.2). A *hermitian Yang–Mills–Higgs connection* on $(E_G\,, E\,, \psi\,, \varphi\,, \widehat{\varphi})$ is a Chern connection ∇ on the principal G–bundle E_G over U satisfying the following two conditions:

(1) the connection on the principal Higgs Z–bundle $(E_Z\,,\varphi^z)$ (defined in (4.8)) induced by ∇ coincides with the unique hermitian Yang–Mills–Higgs connection on the extension of $(E_Z\,,\varphi^z)$ to X (recall that $(E_Z\,,\varphi^z)$ extends holomorphically to M, and the extension has a unique hermitian Yang–Mills–Higgs connection); and
(2) the connection on $E|_U$ induced by ∇ and ψ is an admissible hermitian Yang–Mills–Higgs connection on the reflexive Higgs sheaf $(E^{\vee\vee}\,,\widehat{\varphi})$ (the connection ∇ induces a connection on the associated vector bundle $E_G(\mathfrak{g}')$ in (4.3), and using the isomorphism ψ in (4.3), this induced connection gives a connection on $E|_U$).

Let $(\mathcal{E}\,,\varphi)$ and $(\mathcal{E}',\varphi')$ be two Higgs sheaves on X. Define

$$\mathcal{V} := (\mathcal{E}\otimes\mathcal{E}')/\text{torsion}\,. \tag{4.9}$$

The Higgs fields φ and φ' together induce a Higgs field θ on $\mathcal{V}$. The description of θ is the following:

$$\theta = \varphi\otimes \mathrm{Id}_{\mathcal{E}'} + \mathrm{Id}_{\mathcal{E}}\otimes\varphi'\,. \tag{4.10}$$

Lemma 4.4. *Assume that the two Higgs sheaves $(\mathcal{E}\,,\varphi)$ and $(\mathcal{E}',\varphi')$ are both polystable. Then the Higgs sheaf $(\mathcal{V}\,,\theta)$, defined in (4.9) and (4.10), is also polystable.*

Proof. Consider the double dual $\mathcal{E}^{\vee\vee}$ equipped with the Higgs field induced by φ. This induced Higgs field on $\mathcal{E}^{\vee\vee}$ will be denoted by $\varphi^{\vee\vee}$. Since $(\mathcal{E}\,,\varphi)$ is polystable, it follows that $(\mathcal{E}^{\vee\vee}\,,\varphi^{\vee\vee})$ is also polystable (see Lemma 2.4). Let ∇ be the unique hermitian Yang–Mills–Higgs connection on $(\mathcal{E}^{\vee\vee}\,,\varphi^{\vee\vee})$ given by Corollary 3.5. Similarly, let ∇' be the unique hermitian Yang–Mills–Higgs connection on $((\mathcal{E}')^{\vee\vee}\,,(\varphi')^{\vee\vee})$; as before, the Higgs field on $(\mathcal{E}')^{\vee\vee}$ induced by φ' is denoted by $(\varphi')^{\vee\vee}$.

Now it is easy to see that the connection

$$\nabla^{\mathcal{V}} := \nabla\otimes\mathrm{Id}_{(\mathcal{E}')^{\vee\vee}} + \mathrm{Id}_{\mathcal{E}}\otimes\nabla'$$

induces a hermitian Yang–Mills–Higgs connection on $(\mathcal{V}\,,\theta)$. Consequently, the Higgs sheaf $(\mathcal{V}\,,\theta)$ is polystable (Corollary 3.5). This completes the proof of the lemma. $\square$

Proposition 4.5. *Assume that $(\mathcal{E}\,,\varphi)$ and $(\mathcal{E}',\varphi')$ are both semistable. Then the Higgs sheaf $(\mathcal{V}\,,\theta)$, defined in (4.9) and (4.10), is also semistable.*

Proof. Since $(\mathcal{E}\,,\varphi)$ is semistable, there is filtration of coherent subsheaves

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{\ell-1} \subsetneq E_\ell = \mathcal{E} \tag{4.11}$$

such that the following hold:

- E_i/E_{i-1} is torsionfree with
$$\frac{\deg(E_i/E_{i-1})}{\mathrm{rk}(E_i/E_{i-1})} = \frac{\deg\mathcal{E}}{\mathrm{rk}\mathcal{E}}$$
for all $i\in[1\,,\ell]$,
- $\varphi(E_i)\subset\Omega^1_X\otimes E_i$ for all $i\in[0\,,\ell]$, and
- for all $i\in[1\,,\ell]$, the quotient E_i/E_{i-1} equipped with the Higgs field induced by φ is polystable.

