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A CORRESPONDENCE BETWEEN THE MODULI SPACES OF VECTOR BUNDLES OVER A CURVE

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Abstract. After establishing a correspondence between a smooth moduli space of vector bundles on a curve and a self-product of the Jacobian, the nontriviality of the Griffiths group of the moduli space for a general curve is proved.

1. Introduction. Let $\mathcal{N}$ denote the moduli space of isomorphism classes of stable vector bundles, over a compact Riemann surface X, of rank r and of a fixed determinant of degree d, with r and d being mutually coprime. Let $P(\mathcal{E})$ be the universal projective bundle over the Cartesian product $X \times \mathcal{N}$. The characteristic classes $a_k \in H^{2k}(X \times \mathcal{N}, \boldsymbol{Q})$, where $k = 2, 3, \dots, r$, of $P(\mathcal{E})$, give rise to homomorphisms, from $H_1(X, \boldsymbol{Q})$ to $H^{2k-1}(\mathcal{N}, \boldsymbol{Q})$, by the "slant product" operation. Since the homology algebra, $H_*(J(X), \boldsymbol{Q})$, of the Jacobian $J(X)$ of X is the exterior algebra $\bigwedge H_1(X, \boldsymbol{Q})$, combining all these homomorphisms given by the characteristic classes of $P(\mathcal{E})$, we get an algebra homomorphism from $H_*(J^{r-1}, \boldsymbol{Q})$ (the homology algebra of the $(r-1)$-fold self-product of J) to the cohomology algebra $H^*(N, \boldsymbol{Q})$. The details of this construction are given in Section 2.

We show that the above formally constructed homomorphism has a geometric origin. More precisely, there is a correspondence cycle on the product $\mathcal{N} \times J^{r-1}$, which is canonical as an element of the Chow ring (cycles modulo rational equivalence) of $\mathcal{N} \times J^{r-1}$, such that the above homomorphism is induced by this cycle (Theorem 2.5). As an application of this result, we construct nonzero elements of the Griffiths group of the moduli space $\mathcal{N}$ for a general Riemann surface (Theorem 4.8). These elements are constructed from the nonzero elements of the Jacobian of a general curve discovered by Ceresa [C].

The above results extend to the more general context of smooth moduli spaces of parabolic bundles.

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2. Map between Hodge cycles. Let X be a compact Riemann surface, or equivalently, a connected smooth projective curve over $\boldsymbol{C}$. Assume that the genus $g := \text{genus}(X) \geq$

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2. Choose and fix a point $x_0 \in X$. Fix a pair of mutually coprime integers r and d with $r > 1$. Let $\mathcal{N}$ denote the moduli space of isomorphism classes of stable vector bundles E on X of rank r and with the top exterior product $\bigwedge^r E$ isomorphic to $\mathcal{O}_X(dx_0)$. The space $\mathcal{N}$ has a structure of a connected smooth projective variety over $\boldsymbol{C}$ of dimension $(r^2-1)(g-1)$.

Let $\mathcal{E} \to X \times \mathcal{N}$ be a universal vector bundle; which means that for any $z \in \mathcal{N}$, the vector bundle on X, given by the restriction $\mathcal{E}|_{X \times z}$, is in the isomorphism class represented by the point z. If $\mathcal{E}'$ is another universal bundles on $X \times \mathcal{N}$, then from the projection formula we have the equality $\mathcal{E}' = \mathcal{E} \otimes p_2^* \mathcal{L}$, where $\mathcal{L}$ is a line bundle on $\mathcal{N}$ and $p_2 : X \times \mathcal{N} \to \mathcal{N}$ is the obvious projection. Thus the projective bundle $P(\mathcal{E})$ is independent of the choice of the universal bundle.

Let $a_k \in H^{2k}(X \times \mathcal{N}, \boldsymbol{Q})$, $2 \leq k \leq r$, be the characteristic classes of the principal $PGL(r)$ bundle $P(\mathcal{E}) \to X \times \mathcal{N}$ given by the space of lines in $\mathcal{E}$. We will recall a description of a_k in terms of the Chern classes of $\mathcal{E}$. Consider the k-th degree polynomial, P_k, of r variables $\{x_1, \ldots, x_r\}$, defined by the identity

$$\det(\lambda + D_{\underline{x}}) = \sum_{j=0}^{r} \lambda^{r-j} P_j ,$$

where $D_{\underline{x}}$ is the $r \times r$ diagonal matrix with $x_1, x_2, \ldots, x_r$ as the diagonal entries. So the i-th Chern class $c_i(\mathcal{E}) = P_i(\alpha_1, \ldots, \alpha_r)$, where $\{\alpha_1, \ldots, \alpha_r\}$ are the Chern roots of $\mathcal{E}$. Define

$$\beta_i := r\alpha_i - \sum_{j=1}^{r} \alpha_j .$$

With this notation we have

$$a_i := P_i(\beta_1, \beta_2, \ldots, \beta_r) . \tag{2.1}$$

Note that $a_1 = 0$.

For a cohomology class $c \in H^j(X \times \mathcal{N}, \boldsymbol{Q})$, using the Künneth decomposition of the $\boldsymbol{Q}$-vector space $H^j(X \times \mathcal{N}, \boldsymbol{Q})$, as well as the obvious duality between $H^k(X, \boldsymbol{Q})$ and $H_k(X, \boldsymbol{Q})$, we have a homomorphism

$$S(c, k) : H_k(X, \boldsymbol{Q}) \to H^{j-k}(\mathcal{N}, \boldsymbol{Q}) , \tag{2.2}$$

which is usually called the "slant product". Taking c to be the class a_i, we get

$$S(a_i, k) : H_k(X, \boldsymbol{Q}) \to H^{2i-k}(\mathcal{N}, \boldsymbol{Q}) .$$

