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Principal G-bundles on nodal curves

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Abstract. Let G be a connected semisimple affine algebraic group defined over **C**. We study the relation between stable, semistable G-bundles on a nodal curve Y and representations of the fundamental group of Y. This study is done by extending the notion of (generalized) parabolic vector bundles to principal G-bundles on the desingularization C of Y and using the correspondence between them and principal G-bundles on Y. We give an isomorphism of the stack of generalized parabolic bundles on C with a quotient stack associated to loop groups. We show that if G is simple and simply connected then the Picard group of the stack of principal G-bundles on Y is isomorphic to $\oplus_m \mathbf{Z}$, m being the number of components of Y.

Keywords. Principal bundles; loop groups; parabolic bundles.

0. Introduction

Let G be a connected semisimple affine algebraic group defined over **C**. Let Y be a reduced curve with only singularities ordinary nodes y_j, $j = 1, \ldots, J$. Let Y_i, $i = 1, \ldots, I$ be the irreducible components of Y and C_i the desingularization of Y_i. Let C denote the disjoint union of all C_i. We introduce the notions of stability and semistability for principal G-bundles on Y (§2). If Y is reducible these notions depend on parameters $a = (a_1, \ldots, a_I)$. The study of G-bundles on Y is done by extending the notion of (generalized) parabolic vector bundles [U1] to generalized parabolic principal G-bundles (called GPGs in short) on the curve C and using the correspondence between them and principal G-bundles on Y (2.4, 2.11). We study the relation between stable, semistable G-bundles and representations of the fundamental group of Y. Let $\rho : \pi_1(Y) \to G$ be a representation of the fundamental group $\pi_1(Y)$ of Y in G. For $i = 1, \ldots, I$, let $f_i : \pi_1(Y_i) \to \pi_1(Y)$ be the natural maps, $\rho_i = \rho \circ f_i$.

Theorem 1. (I) *If Y is irreducible and $\rho \mid \pi_1(C)$ is unitary* (*resp. irreducible unitary*) *then the principal G-bundle on Y associated to ρ is semistable* (*resp. stable*). *The converse is not true.*
(II) *If Y is reducible then there exist infinitely many I-tuples of positive rational numbers $a_1, \ldots, a_I$ with $\sum a_i = 1$, depending only on the graph of Y and $g(C_i)$ such that for $a = (a_1, \ldots, a_I)$ the following statements are true.*

(1) *If $\rho_{C_i} = \rho_i \mid \pi_1(C_i)$ are unitary representations for all i, then the principal G-bundle $\mathcal{F}$ on Y associated to ρ is a-semistable.*

(2) *If ρ_{C_i} are irreducible unitary representations for all i, then the principal G-bundle $\mathcal{F}$ associated to ρ is a-stable.*

Let Aff/k be the flat affine site over the base field $k = \mathbf{C}$, i.e. the category of k-algebras equipped with $fppf$ topology. Let R denote a k-algebra, $C_{i,R} := C_i \times$ spec R and $C^*_R = C^* \times$ spec R. For each i, fix a point $p_i \in C_i$ such that p_i maps to a smooth point of Y. Let q_i be a local parameter at the point $p_i, i = 1, \ldots, I$. Let $L_{G,i}$ denote the k-group defined by associating to R the group $G(R(q_i))$. Let $L^+_{G,i}$ (resp. $L^{C_i}_G$) be the k-group defined by associating to R the group $G(R[[q_i]])$ (resp. $G(\Gamma(C^*_{i,R}, \mathcal{O}_{C^*_{i,R}})))$. Define $L_G = \prod_i L_{G,i}$, $L^+_G = \prod_i L^+_{G,i}$, $L^C_G = \prod_i L^{C_i}_G$. Let

$$Q_{G,C} = L_G/L^+_G = \prod_i L_{G,i}/L^+_{G,i}, \quad Q^{\mathrm{gpar}}_{G,C} = Q_{G,C} \times \prod_j G(\mathbf{C}).$$

The indgroup L^C_G acts on $Q^{\mathrm{gpar}}_{G,C}$. Let $L^C_G \backslash Q^{\mathrm{gpar}}_{G,C}$ be the quotient stack. Let $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$ denote the stack of GPGs on C (this is isomorphic to the stack of principal G-bundles on Y.)

Theorem 2. *There exists a canonical isomorphism of stacks*

$$\overline{\pi}_{\mathrm{par}} : L^C_G \backslash Q^{\mathrm{gpar}}_{G,C} \tilde{\to} \mathrm{Bun}^{\mathrm{gpar}}_{G,C}.$$

Moreover the projection map $Q^{\mathrm{gpar}}_{G,C} \to \mathrm{Bun}^{\mathrm{gpar}}_{G,C}$ is locally trivial for etale topology.

Theorem 3. *If G is a simple, connected and simply connected affine algebraic group then*
(1)

$$\mathrm{Pic}(\mathrm{Bun}^{\mathrm{gpar}}_{G,C}) \approx \oplus_i \mathbf{Z}.$$

(2) *If Y is irreducible and C has genus ≥ 2, then*

$$\mathrm{Pic}(\mathrm{Bun}^{\mathrm{gpar}}_{G,C})^{\mathrm{ss}} \approx \mathbf{Z},$$

where $^{\mathrm{ss}}$ denotes semistable points.