Let

$$0 = E'_0 \subsetneq E'_1 \subsetneq \cdots \subsetneq E'_{\ell'-1} \subsetneq E'_{\ell'} = \mathcal{E}'$$

be the filtration constructed as in (4.11) for the semistable Higgs sheaf $(\mathcal{E}',\varphi')$. Therefore, the Higgs fields φ and φ' together induce a Higgs field on

$$\mathcal{V}_{i,j} := ((E_i/E_{i-1})\otimes(E'_j/E'_{j-1}))/\text{torsion}$$

for all $i\in[1\,,\ell]$ and $j\in[1\,,\ell']$. This Higgs field on $\mathcal{V}_{i,j}$ will be denoted by $\theta_{i,j}$. From Lemma 4.4 it follows that the Higgs sheaf $(\mathcal{V}_{i,j}\,,\theta_{i,j})$ is polystable for all i,j. Since

$\deg(E_i/E_{i-1})/\text{rk}(E_i/E_{i-1}) = \deg \mathcal{E}/\text{rk}\mathcal{E}$ and $\deg(E'_j/E'_{j-1})/\text{rk}(E'_j/E'_{j-1}) = \deg \mathcal{E}'/\text{rk}\mathcal{E}'$, we conclude that

$$\frac{\deg \mathcal{V}_{i,j}}{\text{rk}\mathcal{V}_{i,j}} = \frac{\deg \mathcal{V}}{\text{rk}\mathcal{V}}. \tag{4.12}$$

Let $U \subset X$ be the dense open subset over which all E_i/E_{i-1}, $i \in [1\,,\ell]$, and all E'_j/E'_{j-1}, $j \in [1\,,\ell']$, are locally free. The complement $X \setminus U$ is a complex analytic subset of complex codimension at least two (recall that all E_i/E_{i-1} and E'_j/E'_{j-1} are torsionfree).

We note that over U, the Higgs sheaf $(\mathcal{V}\,,\theta)$ admits a filtration such that each successive quotient is a Higgs sheaf of the form $(\mathcal{V}_{i,j}\,,\theta_{i,j})$ for some i, j. We already noted that each $(\mathcal{V}_{i,j}\,,\theta_{i,j})$ is polystable satisfying (4.12). Consequently, the Higgs sheaf $(\mathcal{V}\,,\theta)$ is semistable. This completes the proof of the proposition. □

Theorem 4.6. *A principal Higgs G–sheaf $(E_G\,, E\,, \psi\,, \varphi\,, \widehat{\varphi})$ over a compact connected Kähler manifold X admits an admissible hermitian Yang–Mills–Higgs connection if and only if $(E_G\,, E\,, \psi\,, \varphi\,, \widehat{\varphi})$ is polystable. Furthermore, a polystable principal Higgs G–sheaf admits a unique hermitian Yang–Mills–Higgs connection.*

Proof. Since we have proved Proposition 4.5, the proof of Theorem 4.10 in [AB, p. 227] goes through to give a proof of the above theorem. See the final paragraph in [AB, p. 227] explaining the issue. We give some details of the arguments.