Let the direct sum $H_1(X, \boldsymbol{Q})^{\oplus(r-1)}$ be denoted by W. Taking the direct sum of the above maps, we have a $\boldsymbol{Q}$-vector space homomorphism

$$\sum_{i=2}^{r} S(a_i, 1) : W = H_1(X, \boldsymbol{Q})^{\oplus(r-1)} \to H^{\mathrm{odd}}(\mathcal{N}, \boldsymbol{Q}) ,$$

which, in turn, induces an algebra homomorphism

$$F : \bigwedge W \to H^*(\mathcal{N}, \boldsymbol{Q}) , \tag{2.3}$$

where $\bigwedge W$ is the exterior algebra for W. Consider the complex vector space $W_C := W \otimes_Q C$. Tensoring (2.3) with C, we get a C-algebra homomorphism

$$F_C : \bigwedge W_C \to H^*(\mathcal{N}, C) .$$

Let $J := \mathrm{Pic}^0(X)$ be the Jacobian of X. The homology algebra $H_*(J, Q)$ (the algebra structure is given by the cap product) is canonically isomorphic to the exterior algebra $\bigwedge H_1(X, Q)$. Indeed, consider the map $X \to J$, defined by $x \mapsto \mathcal{O}_X(x - x_0)$. The induced map of the first homology is extended as an algebra homomorphism from the exterior algebra $\bigwedge H_1(X, Q)$ to $H_*(J, Q)$. It is easy to see that this homomorphism is actually an isomorphism, and that it does not depend upon the choice of the base point x_0. The canonical isomorphism between $H_*(J, Q)$ and $\bigwedge H_1(X, Q)$ induces an isomorphism between the homology algebra

$$H_*(J^{r-1}, Q) = \bigwedge H_1(J^{r-1}, Q) = \bigwedge W .$$

Let $\mathcal{J}$ denote the $(r-1)$-fold self-product, namely J^{r-1}, of J with itself. Let

$$\gamma : H_*(\mathcal{J}, Q) \to \bigwedge W$$

be the isomorphism obtained above. Combining this with the homomorphism F in (2.3), we have

$$\bar{F} : H_*(\mathcal{J}, Q) \to H^*(\mathcal{N}, Q) \tag{2.4}$$

defined by $c \mapsto F(\gamma(c))$. On account of the obvious duality, namely $H_i(\mathcal{J}, Q)^* = H^i(\mathcal{J}, Q)$, the homomorphism $\bar{F}$ gives an element in $H^*(\mathcal{N}, Q) \otimes H^*(\mathcal{J}, Q)$. The cohomology algebra $H^*(\mathcal{N} \times \mathcal{J}, Q)$ of $\mathcal{N} \times \mathcal{J}$ is canonically isomorphic to the graded tensor product $H^*(\mathcal{N}, Q) \otimes H^*(\mathcal{J}, Q)$. Thus, $\bar{F}$ gives an element in $H^*(\mathcal{N} \times \mathcal{J}, Q)$; this element will be denoted by Φ.

For a smooth projective variety Y over C, an element of the Q-vector space

$$\bigoplus_p \left(H^{p,p}(Y) \cap H^*(Y, Q)\right)$$

will be called, following [D], a *Hodge cycle* on Y. There is a natural homomorphism, known as the *cycle class map*, from the space of cycles on X to the space of Hodge cycles Y. The cycle class of a cycle of codimension p on Y is in $H^{p,p}(Y) \cap H^{2p}(Y, Q)$. Let $\mathrm{CH}^*(Y)$ denote the Chow group of cycles on Y modulo rational equivalence. The image of $\mathrm{CH}^*(Y) \otimes_Z Q$ in the space of Hodge cycles (by the cycle class map) is called the space of *algebraic cohomology classes* on Y.

From the definition of the cohomology class Φ it is clear that Φ is actually an element of the image of $H^*(\mathcal{N} \times \mathcal{J}, Z)$ in $H^*(\mathcal{N} \times \mathcal{J}, Q)$ by the obvious map. Indeed, this is a consquence of the fact that the homomorphism $\bar{F}$ in (2.4) maps $H_*(\mathcal{J}, Z)$ into $H^*(\mathcal{N}, Z)$.

THEOREM 2.5. *The cohomology class Φ is algebraic, i.e., it is the cycle class of an algebraic cycle on $\mathcal{N} \times \mathcal{J}$.*

PROOF. Let G_Z denote the group of all automorphisms of $H_1(X, Z)$ preserving the cap product. Choosing a symplectic basis of $H_1(X, Z)$, this group G_Z can be identified with $Sp(g, Z)$, the group of $2g \times 2g$ symplectic matrices with entries in Z. The proof of Theorem 2.5

consists of three steps. First, the group G_Z acts on the cohomology algebra $H^*(\mathcal{N} \times \mathcal{J}, Q)$. Second, any invariant class in $H^*(\mathcal{N} \times \mathcal{J}, Q)$, for the action of G_Z, is algebraic. Third, the class Φ is an invariant for the action of G_Z on $H^*(\mathcal{N} \times \mathcal{J}, Q)$.

We will first describe the action of G_Z on $H^*(\mathcal{N} \times \mathcal{J}, Q)$. Let $\bar{W}$ denote the direct sum $H_0(X, Q) \oplus H_2(X, Q)$. Setting the class c to be a_i in (2.2), we have

$$S(a_i) := S(a_i, 0) \oplus S(a_i, 2) : \bar{W} \to H^{\text{even}}(\mathcal{N}, Q)\,. \tag{2.6}$$

These homomorphisms combine together to induce a homomorphism from the symmetric algebra

$$S : S(\bar{W}^{\oplus(r-1)}) \to H^{\text{even}}(\mathcal{N}, Q)\,.$$

Consider the tensor product of the homomorphism S above with F defined in (2.3):

$$S \otimes F : S(\bar{W}^{\oplus(r-1)}) \otimes \bigwedge W \to H^*(\mathcal{N}, Q) \tag{2.7}$$

The group G_Z acts on the direct sum W by the diagonal action corresponding to the natural action on $H_1(X, Q)$. Let G_Z act on $\bar{W}$ as the trivial action. In [BR] and [B1] it is proved that the homomorphism $S \otimes F$ in (2.7) is surjective and the kernel is left invariant by the induced action of G_Z on $S(\bar{W}^{\oplus(r-1)}) \otimes \bigwedge W$. Thus we have an action of G_Z on $H^*(\mathcal{N}, Q)$ induced by the action of G_Z on $S(\bar{W}^{\oplus(r-1)}) \otimes \bigwedge W$.

We will recall a theorem proved in [B1]. Take a smooth family of one pointed curves parametrized by T. Consider the corresponding family of moduli spaces of vector bundles parametrized by T. The holonomy of the Gauss-Manin connection for this family gives rise to a representation of the fundamental group $\pi_1(T)$ into the cohomology algebra of the typical fiber. This representation factors through the homomorphism of $\pi_1(T)$ into G_Z, given by the holonomy of the Gauss-Manin connection for the first homology of the family of curves. The action of G_Z on the cohomology of the typical fiber of the family of moduli spaces is precisely the action of G_Z on $H^*(\mathcal{N}, Q)$ obtained above.

Since $H^*(J, Q) = \bigwedge H^1(X, Q) = \bigwedge H_1(X, Q)^*$ (Poincaré duality on X), we have the equality $H^*(\mathcal{J}, Q) = \bigwedge W^*$. Thus the cohomology algebra $H^*(\mathcal{N} \times \mathcal{J}, Q)$ is the graded tensor product $H^*(\mathcal{N}, Q) \otimes \bigwedge W^*$. The group G_Z acts on $H_1(X, Z)^*$ by the adjoint action and on W^* by the diagonal action. Consider the induced action of G_Z on $\bigwedge W^*$. The actions of G_Z on $H^*(\mathcal{N}, Q)$ (obtained above) and on $\bigwedge W^*$ combine together to induce an action of G_Z on $H^*(\mathcal{N} \times \mathcal{J}, Q)$.