The moduli spaces of principal G-bundles on singular curves are not complete. In case $G = GL(n)$ (resp. $G = O(n), Sp(2n)$) the compactifications of these moduli spaces were constructed as moduli spaces of torsionfree sheaves (resp. orthogonal or symplectic sheaves) on Y. For a general reductive group G neither the moduli spaces nor the compactifications have been constructed on Y yet. One way to construct (normal) compactifications of these moduli spaces is to use GPGs on C, for this one needs a good compactification of G. In case G is $GL(n)$, $SL(n)$, $O(n)$ or $Sp(2n)$ we use a compactification F of G obtained by using the natural representation and construct the normal compactifications of moduli spaces ([U1, U2, U4]). In case G is of adjoint type we use the good compactification F of G defined by Deconcini and Procesi. We define 'a compactification' $\overline{\mathrm{B}}\mathrm{un}^{\mathrm{gpar}}_{G,C}$ of $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$ using F and show that it is isomorphic to the quotient stack $L^C_G \backslash Q_{G,C} \times \prod_j F$. We prove that if further G is simple and simply connected then Pic $\overline{\mathrm{B}}\mathrm{un}^{\mathrm{gpar}}_{G,C} \approx \oplus_i \mathbf{Z} \oplus \oplus_j \mathrm{Pic}\ F$ (Theorem 4).

1. Quasiparabolic bundles

1.1. *Notations*. Let the base field be $\mathbf{C}$ (or an algebraically closed field of characteristic 0). Let I, J be natural numbers. Let Y be a connected reduced (projective) curve with ordinary nodes as singularities. Let Y_i, $i = 1, \ldots, I$ be the irreducible components of Y. Let $Y' = Y - \{\text{singular set of } Y\}$, $Y'_i = Y' \cap Y_i$ for all i. Let C be the partial desingularization of Y obtained by blowing up nodes y_j, $j = 1, \ldots, J$. Assume that $C = \coprod_1^I C_i$ (a disjoint union). Let $C'_i = C_i - \text{sing}(C_i)$. Fix an orientation of the (dual) graph of Y. In the graph of Y, y_j corresponds to an edge. The initial and terminal points of the edge correspond to curves $Y_{i(j)}$ and $Y_{t(j)}$ respectively, one has $i(j) = t(j)$ if the edge is a loop. Let $x_j \in C_{i(j)}$ and $z_j \in C_{t(j)}$ be the two points of C mapping to $y_j \in Y$ and $D_j = x_j + z_j$, $j = 1, \ldots, J$. For each j, D_j is an effective Cartier divisor on C supported outside the singular set of C. We remark that the parabolic structure we shall define in 1.2, 1.4 depends only on these divisors and not on the choice of orientation. Let G denote an affine connected semisimple algebraic group over $\mathbf{C}$ (or an algebraically closed field of characteristic zero). Let $\mathbf{g}$ denote the Lie algebra of G, $n = \dim \mathbf{g}$. A principal G-bundle E on C is an I-tuple (E_i), E_i being a principal G-bundle on C_i.

DEFINITION 1.2

A quasiparabolic structure σ_j on E over the divisor D_j consists of a G-isomorphism $\sigma_j : E_{i(j),x_j} \to E_{t(j),z_j}$ where $E_{i,x}$ denotes the fibre of E_i at x. Let σ be the J-tuple $(\sigma_j)_j$, then (E, σ) is called a quasiparabolic G-bundle, called a QPG in short.

Remark 1.3. A family $(\mathcal{E}, (\sigma_j))$ of QPGs consists of a family of principal G-bundles $\mathcal{E} \to C \times T$ together with an isomorphism of G-bundles $\sigma_j : \mathcal{E} \mid_{x_j \times T} \to \mathcal{E} \mid_{z_j \times T}$ for each $j = 1, \ldots, J$. Given a family of QPGs $(\mathcal{E}, (\sigma_j)) \to C \times T$ and a representation $\rho : G \to GL(V)$ one can associate to it a family $(\mathcal{E}(V), F_j(V)) \to C \times T$ of generalized parabolic vector bundles [U1] as follows. $\mathcal{E}(V) = \mathcal{E} \times_\rho V$ is a family of vector bundles. For each j, σ_j induces $\sigma_{V,j} : \mathcal{E}(V) \mid x_j \times T \to \mathcal{E}(V) \mid z_j \times T$. Let $F_j(V) =$ graph of $\sigma_{V,j}$ in $\mathcal{E}(V) \mid x_j \times T \oplus \mathcal{E}(V) \mid z_j \times T$. Then $F_j(V)$ and $Q_j(V) = (\mathcal{E}(V) \mid x_j \times T \oplus \mathcal{E}(V) \mid z_j \times T)/F_j(V)$ are vector bundles on T of rank = dim V.

1.4. Let α be a real number, $0 \leq \alpha \leq 1$. Taking ρ the adjoint representation of G we get the associated vector bundle $E(\mathbf{g})$. Then $E(\mathbf{g})$ is the adjoint bundle of E and we often denote it by $\text{Ad}E$. The isomorphism σ_j gives an isomorphism $E(\mathbf{g})_{x_j} \to E(\mathbf{g})_{z_j}$ and hence determines an n-dimensional subspace of $E(\mathbf{g})_{x_j} \oplus E(\mathbf{g})_{z_j} = \mathbf{g} \oplus \mathbf{g}$ again denoted by σ_j. Let $\tau_j \in \text{End}_{\mathbf{C}}(\mathbf{g} \oplus \mathbf{g})$ such that τ_j acts on σ_j by $\alpha.Id$ and τ_j restricted to a complement of σ_j in $\mathbf{g} \oplus \mathbf{g}$ is zero. With respect to a suitable basis, $\tau_j = \begin{pmatrix} \alpha I_n & 0 \\ 0 & 0 \end{pmatrix}$, I_n being the unit matrix of rank n. We fix a conjugacy class of τ_j. (This is an analogue of weights in case of (generalized) parabolic vector bundles, the weights in this case being $(0, \alpha)$ for the vector bundle $E(\mathbf{g})$ with induced (generalized) parabolic structure).