Let U be the large open subset of X over which E_G is holomorphic principal G–bundle (see Remark 4.2). Consider the reflexive sheaf $\iota_*\text{ad}(E_G|_U)$ over X, where $\iota : U \hookrightarrow X$ is the inclusion map. We note that using ψ, the direct image $\iota_*\text{ad}(E_G|_U)$ is identified with the direct sum $E^{\vee\vee} \oplus (X \times \mathfrak{z}(\mathfrak{g}))$, where $X \times \mathfrak{z}(\mathfrak{g})$ is the trivial holomorphic vector bundle over X with fiber $\mathfrak{z}(\mathfrak{g})$. For notational convenience, the sheaf $\iota_*\text{ad}(E_G|_U)$ will be denoted by $\text{ad}(E_G)$. The Higgs field φ clearly defines a Higgs field on the reflexive sheaf $\text{ad}(E_G)$; this induced Higgs field on $\text{ad}(E_G)$ will be denoted by φ'.

If ∇ is an admissible hermitian Yang–Mills–Higgs connection on $(E_G\,, E\,, \psi\,, \varphi\,, \widehat{\varphi})$, then it is straight–forward to check that the connection on $\text{ad}(E_G)$ induced by ∇ is an admissible hermitian Yang–Mills–Higgs connection for the Higgs sheaf $(\text{ad}(E_G)\,, \varphi')$. Now from Corollary 3.5 it follows that $(\text{ad}(E_G)\,, \varphi')$ is polystable. From this it is easy to deduce that $(E_G\,, E\,, \psi\,, \varphi\,, \widehat{\varphi})$ is polystable.

To prove the converse, assume that $(E_G\,, E\,, \psi\,, \varphi\,, \widehat{\varphi})$ is polystable. Using this assumption it can be shown that the above defined Higgs sheaf $(\text{ad}(E_G)\,, \varphi')$ is polystable; the details are in [AB]. Let ∇' be the admissible hermitian Yang–Mills–Higgs connection for the Higgs sheaf $(\text{ad}(E_G)\,, \varphi')$ given by Corollary 3.5. This connection ∇' on $\text{ad}(E_G)$ induces a connection on $E_G|_U$, where $U \subset X$ is the open subset over which E_G is a holomorphic principal G–bundle. It can be shown that this induced connection on $E_G|_U$ is an admissible hermitian Yang–Mills–Higgs connection on $(E_G\,, E\,, \psi\,, \varphi\,, \widehat{\varphi})$. □

We have the following analog of the Bogomolov inequality.

Let $(E_G\,, E\,, \psi\,, \varphi\,, \widehat{\varphi})$ be a polystable principal Higgs G–sheaf over a compact connected Kähler manifold X equipped with the Kähler form ω. Let U be the large open subset of X over which E_G is holomorphic principal G–bundle (see Remark 4.2). Consider the reflexive sheaf $\text{ad}(E_G) := \iota_*\text{ad}(E_G|_U)$ over X, where $\iota : U \hookrightarrow X$ is the inclusion map. Let φ'

be the Higgs field on $\text{ad}(E_G)$ induced by φ. Let ∇' be the (singular) connection on $\text{ad}(E_G)$ induced by the admissible hermitian Yang–Mills–Higgs connection on $(E_G\,, E\,, \psi\,, \varphi\,, \widehat{\varphi})$. (It was noted in the proof of Theorem 4.6 that ∇' coincides with the admissible hermitian Yang–Mills–Higgs connection on the polystable Higgs sheaf $(\text{ad}(E_G)\,, \varphi')$ given by Corollary 3.5.)

Proposition 4.7. *With the above notation,*

$$(2\dim_{\mathbb{C}}\mathfrak{g}\cdot(c_2(\text{ad}(E_G))) - (\dim_{\mathbb{C}}\mathfrak{g} - 1)c_1(\text{ad}(E_G))^2)\omega^{d-2} \,\geq\, 0\,,$$

where $\mathfrak{g}$ is the Lie algebra of G. Furthermore, the equality holds if and only if $U = X$ and the connection $\nabla' + \varphi' + (\varphi')^*$ on $\text{ad}(E_G)$ is projectively flat.

Proof. Since the connection ∇' is the admissible hermitian Yang–Mills–Higgs connection on the polystable Higgs sheaf $(\text{ad}(E_G)\,, \varphi')$, this proposition follows from the proof of Proposition 3.4 in [Si, p. 878]. See also Corollary 3 in [BS, p. 40]. □

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