As the next step we want to show that any invariant class in $H^*(\mathcal{N} \times \mathcal{J}, Q)$ for the action of G_Z is an algebraic class.

The action of G_Z on $H_1(X, Z)$ extends naturally to an action of its complexification G_C on $H_1(X, C)$. The Borel density theorem asserts that the subgroup G_Z is Zariski dense in G_C. This implies that the space of invariants in $H^*(\mathcal{N} \times \mathcal{J}, Q)$ for the action of G_Z is precisely the image of $S(\bar{W}^{\oplus(r-1)}) \otimes (\bigwedge W \otimes \bigwedge W^*)^{G_Z}$ (in $H^*(\mathcal{N} \times \mathcal{J}, Q)$) by the map $S \otimes \bar{F} \otimes \mathrm{Id}$, where $(\bigwedge W \otimes \bigwedge W^*)^{G_Z}$ denotes the space of all G_Z invariants in $\bigwedge W \otimes \bigwedge W^* = \mathrm{End}(\bigwedge W)$. (Note that the action of G_Z on $S(\bar{W}^{\oplus(r-1)})$ is the trivial action.) From Theorem 2 of [Ho] we know that the space of G_C invariants in $\mathrm{End}(\bigwedge W_C)$ is generated by a subclass of degree two elements (when considered as elements of $\bigwedge(W \oplus W)$ using the symplectic form). The subclass in question is described as follows: identify $H_1(X, \mathbb{Q})$ with $H^1(X, \mathbb{Q})$ using the Poincaré duality on X. Thus

$$\bigwedge W \otimes \bigwedge W^* = \bigwedge(W \oplus W) = \bigwedge(H_1(X, \mathbb{Q})^{\oplus 2(r-1)}).$$

For $1 \leq i \leq j \leq 2(r-1)$, let $f_{i,j}$ denote the diagonal inclusion of $H_1(X, \mathbb{Q})$ in $H_1(X, \mathbb{Q})^{\oplus 2(r-1)}$ along the (i, j)-th coordinates, and let $\bigwedge^2 f_{i,j}$ denote the induced homomorphism of the second exterior products. Let

$$\tau \in \bigwedge^2 H_1(X, \mathbb{Q})$$

be the element corresponding to the Poincaré duality pairing, namely

$$\omega_1 \otimes \omega_2 \mapsto \int_X \omega_1 \wedge \omega_2 ,$$

where $\omega_1, \omega_2 \in H^1(X, \mathbb{Q})$. Now define

$$\tau_{i,j} := (\bigwedge^2 f_{i,j})(\tau).$$

The space of all $G_{\mathbb{Z}}$ invariants in $\bigwedge W \otimes \bigwedge W$ is generated, as an algebra, by the collection of elements $\{\tau_{i,j}\}$.

Since the class $a_k \in H^{2k}(\mathcal{N} \times \mathcal{J}, \mathbb{Q})$ is algebraic (any Chern class of an algebraic vector bundle is algebraic), the image of the homomorphism S is contained in the algebraic classes in $H^*(\mathcal{N}, \mathbb{Q})$. Indeed, $S(a_k, 0)(1)$ is the restriction of a_k to $\mathcal{N}$ (1 is the canonical generator of $H_0(X, \mathbb{Q})$); and, $S(a_k, 2)([X])$ is the image of a_k by the Gysin map for the projection of $X \times \mathcal{N}$ onto $\mathcal{N}$ ($[X]$ is the oriented generator of $H_2(X, \mathbb{Z})$). We note that the proof of the assertion that any $G_{\mathbb{Z}}$ invariant class in $H^*(\mathcal{N} \times \mathcal{J}, \mathbb{Q})$ is algebraic will be completed once we are able to show that any $\bar{F} \otimes \mathrm{Id}(\tau_{i,j})$ in $H^*(\mathcal{N} \times \mathcal{J}, \mathbb{Q})$ is actually algebraic.

For notational simplicity we will also use $\tau_{i,j}$ to denote $\bar{F} \otimes \mathrm{Id}(\tau_{i,j})$. From the context it will be clear which one is being used.

If $1 \leq i, j \leq r-1$, then $\tau_{i,j}$ is the pullback of a cohomology class on $\mathcal{N}$ (by the projection of $\mathcal{N} \times \mathcal{J}$ onto $\mathcal{N}$); we will denote this class in $\mathcal{N}$ by t. Moreover, t is an invariant for the action of $G_{\mathbb{Z}}$ (since $\tau_{i,j}$ is, by definition, an invariant). Hence from Proposition 2.4 of [BN] it follows that t is algebraic. Thus $\tau_{i,j}$ is algebraic.

If $r \leq i < j \leq 2(r-1)$, then $\tau_{i,j}$ is the pullback of a cohomology class in $J \times J$, denoted by t, using the projection of $\mathcal{N} \times \mathcal{J}$ along the i-th and j-th factors in $\mathcal{J}$. If $i = j$, then it is the pullback of a cohomology class on J, denoted by t, using the projection onto the i-th factor of $\mathcal{J}$. Since $\tau_{i,j}$ is invariant under the action of $G_{\mathbb{Z}}$, the cohomology class t, in either case, is also invariant under the action of $G_{\mathbb{Z}}$. If $i = j$, then t is the cycle class of the theta divisor in J. (The theta divisor on J is the pullback of the theta divisor on $\mathrm{Pic}^{g-1}(X)$ using the map defined by $L \mapsto L \otimes \mathcal{O}_X((g-1)x_0)$.) Let Θ denote this theta divisor on J. Let

$$A : J \times J \to J$$

be the addition map defined by $(L, L') \mapsto L \otimes L'$. If $i \neq j$, then t is the cycle class of the divisor

$$A^*\Theta - p_1^*\Theta - p_2^*\Theta$$

on $J \times J$, where p_1 (respectively, p_2) is the projection of $J \times J$ to the first (respectively, second) factor. It is a straightforward calculation to check the above statement. Thus the cohomology class $\tau_{i,j}$ is algebraic for $r \leq i \leq j \leq 2(r-1)$.

Finally, consider a cohomology class $\tau_{i,j}$ with $i \leq r-1$ and $j \geq r$. Let f denote the projection of $\mathcal{N} \times \mathcal{J}$ onto $\mathcal{N} \times J$ which maps any

$$(n, L_1, \ldots, L_{r-1}) \in \mathcal{N} \times J^{r-1} = \mathcal{N} \times \mathcal{J}$$

to $(n, L_j) \in \mathcal{N} \times \mathcal{J}$. There is a unique cohomology class on $\mathcal{N} \times J$, say t, such that $f^*t = \tau_{i,j}$.