We want to define the notions of stability and semistability for QPGs. Since the definitions are rather complicated in the general case, we first define these notions on an irreducible smooth curve C (1.5, 1.6) and later extend these notions to the general case (1.7, 1.8, 1.9).

Assume that C is a nonsingular irreducible curve. Let P be a maximum parabolic subgroup of G and $\mathbf{p}$ its Lie algebra. Let $E/P = E(G/P)$ be the associated fibre bundle with fibres isomorphic to G/P. Let $s : C \to E/P$ be a section i.e. a reduction of the structure group to the maximum parabolic subgroup P. Let Q_j be the stabilizer in

$GL(\mathbf{g}\oplus\mathbf{g})$ of the subspace $E(\mathbf{p})_{x_j}\oplus E(\mathbf{p})_{z_j}=\mathbf{p}\oplus\mathbf{p}\subset\mathbf{g}\oplus\mathbf{g}=E(\mathbf{g})_{x_j}\oplus E(\mathbf{g})_{z_j}$. Let μ_j denote the determinant of the action of Q_j on $\mathbf{g}/\mathbf{p}\oplus\mathbf{g}/\mathbf{p}$. Let $\overline{\mu}_j$ be the form on the Lie algebra $L(Q_j)$ of Q_j corresponding to μ_j. Let $\overline{\tau}_j$ be a conjugate of τ_j in $L(Q_j)$.

DEFINITION 1.5

A QPG $(E,(\sigma_j))$ is α-stable (resp. α-semistable) if for every maximum parabolic P of G and every reduction $s: C\to E/P$, one has

$$\text{degree } s^*T(G/P)+\sum_j\overline{\mu}_j(\overline{\tau}_j) > (\text{resp. } \geq)\,\alpha J.\ \text{rank } s^*T(G/P). \qquad (*1)$$

Here $T(G/P)$ is the tangent bundle along the fibres of $E/P\to C$.

Lemma 1.6. *With the above notations, the condition* $(*1)$ *is equivalent to the following*

$$\text{par deg } E(\mathbf{p}) < (\text{resp. } \leq)\,\alpha J.\ \text{rank } E(\mathbf{p}), \qquad (*2)$$

where par deg $E(\mathbf{p})$ *denotes the parabolic degree of the subbundle* $E(\mathbf{p})$ *of the* (*generalized*) *parabolic vector bundle* $(E(\mathbf{g}),(\sigma))$ *with weights* $(0,\alpha)$, *each weight being of multiplicity* n.

Proof. One has $s^*T(G/P)=E(\mathbf{g}/\mathbf{p}),\sum_j\overline{\mu}_j(\overline{\tau}_j)$ = parabolic weight of the quotient bundle $E(\mathbf{g}/\mathbf{p})$ of $(E(\mathbf{g}),(\sigma_j))$. Thus $(*1)$ can be restated as par deg$E(\mathbf{g}/\mathbf{p}) >$ (resp. $\geq$) αJ rank$E(\mathbf{g}/\mathbf{p})$. Since G is semisimple, deg$E(\mathbf{g})=0$ ([R1], Remark 2.2) and hence par deg$E(\mathbf{g})=\alpha J$ rank $E(\mathbf{g})$. The result now follows from the exact sequence $0\to E(\mathbf{p})\to E(\mathbf{g})\to E(\mathbf{g}/\mathbf{p})\to 0$ using the additivity of parabolic degrees for exact sequences.

1.7. *Semistable QPGs on reducible curves*. Let the notation be as in 1.1. We consider QPGs $(E,(\sigma_j))$ on C with parabolic structure over $D_j=x_j+z_j$, $j=1,\dots,J$. Let $\{\sigma_j\},\{\tau_j\},a,\alpha$ be as in 1.4. For $i=1,\dots,I$ let P_i denote either a maximum parabolic subgroup of G or the trivial group e or the group G itself. We need to consider the cases $P=\{e\}$ or G also, because a sub-object $N=(N_i)$ of $E=(E_i)$ may have the property that for some i, $N_i=E_i$ or N_i is trivial. For an I-tuple $P=(P_1,\dots,P_I)$, let $r_i=\dim\ \mathbf{p}_i, n_i=\dim\ \mathbf{g}/\mathbf{p}_i$ for all i. For $j=1,\dots,J$ denote by Q_j the stabilizer in $GL(\mathbf{g}\oplus\mathbf{g})$ of the subspace $\mathbf{p}_{i(j)}\oplus\mathbf{p}_{t(j)}\subseteq\mathbf{g}\oplus\mathbf{g}$. Let μ_j be the determinant of the action of Q_j on $\mathbf{g}/\mathbf{p}_{i(j)}\oplus\mathbf{g}/\mathbf{p}_{t(j)}$ and $\overline{\mu}_j$ the form on the Lie algebra $L(Q_j)$ of Q_j corresponding to μ_j. Let $\overline{\tau}_j$ be a conjugate of τ_j in $L(Q_j)$. Let $C'_i=C_i$-sing(C_i), $s_i: C'_i\to E(G/P_i)\mid C'_i$ any section, $s=(s_1,\dots,s_I)$. Let S_{s_i} = the largest subsheaf of Ad $E\mid C_i$ such that $S_{s_i}\mid C'_i=s_i^*(E(\mathbf{p}_i))\mid C'_i$. Let $S_s=(S_{s_1},\dots,S_{s_I})$, $\chi(S_s)=\sum_i\chi(S_{s_i})$, $\chi(\text{Ad } E)=\sum_i\chi(\text{Ad } E\mid C_i)$. Let Q_{s_i} be the (smallest) torsion free quotient sheaf of Ad $E\mid C_i$ with $Q_{s_i}\mid C'_i=s_i^*(E(\mathbf{g}/\mathbf{p}_i))\mid C'_i$. Let $Q_s=(Q_{s_i})_i$, $\chi(Q_s)=\sum_i\chi(Q_{s_i})$.

DEFINITION 1.8.

A QPG $(E,(\sigma_j))$ is (a,α)-semistable (resp. (a,α)-stable) if for every reduction s of the structure group to P such that $P_i\neq G$ for all i and $P_i\neq\{e\}$ for all i one has

$$\left[\chi(Q_s) + \sum_j \overline{\mu}_j(\overline{\tau}_j) - \alpha \sum_j (n_{i(j)} + n_{t(j)})\right] \Bigg/ \sum_i a_i n_i \geq (>) \chi(\mathrm{Ad}\, E)/n - \alpha J. \quad (*1')$$

Lemma 1.9. (a) *The condition* $(*1')$ *above is equivalent to the following condition*

$$\left[\chi(S_s) - \alpha \sum_j q_j(S_s)\right] \Bigg/ \sum_i a_i r_i \leq (<) \chi(\mathrm{Ad}\ E)/n - \alpha J, \qquad (*2')$$

where $q_j(S_s) = r_{i(j)} + r_{t(j)} - \dim\left(\sigma_j \cap \left((S_s)_{i(j),x_j} \oplus (S_s)_{t(j),z_j}\right)\right)$.
(b) *If* C *is irreducible and smooth, then* $(*2')$ *is same as* $(*2)$.

Proof. (a) The quotient Q_s of $E(\mathbf{g})$ has induced parabolic structure over D_j, $j = 1, \ldots, J$ given by $(Q_s)_{x_j} \oplus (Q_s)_{z_j} \supset F_j(Q_s) \supset 0$ with weights $(0, \alpha)$, where $F_j(Q_s)$ is the image of the n-dimensional subspace σ_j of $(E(\mathbf{g})_{x_j} \oplus E(\mathbf{g})_{z_j})$ in $((Q_s)_{x_j} \oplus (Q_s)_{z_j})$. Let $f_j(Q_s) = \dim F_j(Q_s)$. By definition, the parabolic weight of $Q_s = \alpha \sum_j f_j(Q_s)$. Define $q_j(Q_s) = n_{i(j)} + n_{t(j)} - f_j(Q_s)$, it is additive for exact sequences. Then one has

$$\begin{aligned} \text{parabolic weight of } Q_s &= \alpha \sum_j (n_{i(j)} + n_{t(j)} - q_j(Q_s)) \\ &= \alpha \sum_j (n_{i(j)} + n_{t(j)} - q_j(\mathrm{Ad}\ E) + q_j(S_s)) \\ &= \alpha \sum_j (n_{i(j)} + n_{t(j)} - n + q_j(S_s)). \end{aligned}$$

Note that since $Q_s \approx E(\mathbf{g}/\mathbf{p})$ outside $\mathrm{sing}(C)$ and all D_j avoid $\mathrm{sing}(C)$, one has parabolic weight of Q_s = the parabolic weight of $E(\mathbf{g}/\mathbf{p}) = \sum_j \overline{\mu}_j(\overline{\tau}_j)$. Hence, $\sum_j \overline{\mu}_j(\overline{\tau}_j) - \sum_j \alpha(n_{i(j)} + n_{t(j)}) = \alpha \sum_j q_j(S_s) - \alpha J n$. Using this equality and $\sum a_i r_i = n - \sum a_i n_i$ the first part of the Lemma follows.
(b) If C is a smooth irreducible curve then one has $I = 1$, $S_s = E(\mathbf{p})$, $Q_s = E(\mathbf{g}/\mathbf{p})$, $\Sigma a_i r_i = r_1$, $\chi(S_s) = \deg(E(\mathbf{p})) + r_1(1-g)$, $\sum_j \alpha q_j(S_s) = \sum_j \alpha(2r_1 - \dim F_j(S_s) = 2\alpha J r_1 -$ parabolic weight (S_s). Hence the left hand side of $(*2')$ becomes equal to par deg $E(\mathbf{p})/\mathrm{rank}\ E(\mathbf{p}) - 2\alpha J + (1-g)$. The right hand side of $(*2') = (1-g) - \alpha J$. Hence the result follows.