Consider the characteristic class $a_k \in H^{2k}(X \times \mathcal{N}, \mathbf{Q})$ defined above. Let

$$a_{i+1} = a \otimes 1 + b + c \otimes [X] \tag{2.8}$$

be the Künneth decomposition of the cohomology class a_{i+1}, where the cohomology class $[X]$ is the oriented generator of $H^2(X, \mathbf{Z})$. The classes $a, c \in H^*(\mathcal{N}, \mathbf{Q})$ are algebraic. Indeed, a is the restriction of a_{i+1} to $x_0 \times \mathcal{N}$, and c is the push-forward of a_{i+1} by the projection of $X \times \mathcal{N}$ onto $\mathcal{N}$. Thus the cohomology class b in (2.8) is also algebraic.

Let $\mathcal{L}$ be a Poincaré line bundle over $X \times J$. Let θ be the component of $c_1(\mathcal{L})$ in $H^1(X, \mathbf{Q}) \otimes H^1(J, \mathbf{Q})$ for the Künneth decomposition of $H^2(X \times J, \mathbf{Q})$. Note that, since any two choices of the Poincaré line bundle differ only by the pullback of a line bundle on J, this Künneth component of $c_1(\mathcal{L})$ does not depend upon the choice of the Poincaré bundle. Let q_1 and q_2 denote the obvious projections of $X \times \mathcal{N} \times J$ onto $X \times \mathcal{N}$ and $X \times J$, respectively. The class θ is algebraic for the same reason as for b. Consider

$$\omega := q_1^*b \cup q_2^*\theta \in H^{2i+4}(X \times \mathcal{N} \times J, \mathbf{Q}).$$

Now, since both b and θ are algebraic, so is the class ω. Let

$$p : X \times \mathcal{N} \times J \to \mathcal{N} \times J$$

be the obvious projection, and p_* the corresponding Gysin map of cohomologies. It is a straightforward calculation (the details are in the proof of the Proposition 2.4 of [BN]) to check that $t = p_*\omega$. In other words, t is the cycle class of the push-forward of a cycle whose cycle class is ω. Thus the cohomology class $\tau_{i,j}$ must be algebraic. This completes the proof of the assertion that any invariant class is algebraic.

To show that Φ is an invariant for the action of $G_{\mathbf{Z}}$, first note that, since the classes a_i are canonical in the sense that they do not depend upon the choice of the universal bundle, these classes are invariant under the monodromy action on $H^*(X \times \mathcal{N}, \mathbf{Q})$ for a family of curves. This, in turn, implies that the class Φ is an invariant for the monodromy action on $H^*(\mathcal{N} \times \mathcal{J}, \mathbf{Q})$. The monodromy action factors through $G_{\mathbf{Z}}$ [B1]. Now, since the mapping class group projects onto $G_{\mathbf{Z}}$, the class Φ must be an invariant for the action of $G_{\mathbf{Z}}$. This completes the proof of the theorem. $\square$

Theorem 2.5 implies the following: by virtue of the natural identification of $H^*(\mathcal{J}, \boldsymbol{Q})$ with $H_*(\mathcal{J}, \boldsymbol{Q})$ using the Poincaré duality on X, the homomorphism $\bar{F}$ in (2.4) maps any algebraic class in $H^*(\mathcal{J}, \boldsymbol{Q})$ into an algebraic class in $H^*(\mathcal{N}, \boldsymbol{Q})$. In the case where $r = 2$, this is in fact a consequence of Proposition 6.1 of [BKN].

Fix, once and for all, a polynomial $P(\{\tau_{i,j}\})$ of $\binom{2r-2}{2}$ variables (same as the number of $\tau_{i,j}$) and with coefficients in $\boldsymbol{Q}$ such that

$$P(\{\tau_{i,j}\}) = \Phi\,.$$

We will show that there is a natural element in the Chow group of $\mathcal{N} \times \mathcal{J}$ whose cycle class is Φ. Before proceeding, we note that the Chern classes of an algebraic vector bundle can be defined in the Chow group [F]. Let $\hat{a}_i \in \mathrm{CH}^i(X \times \mathcal{N})$ be the element of the Chow group given by (2.1), with any β_j being considered as an element of $\mathrm{CH}^j(X \times \mathcal{N})$. So the cycle class of $\hat{a}_i$ is a_i.

There is a natural choice of an element in the Chow group of $\mathcal{N} \times \mathcal{J}$, namely

$$E_{i,j} \in \mathrm{CH}^*(\mathcal{N} \times \mathcal{J})$$

such that the cycle class of $E_{i,j}$ is the cohomology class $\tau_{i,j}$. Indeed, for the two cases $r \leq i, j \leq 2(r-1)$ and $i \leq r-1 < j$, in the proof of Theorem 2.5, we have produced an explicit element of the Chow group of $\mathcal{N} \times \mathcal{J}$ (depending only upon the point x_0) whose cycle class is $\tau_{i,j}$. We consider the remaining case, namely $1 \leq i \leq j \leq r-1$. As in the proof of Theorem 2.5, let t denote the cohomology class on $\mathcal{N}$ which pulls back to $\tau_{i,j}$ by the projection of $\mathcal{N} \times \mathcal{J}$ onto $\mathcal{N}$. Let a' denote the cycle on $\mathcal{N}$ obtained by taking the intersection of the cycle $\hat{a}_{j+1}$ with $x_0 \times \mathcal{N}$, and let c' be the cycle on $\mathcal{N}$ given by the push-forward of $\hat{a}_{j+1}$ using the projection of $X \times \mathcal{N}$ onto $\mathcal{N}$. Imitating the Künneth decomposition (2.8), define the cycle b_j on $\mathcal{N}$ by

$$b_j := \hat{a}_{j+1} - a' \otimes 1 - c' \otimes [X]\,.$$

Similarly, construct b_i from $\hat{a}_{i+1}$. Now consider the image of the intersection cycle $b_i \cap b_j (\in \mathrm{CH}^*(X \times \mathcal{N}))$ in $\mathrm{CH}^*(\mathcal{N})$ by the push-forward map for the projection of $X \times \mathcal{N}$ onto $\mathcal{N}$. The cycle class of this cycle on $\mathcal{N}$ is the cohomology class t. This is a straightforward computation, which is already done in the proof of Proposition 2.4 of [BN]. Thus the cycle class of the pullback of this cycle to $\mathcal{N} \times \mathcal{J}$, which we will denote by $E_{i,j}$, is actually $\tau_{i,j}$.

Define

$$\Gamma := P(\{E_{i,j}\}) \in \mathrm{CH}^*(\mathcal{N} \times \mathcal{J})\,, \tag{2.9}$$

where $E_{i,j}$ and the polynomial P are defined as above. Thus we have that the cycle class of Γ is Φ.