2. Principal *G*-bundles on a singular curve *Y*

2.1. We want to introduce the notions of stability and semistability for principal G-bundles on singular curves. On a smooth curve there are different definitions of stability and semistability of a principal G-bundle, but they all coincide [R1]. The problem is that this is not true on a singular curve. The choice of a representation of G used to define semistability does not matter on a smooth curve essentially because the associated bundles (tensor products etc.) of semistable vector bundles (in characteristic 0) are semistable. This fails if the curve has singularities. For example, if F_1 is the semistable vector bundle of rank 2, degree 0 (on an irreducible nodal curve Y) constructed in Proposition 2.7 of

[U3] then $F_1 \otimes F_1$ and $S^2 F_1$ are not semistable [U5]. This is seen by checking that the corresponding generalized parabolic vector bundles on C are not semistable. Similarly one can show that if F_2 is the stable vector bundle of rank $2m$ constructed in Proposition 2.9, [U3] then $F_1 \otimes F_2$ is not semistable for all $m \geq 2$ [U5].

We give here a notion of semistability for principal G-bundles on singular curves (see Definitions 2.2, 2.3, 2.9, 2.10) which is intrinsic and seems most useful. We first assume that Y is irreducible (the case of a reducible curve will be dealt with later). Let $Y' = Y - \{\text{singular set of } Y\}$, $i : Y' \to Y$ inclusion map. Let G be a connected reductive algebraic group. Let P be a maximum parabolic subgroup of G and $\mathbf{p}$ the Lie algebra of P. Let $\mathcal{F}$ be a principal G-bundle on Y and $\mathcal{F}/P = \mathcal{F}(G/P)$ the associated fibre bundle with fibres isomorphic to G/P. Let $s' = Y' \to (\mathcal{F}/P) \mid Y'$ be a reduction of the structure group to P (i.e. a section of $\mathcal{F}/P$ restricted to Y'). Let $T(G/P)$ denote the tangent bundle along the fibres of $\mathcal{F}/P \to Y$. Let $Q_{s'}$ be a torsion free quotient of $\mathcal{F}(\mathbf{g})$ such that $Q_{s'} \mid Y' \cong (s')^*(T(G/P)) \mid Y'$ and no further quotient of $Q_{s'}$ has this property. Let $S_{s'}$ be the maximum subsheaf of $\mathcal{F}(\mathbf{g})$ containing $(s')^*\mathcal{F}(\mathbf{p})$.

DEFINITION 2.2

$\mathcal{F}$ is *stable* (resp. *semistable)* if for every reduction s' of the structure group to a maximum parabolic P (over Y'), one has degree $Q_{s'} > 0$ (resp. ≥ 0).

Lemma 2.3. *The above definition is equivalent to the following: $\mathcal{F}$ is stable* (*resp. semistable*) *if for every s' as above,* degree $\mathrm{S}_{s'} < 0$ (*resp.* ≤ 0).

Proof. The exact sequence $0 \to \mathbf{p} \to \mathbf{g} \to \mathbf{g}/\mathbf{p} \to 0$ gives an exact sequence $0 \to s'^*\mathcal{F}(\mathbf{p}) \to \mathrm{Ad}\mathcal{F} \mid Y' \to s'^*T(G/P) \to 0$ and hence $0 \to S_{s'} \to \mathrm{Ad}\mathcal{F} \to Q_{s'} \to 0$. Noting that Ad $\mathcal{F}$ has degree zero, the lemma follows.

We now assume that Y has only ordinary nodes $y_1, \ldots, y_J$ as singularities and $p : C \to Y$ is the normalization map, $D_j = p^{-1}(y_j) = x_j + z_j$, $j = 1, \ldots, J$. Then giving a principal G-bundle $\mathcal{F}$ on Y is equivalent to giving the principal G-bundle $p^*\mathcal{F} = E$ on C together with a G-isomorphism σ_j of the fibres E_{x_j} and E_{z_j} of E for each j. The isomorphisms σ_j induce isomorphisms $E(\mathbf{g})_{x_j} \to E(\mathbf{g})_{z_j}$. We denote the graph of these isomorphisms also by σ_j.

PROPOSITION 2.4

$(E, (\sigma_j))$ *is* 1-*stable* (*resp.* 1-*semistable*) *if and only if the corresponding G-bundle $\mathcal{F}$ on Y is stable* (*resp. semistable*).

Proof. Suppose that $\mathcal{F}$ is stable (resp. semistable). Let $s : C \to E/P$ be a reduction to a maximum parabolic subgroup P. Since $C - \cup_j D_j \approx Y - \cup_j y_j$, under p and $E \approx p^*\mathcal{F}$, the section s gives a reduction $s' : Y' = Y - \cup_j y_j \to (\mathcal{F}/P) \mid_{Y'}$. One has the exact sequences $0 \to \mathcal{F}(\mathbf{g}) \to p_* s^* E(\mathbf{g}) \to \oplus_j Q_j E(\mathbf{g}) \to 0$, $0 \to S_{s'} \to p_* s^* E(\mathbf{p}) \to \oplus_j Q_j E(\mathbf{p}) \to 0$ where $Q_j(E(\mathbf{g})) = (s^*E(\mathbf{g})_{x_j} \oplus s^*E(\mathbf{g})_{z_j})/\sigma_j$, $Q_j E(\mathbf{p}) = (s^*E(\mathbf{p})_{x_j} \oplus s^*E(\mathbf{p})_{z_j})/(\sigma_j \cap (s^*E(\mathbf{p})_{x_j} \oplus s^*E(\mathbf{p})_{z_j}))$. Note that the quotient of $\mathcal{F}(\mathbf{g})$ by $S_{s'}$ is the torsion free sheaf obtained from $s^*E(\mathbf{g}/\mathbf{p})$ with induced parabolic structure (viz. the image of σ_j in $E(\mathbf{g}/\mathbf{p})_{x_j} \oplus E(\mathbf{g}/\mathbf{p})_{z_j}, \forall j$). The second sequence implies that par deg $s^*E(\mathbf{p}) - J$ rank

$s^*E(\mathbf{p}) = \deg(S_{s'})$. Since $\mathcal{F}$ is stable, $\deg(S_{s'}) < 0$. The result follows from Lemma 1.6. The converse follows similarly working backwards in the above argument. One has only to note that if $s' : Y' \to (\mathcal{F}/P)\mid_Y'$ is a reduction to a maximum parabolic P, then s' gives a reduction $s : C \to E/P$ (as G/P is complete). In case of semistability one has to replace strict inequalities in the above proof by inequalities.