3. Map between cycles. Using the cycle Γ defined in Section 2, we get a map (correspondence) from the Chow group $\mathrm{CH}^*(\mathcal{J})$ of $\mathcal{J}$ to $\mathrm{CH}^*(\mathcal{N})$ [F, Definition 16.1.2], which we will now describe. Let q_1 (respectively, q_2) be the projection of $\mathcal{N} \times \mathcal{J}$ onto $\mathcal{N}$ (respectively, $\mathcal{J}$). For a cycle c on $\mathcal{J}$ consider the intersection cycle, $q_2^* c \cap \Gamma$, on $\mathcal{N} \times \mathcal{J}$. Let $q_{1*}(q_2^* c \cap \Gamma)$

be the cycle on $\mathcal{N}$ obtained by taking the push-forward of $q_2^* c \cap \Gamma$ by the projection q_1 [F, §1.4]. Now let

$$\Pi : \mathrm{CH}^*(\mathcal{J}) \to \mathrm{CH}^*(\mathcal{N}) \tag{3.1}$$

be the homomorphism defined by $c \mapsto q_{1*}(q_2^* c \cap \Gamma)$.

With a slight abuse of notation, the Gysin map $H^*(\mathcal{N} \times \mathcal{J}, \boldsymbol{C}) \to H^*(\mathcal{N}, \boldsymbol{C})$, for the projection q_1, will also be denoted by q_{1*}. Consider the homomorphism

$$[\Phi] : H^*(\mathcal{J}, \boldsymbol{C}) \to H^*(\mathcal{N}, \boldsymbol{C}) \tag{3.2}$$

defined by $a \mapsto q_{1*}((q_2^* a) \cup \Phi)$. Note that $[\Phi]$ maps $H^*(\mathcal{J}, \boldsymbol{Q})$ into $H^*(\mathcal{N}, \boldsymbol{Q})$. Moreover, $[\Phi]$ maps the space of Hodge cycles on $\mathcal{J}$ into that on $\mathcal{N}$. The cycle class $[\Pi(c)] \in H^*(\mathcal{N}, \boldsymbol{Q})$ is actually $[\Phi]([c])$.

Using the Poincaré duality on $\mathcal{J}$, we have an isomorphism

$$\rho : H^*(\mathcal{J}, \boldsymbol{Q}) \to H_*(\mathcal{J}, \boldsymbol{Q})\,. \tag{3.3}$$

It is a routine calculation to check that for any $\omega \in H^*(\mathcal{J}, \boldsymbol{Q})$, the equality

$$[\Phi](\omega) = \bar{F}(\rho(\omega)) \tag{3.4}$$

holds. (The homomorphism $\bar{F}$ was defined in (2.4).) From (3.4) it follows that for a cycle c on $\mathcal{J}$,

$$[\Pi(c)] = (\bar{F} \circ \rho)([c])\,.$$

We give an application of the correspondence using Γ.

It is easy to construct a curve X of genus g such that the rank of the Neron-Severi group of its Jacobian, J, is greater than one. (Take a curve X with a nontrivial automorphism, f, such that the genus of the quotient curve X/f is at least one. Then pullback the theta line bundle on the Jacobian $J(X/f)$ to J using the map defined by $L \mapsto \det(\pi_* L)$, where π is the quotient map. The first Chern class of this pullback bundle and that of the theta line bundle on J are linearly indpendent in $\mathrm{NS}(X) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$.) Let $a \in H^2(J, \boldsymbol{Q})$ be an element of the Neron-Severi group of J which is linearly independent with the theta bundle on J. Since $H^1(X, \boldsymbol{Q}) = H_1(X, \boldsymbol{Q})$ by the Poincaré duality on X, we get that

$$H^2(J, \boldsymbol{Q}) = \bigwedge^2 H^1(X, \boldsymbol{Q}) = \bigwedge^2 H_1(X, \boldsymbol{Q}) = H_2(J, \boldsymbol{Q})\,.$$

Let $\bar{a} \in H_2(J, \boldsymbol{Q})$ be the element corresponding to a. Define

$$b_k := 1 \otimes \cdots \otimes \bar{a} \otimes 1 \otimes \cdots \otimes 1 \in H_2(\mathcal{J}, \boldsymbol{Q})$$

to be the element of the Künneth decomposition of $H_2(\mathcal{J}, \boldsymbol{Q})$, where $\bar{a}$ is at the k-th position.

It is easy to check that $\rho^{-1}(b_k)$ is a Hodge cycle on $\mathcal{J}$. Since any Hodge cycle of degree $2d-2$ on a projective manifold of complex dimension d is actually algebraic, the cohomology class $\rho^{-1}(b_k)$ must be algebraic.

Consider $\bar{F}(b_k) \in H^{4k+2}(\mathcal{N}, \boldsymbol{Q})$, which is a Hodge cycle on $\mathcal{N}$. The class $\bar{F}(b_k)$ is not zero, for example, when $r = 2$ and $k = 1$ ([N], [KN]). Let C be a cycle on $\mathcal{J}$ whose cycle class is $\rho^{-1}(b_k)$. The equality (3.4) implies that the cohomology class $\bar{F}(b_k)$ is the cycle class of $\Pi(C)$. Thus the Hodge cycle $\bar{F}(b_k)$ must be algebraic.

The cohomology class $\rho^{-1}(b_k) \in H^*(\mathcal{J}, \boldsymbol{Q})$ is not an invariant for the action of $G_{\boldsymbol{Z}}$. Indeed, the invariants in $H^2(J, \boldsymbol{Q})$ are spanned by the class of the theta line bundle. Thus the cohomology class a, and hence b_k, is not an invariant for the action of $G_{\boldsymbol{Z}}$. If $\bar{F}(b_k)$ is not zero, then it is not an invariant for the action of $G_{\boldsymbol{Z}}$, since the homomorphism $\bar{F}$ is equivariant for the action of $G_{\boldsymbol{Z}}$. In [BN] it was proved that the space of Hodge cycles on $\mathcal{N}$ for a general Riemann surface is exactly the space of invariants for the action of $G_{\boldsymbol{Z}}$. The above construction shows that the "generality" is essential. In other words, for special $\mathcal{N}$ there are more Hodge cycles (and also algebraic classes) than those simply being the invariants.

4. Map between Deligne-Beilinson cohomologies. Let Y be a connected smooth projective variety over $\boldsymbol{C}$. For $p \geq 0$, let $\boldsymbol{Z}(p)$ denote the constant sheaf $\boldsymbol{Z} \cdot (2\pi\sqrt{-1})^p$ on Y. The following complex

$$\mathcal{D}(p)Y : 0 \to \boldsymbol{Z}(p) \to \mathcal{O} \xrightarrow{d} \Omega^1_Y \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_Y \to 0\,,$$

where the sheaf $\boldsymbol{Z}(p)$ is at the 0-th position, is called the p-th *Deligne complex.* Since we are working over the field of complex numbers, the distinction between $\boldsymbol{Z}(p)$ and $\boldsymbol{Z}$ is not important for us. The j-th hypercohomology group of $\mathcal{D}(p)Y$ is called the j-th *Deligne-Beilinson cohomology group*, and it is denoted by $H^j_{\mathcal{D}}(Y, p)$ [EV, 1.1].

Consider the following complex, $\mathcal{C}(p)$, of $\mathcal{O}_Y$-modules

$$\mathcal{C}(p) : 0 \to \mathcal{O} \xrightarrow{d} \Omega^1_Y \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1}_Y \to 0\,.$$

The following exact sequences of complexes, that is,

$$0 \to \mathcal{C}(p)[1] \to \mathcal{D}(p) \to \boldsymbol{Z}(p) \to 0\,,$$

where $\mathcal{C}(p)[1]$ denotes the one step right translation of $\mathcal{C}(p)$ (i.e., $\mathcal{O}$ is in the first position), induces the following exact sequence [EV, 7.9]

$$0 \to J^p(Y) \to H^{2p}_{\mathcal{D}}(Y, p) \xrightarrow{\gamma} \mathrm{Hg}^p(Y) \to 0\,, \tag{4.1}$$

where $J^p(Y)$ is the p-th Griffiths intermediate Jacobian, and $\mathrm{Hg}^p(Y)$ is the subgroup of $H^{2p}(Y, \boldsymbol{Z}(p))$ consisting of those elements in $H^{2p}(Y, \boldsymbol{Z}(p))$ which are of type (p, p).

Let $H^*_{\mathcal{D}}(Y)$ denote the direct sum $\bigoplus_{p\geq 0} H^{2p}_{\mathcal{D}}(Y, p)$. The cycle map in the Deligne-Beilinson cohomology

$$\psi : \mathrm{CH}^*(Y) \to H^*_{\mathcal{D}}(Y) \tag{4.2}$$

is a ring homomorphism [EV, Corollary 7.7]. For $c \in \mathrm{CH}^p(Y)$, the class

$$(\gamma \circ \psi)(c) \in \mathrm{Hg}^p(Y)$$

(γ is defined in (4.1)) is $(2\pi\sqrt{-1})^p$ times the integral class corresponding to c. If c is homologically equivalent to zero, then $\psi(c)$, as an element of $J^p(Y)$, coincides with the image of c in $J^p(Y)$ under the Abel-Jacobi map [EV, Theorem 7.11].

The direct sum $\bigoplus_p H^{2p}_{\mathcal{D}}(Y, p)$ will be denoted by $H^*_{\mathcal{D}}(Y)$.

There is the pullback homomorphism $q_2^* : H^*_{\mathcal{D}}(\mathcal{J}) \to H^*_{\mathcal{D}}(\mathcal{N} \times \mathcal{J})$ such that for any cycle c on $\mathcal{J}$

$$\psi(q_2^*(c)) = q_2^*(\psi(c)),$$

where ψ is the homomorphism defined in (4.2) [EV, Proposition 7.5].

We know that for a cycle d on $\mathcal{N} \times \mathcal{J}$, the following equality holds:

$$\psi(d \cdot \Gamma) = \psi(d) \cup \psi(\Gamma).$$

The push-forward map, which corresponds to the Gysin map of cohomology,

$$q_{1*} : H^*_{\mathcal{D}}(\mathcal{N} \times \mathcal{J}) \to H^*(\mathcal{N})$$

satisfies the condition that for any cycle d on $\mathcal{N} \times \mathcal{J}$ the following equality holds:

$$\psi(q_{1*}(d)) = q_{1*}(\psi(d)),$$

where $q_{1*}(d)$ is the push-forward of d.

The map Π, defined in (3.1), is the composite of a pullback, the intersection with Γ, and a push-forward. Thus, from the above observations it follows that for any cycle c on $\mathcal{J}$,

$$\psi(\Pi(c)) = [\Phi](\psi(c))$$

($[\Phi]$ was defined in (3.2)).

We want to describe how one can construct, using the map Π, nonzero elements of Griffiths group of $\mathcal{N}$. But before that we need to set up some notation.

For any integer $l \geq 0$, let

$$\Gamma_l \subset \Gamma \tag{4.3}$$

be the union of components of Γ of codimension l. In other words, $\Gamma_l \in \mathrm{CH}^l(\mathcal{J} \times \mathcal{N})$. Let

$$[\Phi_l] : H^*(\mathcal{J}, \mathbf{C}) \to H^*(\mathcal{N}, \mathbf{C}) \tag{4.4}$$

be the induced map of cohomologies obtained by taking Γ_l as the correspondence cycle. Note that $[\Phi_l]$ maps $H^*(\mathcal{J}, \mathbf{Q})$ to $H^*(\mathcal{N}, \mathbf{Q})$.

Let p_i, where $1 \leq i \leq r-1$, denote the projection of $\mathcal{J}$ onto the i-th factor. It is easy to check that for any $\omega \in H^k(J, \mathbf{C})$, the cohomology class $[\Phi](p_i^*\omega) \in H^*(\mathcal{N}, \mathbf{C})$ is actually of degree

$$\sum_{j=2}^{r} 2g(2j-1) - k(2i+1) = 2g(r^2-1) - k(2i+1).$$

On the other hand, we have that

$$[\Phi_l](p_i^*\omega) \in H^{2l+k-2g(r-1)}(\mathcal{N}, \mathbf{C}). \tag{4.5}$$

Since Γ_l is the union of components of codimension l of Γ, when $l = g(r^2+r-2) - k(i+1)$, the equality

$$[\Phi](p_i^*\omega) = [\Phi_l](p_i^*\omega) \tag{4.6}$$

is valid for any $\omega \in H^k(J, \mathbf{C})$.

For any $k \in N$, let $f : X^k \to J$ be the morphism defined by $(x_1, \ldots, x_k) \mapsto \sum_j (x_j - x_0)$, where x_0 is the base point of the pointed curve. Let W_k denote the cycle on J given by the image of f. Define $C_k := W_{g-k} - W_{g-k}^-$ to be the cycle of codimension k, where W_{g-k}^- denotes the image of W_{g-k} under the involution $j \mapsto -j$ of J. Clearly, C_k is homologically equivalent to zero. Ceresa in [C] proved that if $g \geq 3$ and $1 \leq k \leq g-1$, then for a generic curve X the cycle C_k is not algebraically equivalent to zero, i.e., it represents a nonzero element of the Griffiths group of J.