2.5 *Bundles associated to representations*

The fundamental group $\pi_1(Y)$ of Y is isomorphic to $H = \pi_1(C) * \mathbf{Z} * \ldots * \mathbf{Z}$, a free product of $\pi_1(C)$ and J copies of $\mathbf{Z}$ (3.5, [U3]). To a representation $\rho : H \to G$ we associate a QPG $(E_\rho, (\sigma_j))$ as follows. E_ρ is the principal G-bundle on C associated to the representation $\rho_C = \rho \mid \pi_1(C)$. If $\widetilde{C}$ is the universal covering of C, then $E_\rho = \widetilde{C} \times_\rho G$. Fixing suitably points x'_j, z'_j of $\widetilde{C}$ lying over x_j, z_j respectively, the fibres $(E_\rho)_{x_j}$ and $(E_\rho)_{z_j}$ can be identified to G. Let $g_j = \rho(1_j)$, 1_j denoting the generator of the jth factor $\mathbf{Z}$ in H. Then g_j gives an isomorphism $h'_j : (E_\rho)_{x_j} \cong (E_\rho)_{z_j}$ and hence $h_j : (E_\rho(\mathbf{g}))_{x_j} \cong (E_\rho(\mathbf{g}))_{z_j}$. Define σ_j = graph of h_j. If $\mathcal{F}$ is the principal G-bundle on Y obtained by identifying fibres of E_ρ at x_j and z_j by $g_j \forall_j$, then one has $\mathcal{F} = \mathcal{F}_\rho$, the G-bundle associated to the representation ρ of $\pi_1(Y)$ and $E_\rho = p^*\mathcal{F}_\rho$.

PROPOSITION 2.6

If ρ_C is irreducible unitary (resp. unitary) then $\mathcal{F}_\rho$ is stable (resp. semistable).

Proof. If ρ_C is unitary, so is $\mathrm{Ad}\circ\rho_C$ and hence $\mathcal{F}_{\mathrm{Ad}\circ\rho} = \mathcal{F}_\rho(\mathbf{g})$ is semistable ([U3], Proposition 2.5). Therefore $\mathcal{F}_\rho$ is semistable.

If ρ_C is irreducible unitary, then by Theorem 7.1 of [R1] (in our case $E(\rho, c) = E_\rho$, $c = Id) E_\rho$ is a stable G-bundle. We check below that $(E_\rho, (\sigma_j))$ is 1-stable, then $\mathcal{F}_\rho$ is stable by Proposition 2.4. Let s be a reduction of the structure group of E_ρ to a maximum parabolic subgroup P. The stability of E_ρ implies that $\deg(s^*E_\rho(\mathbf{p})) < 0$. Note that σ_j maps isomorphically onto $(E_\rho)_{x_j}$, $j = 1, \ldots, J$. Hence $\sigma_j(E_\rho(\mathbf{p})) = \sigma_j \cap (E_\rho(\mathbf{p})_{x_j} \oplus E_\rho(\mathbf{p}) z_j)$ maps injectively into $E_\rho(\mathbf{p})_{x_j}$. Therefore $\dim \sigma_j(E_\rho(\mathbf{p})) \leq \operatorname{rank}(E_\rho(\mathbf{p}))$ for all j. It follows that par deg $(s^*E_\rho(\mathbf{p}))$ = deg $(s^*E_\rho(\mathbf{p})) + \sum_j \dim \sigma_j(E_\rho(\mathbf{p})) < J \operatorname{rank}(E_\rho(\mathbf{p}))$. Thus $(E_\rho, (\sigma_j))$ is 1-stable.

Remark 2.7. There may exist stable principal G-bundles on Y which are not associated to any representations of $\pi_1(Y)$. For examples in case $G = GL(n)$ see [U3], similar examples can be constructed in case $G = O(n)$, $Sp(2n)$ also.

Principal G-bundles on a reducible curve Y

Notations 2.8. Let the notation be as in 1.1. Assume further that Y has nodes y_j, $j = 1, \ldots, J$ as only singularities. Let Γ be the graph obtained from the (dual) graph of Y by omitting loops. Let $y_1, \ldots, y_K$ be the nodes of Y such that each y_j lies on two different components of Y. Then $K =$ the number of edges of Γ, $I =$ the number of vertices of Γ.