Define $C_{k,i}$ to be the following pullback cycle on $\mathcal{J}$:

$$C_{k,i} := p_i^* C_k ,$$

where p_i is the projection of $\mathcal{J}$ onto the i-th factor. The cycle $C_{k,i}$ is homologically equivalent to zero, since C_k is homologically equivalent to zero.

Let Π_l denote the correspondence map between Chow groups obtained by taking Γ_l as the correspondence cycle (as in (3.1)). In other words, for any $c \in \mathrm{CH}^*(\mathcal{J})$, we have

$$\Pi_l(c) := q_{1*}(q_2^* c \cap \Gamma_l) \in \mathrm{CH}^*(\mathcal{N}) .$$

Since $C_{k,i}$ is homologically equivalent to zero, so is the cycle $\Pi_l(C_{k,i})$.

Let $L \in H^2(J, \mathbf{Z})$ be the polarization on the Jacobian J given by the Poincaré dual of the theta divisor (which is an ample divisor on J). Let $P_k \subset H^k(J, \mathbf{C})$ denote the space of primitive cohomology classes. Consider the Lefschetz decomposition

$$H^{2p-1}(J, \mathbf{Q}) = \bigoplus_{1 \leq k \leq p} L^{p-k} \bigwedge P_{2k-1}^{\mathbf{Q}} , \tag{4.7}$$

where $P_j^{\mathbf{Q}}$ is the space of all rational primitive cohomology classes of degree j, i.e., $P_j^{\mathbf{Q}} = P_j \cap H^j(J, \mathbf{Q})$. Recall that $G_{\mathbf{Z}}$ denotes the group of all automorphisms of $H_1(X, \mathbf{Z})$ preserving the cap product. The subspaces $P_j^{\mathbf{Q}}$ are all irreducible $G_{\mathbf{Z}}$ modules, and (4.7) is the decomposition into irreducible $G_{\mathbf{Z}}$ modules. Indeed, this is a consequence of the fact that the complexification of the decomposition (4.7) is the irreducible decomposition of the $G_{\mathbf{C}}$ module $H^{2p-1}(J, \mathbf{C})$, where $G_{\mathbf{C}}$ is the complexification of $G_{\mathbf{Z}}$, i.e., the group of all automorphisms of $H_1(X, \mathbf{C})$ preserving the cap product. As a $G_{\mathbf{Z}}$ module, the irreducibility of any $P_{2k-1}^{\mathbf{Q}}$ is a consequence of the Borel density theorem which asserts that $G_{\mathbf{Z}}$ is Zariski dense in $G_{\mathbf{C}}$.

Let $\mathcal{M}_g^1$ denote the moduli space of one pointed Riemann surfaces of genus g. In other words, it is the moduli space of pairs of the form (X, p), where p is a point of a Riemann surface X of genus g.

THEOREM 4.8. *Let g and p be integers such that $g - 1 \geq p \geq 1$. Assume that the image $[\Phi_l](p_i^*(L^{p-2} P_3^{\mathbf{Q}}))$ is a nontrivial subspace of $H^*(\mathcal{N}, \mathbf{Q})$. Then there is a countable union of proper subvarieties, say $\bigcup_k S_k$, of the moduli space $\mathcal{M}_g^1$ with the following property: For any $(X, x_0) \in \mathcal{M}_g^1 - \bigcup_k S_k$, the cycle $\Pi_l(C_{p,i})$ on $\mathcal{N}$ is not algebraically equivalent to zero, and it is of infinite order as an element of the Griffiths group of $\mathcal{N}$.*

Before proving the theorem we remark some observations.

REMARK 4.9. (1) The condition that $[\Phi_l](p_i^*(L^{p-2}P_3^{\mathbf{Q}}))$ is a nontrivial subspace is independent of the choice of the curve X. For a smooth family of pointed curves, the subspace $p_i^*(L^{p-2}P_3^{\mathbf{Q}}) \subset H^{2p-1}(\mathcal{J}, \mathbf{Q})$ gives rise to a sub-local system of the local system on the parameters space given by $H^{2p-1}(\mathcal{J}, \mathbf{Q})$. The homomorphism Φ_l actually gives a homomorphism of local systems. Thus, for a smooth family of curves parametrized by a connected space, if $[\Phi_l](p_i^*(L^{p-2}P_3^{\mathbf{Q}}))$ is zero for one curve, then it is zero for all curves in the family.

(2) Set $r = 2$ and $l = 6$. So the equality (4.6) is valid with $k = 2g-3$. Now using the equality (3.4), the image $[\Phi_l](L^{p-2}P_3^{\mathbf{Q}})$ is identified with $\bar{F}$ on $\bigwedge^3 H_1(X, \mathbf{C})$. If $g \geq 4$, then $\bar{F}$ maps $\bigwedge^3 H_1(X, \mathbf{C})$ injectively into $H^9(\mathcal{N}, \mathbf{Q})$ ([N], [KN, Proposition 2.1]). So in this situation Theorem 4.8 implies that, for a general one pointed curve (X, x_0), the cycle $\Pi_l(C_{g-1})$ is not algebraically equivalent to zero.

(3) The cycle $\Pi(C_{p,i})$ is not algebraically equivalent to zero if the component $\Pi_l(C_{p,i})$ is so. However we consider $\Pi_l(C_{p,i})$ to get a cycle of pure dimension.

PROOF OF THEOREM 4.8. The image $[\Phi_l](p_i^*(H^{2p-1}(J, \mathbf{C})))$ is contained in $H^m(\mathcal{N}, \mathbf{Q})$, where m is some odd integer which depends on l, i, p and r. Consider the *Griffiths intermediate Jacobian*

$$J^{(m+1)/2}(\mathcal{N}) := \frac{H^m(\mathcal{N}, \mathbf{C})}{F^{(m+1)/2} + H^m(\mathcal{N}, \mathbf{Z}((m+1)/2))}$$

of $\mathcal{N}$, where $F^{(m+1)/2} = \bigoplus_{j\geq 0} H^{(m+1)/2+j,(m-1)/2-j}(\mathcal{N})$. (Note that m is odd.) Since both $[\Phi_l]$ and the pullback operation on cohomology, namely p_i^*, are given by correspondence cycles, we have a homomorphism

$$\nu : J^p(J) := \frac{H^{2p-1}(J, \mathbf{C})}{F^p + H^{2p-1}(J, \mathbf{Z}(p))} \to J^{(m+1)/2}(\mathcal{N}),$$

where $F^p = \bigoplus_{j\geq 0} H^{p+j,p-1-j}(J)$.

Since the Abel-Jacobi map ψ, defined in (4.2), is compatible with correspondence, the following equality is valid:

$$\nu(\psi(C_{p,i})) = \psi(\Pi_l(C_{p,i})). \tag{4.10}$$

If $\Pi_l(C_{p,i})$ is algebraically equivalent to zero, then there is a closed algebraic subgroup of the complex torus $J^{(m+1)/2}(\mathcal{N})$ which contains $\psi(\Pi_l(C_{p,i}))$ [H1, Proposition 10.1]. Assuming that $\Pi_l(C_{p,i})$ is algebraically equivalent to zero, denote this algebraic subgroup of $J^{(m+1)/2}(\mathcal{N})$ by A.

Consider the following complex torus:

$$J_3 := \frac{L^{p-2}P_3}{(F^p + H^{2p-1}(J, \mathbf{Z}(p))) \cap L^{p-2}P_3}, \tag{4.11}$$

where $L^{p-2}P_3$ is a factor of the Lefschetz decomposition in (4.7). Using the decomposition (4.7), a finite (unramified) covering of the torus $J^p(J)$, say $\hat{J}$, is identified with the Certesian product $J_3 \times B$, where B is a complex torus. Since the decomposition (4.7) is not necessarily over $\mathbf{Z}$, we need to go to a finite covering of $J^p(J)$. Let $\hat{\nu}$ be the homomorphism from $\hat{J}$ to $J^{(m+1)/2}(\mathcal{N})$ given by ν. Let f (respectively g) denote the projection (respectively inclusion) homomorphism between $\hat{J}$ and J_3.

In the statement of Theorem 4.8 it is assumed that the image $[\Phi_l](p_i^*(L^{p-2}P_3^{\mathbf{Q}}))$ is a nonzero subspace of $H^m(\mathcal{N},\mathbf{Q})$. Now, the homomorphism $[\Phi_l]$ in (4.4) is equivariant for the action of $G_{\mathbf{Z}}$ described in Section 2. Indeed, this is a consequence of the fact that the homomorphism $\bar{F}$ in (2.4) is equivariant for the action of $G_{\mathbf{Z}}$ ([B1], [BN]). Since $P_3^{\mathbf{Q}}$ is an irreducible $G_{\mathbf{Z}}$ module, $[\Phi_l](p_i^*(L^{p-2}P_3^{\mathbf{Q}}))$ must be isomorphic to $P_3^{\mathbf{Q}}$ as a $G_{\mathbf{Z}}$ module. This implies that $\hat{\nu}\circ g$ is an embedding of J_3 in $J^{(m+1)/2}(\mathcal{N})$. (The homomorphism $\hat{\nu}\circ g$ coincides with the restriction of ν to the image of J_3 in $J^p(J)$ by the obvious inclusion.) Moreover, since (4.7) is an isotypical decomposition, i.e., all the $P_{2k-1}^{\mathbf{Q}}$ are distinct $G_{\mathbf{Z}}$ modules, we conclude that for any $k \neq 2$, the two subspaces of $H^m(\mathcal{N},\mathbf{Q})$, namely $[\Phi_l](p_i^*(L^{p-2}P_3^{\mathbf{Q}}))$ and $[\Phi_l](p_i^*(L^{p-k}P_{2k-1}^{\mathbf{Q}}))$, belong to two distinct components of the isotypical decomposition of the $G_{\mathbf{Z}}$ module $H^m(\mathcal{N},\mathbf{Q})$.

Consider the natural projection of $G_{\mathbf{Z}}$ module $H^m(\mathcal{N},\mathbf{Q})$ onto its isotypical component corresponding to the $G_{\mathbf{Z}}$ module $P_3^{\mathbf{Q}}$. This projection induces a projection, denoted by h, of the torus $J^{(m+1)/2}(\mathcal{N})$ to another complex torus, say T. Evidently, $h\circ\hat{\nu}\circ g$ is a homomorphism from J_3 with finite kernel (recall that $\hat{\nu}\circ g$ is an embedding). Given this, since $f(\psi(C_{p,i}))$ in J_3 is of infinite order (a consequence of Proposition 8.8 of [H2]), the equality (4.10) implies that the element $h(\psi(\Pi_l(C_{p,i})))$ is of infinite order in T. Since A is an abelian variety, so is $h(A)$. Since $h(A)$ contains an element of infinite order, namely $h(\psi(\Pi_l(C_{p,i})))$, it must be of strictly positive dimension.

Thus $(h\circ\hat{\nu}\circ g)^{-1}(h(A))$ is an algebraic subgroup of J_3 of strictly positive dimension. That the dimension of $(h\circ\hat{\nu}\circ g)^{-1}(h(A))$ is strictly positive is a consequence of the fact that the order of the element $f(\psi(C_{p,i}))$ of $(h\circ\hat{\nu}\circ g)^{-1}(h(A))$ is infinite.

We will now complete the proof of Theorem 4.8 imitating the argument given in the proof of Theorem 10.3 of [H1]. The Lemma 10.2 in [H1, page 125] asserts the following: let S be the set of all curves, X, such that the complex torus J_3 (defined in 4.11) corresponding to X contains a closed algebraic subgroup (abelian variety) of strictly positive dimension. Then the subset of $\mathcal{M}_g^1$ defined by S is actually contained in a countable union of proper closed subvarieties of $\mathcal{M}_g^1$. In view of this lemma, the proof of Theorem 4.8 is completed. □

REMARK 4.12. The product $\mathcal{N}\times J$ is an étale cover of $M(r,d)$, where $M(r,d)$ is the moduli of stable bundle of rank r and degree d. The covering map is given by $(E,L)\mapsto E\times L$. Thus the image in $M(r,d)$, of any cycle on $\mathcal{N}$ (or J), that represents a nontrivial element of the Griffiths group of $\mathcal{N}$ (or J), would represent a nontrivial element of the Griffiths group of $M(r,d)$.

REMARK 4.13. Deligne proved that any Hodge cycle on an abelian variety is an absolute Hodge cycle [D, Main Theorem 2.11]. So, in particular, any Hodge cycle on any self-product (arbitrary number of times) of a Jacobian is an absolute Hodge cycle. Using this and the existence of the correspondence cycle Γ, it is easy to check that any Hodge cycle on $\mathcal{N}$ is an absolute Hodge cycle. In the case of rank two, this was proved in [B2].

Both Theorem 2.5 and Theorem 4.8 generalize to any (smooth) moduli space of parabolic bundles. Indeed, the results of [BR] and [BN] used here are actually proved in that generality. If $\mathcal{N}$ denotes a smooth moduli space of parabolic bundles of rank r and fixed determinant, then the homomorphism $\bar{F}$ in (2.4) remains valid. The monodromy action of the mapping class group, for Riemann surfaces with marked points, on the cohomology of a smooth moduli space of parabolic bundles factors through the symplectic group. Since invariant rational cohomology classes continue to be algebraic, the argument for Theorems 2.4 and 4.8 remains valid.

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