For $i = 1, \ldots, I$, let P_i denote either a maximum parabolic subgroup of G or the trivial group $\{e\}$ or the group G itself. Let $\mathcal{F}$ denote a principal G-bundle on Y. For each i, let $s'_i : Y'_i \to \mathcal{F}(G/P_i)\mid_{Y'_i}$ be a section. Let $P = (P_i)_i$, $s' = (s'_i)_i$ be I-tuples. We call s' a *reduction of the structure group to* P over Y'. Let $T(G/P_i)$ denote the tangent bundle along the fibres of $\mathcal{F}(G/P_i) \mid_{Y_i}$. If $P_i = \{e\}$ then $s_i^{'*}(T(G/P_i)) \approx \mathrm{Ad}\,\mathcal{F} \mid_{Y_i'}$. If $P_i = G$, then $\mathcal{F}(G/P_i) \mid_{Y_i} \approx Y_i$ and the Euler characteristic $\chi(s_i^{'*}(T(G/P_i))) = 0$. Let $Q_{s'}$ be the smallest torsionfree quotient of Ad $\mathcal{F}$ such that $Q_{s'} \mid_{Y_i'} \approx s_i^{'*}(T(G/P)) \mid_{Y_i'}$ for all i. Let $\mathbf{p}_i$ denote the Lie algebra of P_i and $\mathcal{F}(\mathbf{p}_i), \mathcal{F}(\mathbf{g}), \mathcal{F}(\mathbf{g}/\mathbf{p}_i)$ the fibre bundles (with fibres $\mathbf{p}_i, \mathbf{g}, \mathbf{g}/\mathbf{p}_i$ respectively) associated to the P_i-bundle $\mathcal{F} \to \mathcal{F}(G/P_i)$ via the adjoint representation. Thus $s_i^{'*}\mathcal{F}(\mathbf{g}) = \mathrm{Ad}\,\mathcal{F} \mid_{Y_i'}, s_i^{'*}\mathcal{F}(\mathbf{g}/\mathbf{p}_i) = s_i^{'*}T(G/P_i)$. Let $S_{s'}$ be the maximum subsheaf of Ad $\mathcal{F}$ such that $S_{s'} \mid_{Y_i'} \approx s'^*\mathcal{F}(\mathbf{p})$. Let $a = (a_1, \ldots, a_I)$, where $\{a_i\}$ are positive rational numbers with $\sum a_i = 1$. Recall that for a vector bundle V on Y, $a\text{-rank } V = \sum_i a_i \text{ rank}(V \mid_{Y_i})$.

DEFINITION 2.9

The principal G-bundle $\mathcal{F}$ on Y is a-semistable (resp. a-stable) if for every reduction s' of the structure group to P with $P_i \neq \{e\}$ for all i and $P_i \neq G$ for all i one has (in the notations of 2.8)

$$\chi(Q_{s'})/a - \text{rank } Q_{s'} \geq (\text{resp.} >) \chi(\mathrm{Ad}\,\mathcal{F})/a - \text{rank Ad } \mathcal{F}.$$

Lemma 2.10. *$\mathcal{F}$ is a-semistable (resp. a-stable) if for every reduction s' as above,*

$$\chi(S_{s'})/a - \text{rank } S_{s'} \leq (\text{resp.} <) \chi(\mathrm{Ad}\,\mathcal{F})/a - \text{rank Ad } \mathcal{F}.$$

Proof. As in Lemma 2.3, we have the exact sequences $0 \to s_i^{'*}\mathcal{F}(\mathbf{p}) \to \mathrm{Ad}\,\mathcal{F} \mid Y_i' \to s_i^{'*}T(G/P) \to 0$ for all i and so $0 \to S_{s'} \to \mathrm{Ad}\,\mathcal{F} \to Q_{s'} \to 0$. The lemma follows using the fact that both the Euler characteristic and a-rank are additive for an exact sequence.

PROPOSITION 2.11

For $i = 1, \ldots, I$, let C_i be a partial desingularization of Y_i and $C = \coprod C_i$. Suppose that C is obtained by blowing up nodes $y_1, \ldots, y_{J'}$, $J' \leq J$ of Y. Let $(E, (\sigma_j))$ denote a QPG with quasi-parabolic structure σ_j over D_j, $1 \leq j \leq J'$. Then a QPG $(E, (\sigma_j))$ is $(a, 1)$-stable (resp. $(a, 1)$-semistable) if and only if the corresponding principal G-bundle on Y (obtained by identifying fibres of E by σ_j) is a-stable (resp. a-semistable).

Proof. The proof is exactly on same lines as that of Proposition 2.4. Starting with $\mathcal{F}$ a-stable (resp. semistable) and a reduction s' to P, one gets an exact sequence $0 \to S_{s'} \to p_*S_s \to \oplus_j Q_j(S_s) \to 0$, with $q_j(S_s) = \dim\ Q_j(S_s)$. Then Lemma 1.9 gives $(a, 1)$-stability (resp. semistability) of $(E, (\sigma_j))$. The converse is proved by reversing the argument.

2.12. *G-bundles associated to representations*

Let $\rho : \pi_1(Y) \to G$ be a representation of the fundamental group $\pi_1(Y)$ of Y in G. For $i = 1, \ldots, I$, let $f_i : \pi_1(Y_i) \to \pi_1(Y)$ be the natural maps, $\rho_i = \rho \circ f_i$. Let $\mathcal{F}$ be the G-bundle on Y associated to ρ. Let $p^*\mathcal{F} = E = (E_i)_i$. Then E_i is the G-bundle on Y_i associated to ρ_i. The principal G-bundle $\mathcal{F}$ corresponds to a QPG $(E, (\sigma_j))$ on $\coprod_i Y_i$ where $\{\sigma_j\}$, $j = 1, \ldots, K$ are G-isomorphisms of fibres of E. Finally let C_i denote the desingularization of Y_i, g_i = arithmetic genus of Y_i, $g(C_i)$ = genus of C_i, $g(C_i) \geq 1$. Our aim is to prove the following Theorem.

Theorem 1. *There exist positive rational numbers $a_1, \ldots, a_I$ with $\sum a_i = 1$, depending only on Γ and g_i, such that for $a = (a_1, \ldots, a_I)$ the following statements are true.*

(1) *If $\rho_{C_i} = \rho_i \mid \pi_1(C_i)$ are unitary representations for all i, then the principal G-bundle $\mathcal{F}$ on Y associated to ρ is a-semistable.*

(2) *If ρ_{C_i} are irreducible unitary representations for all i, then the principal G-bundle $\mathcal{F}$ associated to ρ is a-stable.*

For the proof of the theorem, we need the following combinatorial result.

PROPOSITION 2.13

Let Γ be a connected graph without loops. Let I be the number of vertices and K the number of edges of Γ, $K = I + m$, $m \geq -1$. Fix integers $r \geq 1$, $g_i \geq 1$, $i = 1, \ldots, I$ and $g = \sum g_i$. Then there exist positive rational numbers a_i, $i = 1, \ldots, I$ with $\sum a_i = 1$ such that for every I-tuple $\underline{r} = (r_1, \ldots, r_I)$ of integers r_i with $0 \leq r_i \leq r$ and for every K-tuple of integers $\underline{q} = (q_1, \ldots, q_K)$ with $\max(r_{i(j)}, r_{t(j)}) \leq q_j \leq r$, $j = 1, \ldots, K$, one has

$$\sum_{i=1}^{I} r_i(g_i - 1) + \sum_{j=1}^{K} q_j \ \geq (g+m)\left(\sum_{i=1}^{I} a_i r_i\right). \tag{SS}$$

If in addition $r_i = 0$ for some i, $0 \neq \sum_i r_i$, then the inequality (SS) *is a strict inequality.*

Proof. We prove the result by induction on m.

Case $m = -1$: Γ is a tree in this case. Let $a_i = g_i/g$, $i = 1, \ldots, I$. Let $r_{i_0} = \min_i r_i$. Since $K = I - 1$ and $q_j \geq \max\,(r_{i(j)}, r_{t(j)})$, one has $\sum_j q_j \geq \sum_{i \neq i_0} r_i = \sum_i r_i - r_{i_0}$.

$$\begin{aligned}
\text{L.H.S. of (SS)} &= \textstyle\sum_i r_i g_i - \sum_i r_i + \sum_j q_j \\
&\geq \textstyle\sum_i r_i g_i - r_{i_0} = \sum_i r_i g_i - \sum_i (g_i r_{i_0})/g \\
&\geq \textstyle\sum_i r_i g_i - \sum_i g_i r_i/g \\
&= \textstyle(g-1)\sum r_i g_i/g \\
&= \textstyle(g-1)\sum_i a_i r_i.
\end{aligned}$$

If $0 \neq \sum r_i$ and $r_i = 0$ for some i, then $r_{i_0} = 0$ and

$$\text{L.H.S. of (SS)} \ \geq \sum r_i g_i = g \sum_i a_i r_i > (g-1) \sum a_i r_i.$$

Case $m \geq 0$: If $m \geq 0$, then Γ contains a cycle. By removing a suitable edge, say e_ℓ, from this cycle in Γ, we get a connected subgraph Γ' of Γ such that $K(\Gamma') - I(\Gamma') = m - 1$. By induction, there exist positive rational numbers a'_i, $i = 1, \ldots, I$ with $\sum a'_i = 1$, such that for all $\underline{r}$ and $\underline{q} = (q_1, \ldots, \widehat{q_\ell}, \ldots, q_j)$ satisfying the given conditions, one has

$$\begin{aligned}
\textstyle\sum_i r_i(g_i - 1) + \sum_{j \neq \ell} q_j &\geq (g + m - 1)(\textstyle\sum_i a_i' r_i) \\
&= \textstyle\sum_i b_i' r_i, \;\; b_i' = a_i'(g + m - 1), \\
\text{L.H.S. of (SS)} &= \textstyle\sum_i r_i(g_i - 1) + \sum_{j=1}^{J} q_j \\
&\geq \textstyle\sum_i b_i' r_i + (r_{i(\ell)} + r_{t(\ell)})/2 \\
&= \textstyle\sum b_i r_i,
\end{aligned}$$

where $b_i = b_i'$ if $i \neq i(\ell), t(\ell)$ and $b_i = b_i' + \frac{1}{2}$ if $i = i(\ell)$ or $t(\ell)$. Take $a_i = b_i/(g+m)$ for all i, then (SS) holds. The assertion about strict inequality follows by induction similarly.

Remark 2.14. (1) Note that if both a, a' satisfy (SS) then for $0 \leq t \leq 1$, $a^t = ta + (1-t)a'$ also satisfies (SS). Thus the set of solutions a of (SS) is a convex set.
(2) Given $i_1, i_2, , 1 \leq i_1, i_2 \leq I$, take $a_i' = (g_i - \frac{1}{2})/(g-1)$ for $i = i_1, i_2$ and $a_i' = g_i/(g-1)$ for $i \neq i_1, i \neq i_2$. Then in case Γ is a tree (i.e. $m = -1$) the inequality (SS) holds (though the strict inequality may not be true eg. for $r_{i_1} = r_{i_2} = 0$). For $K - I \geq 0$, the inductive proof of Proposition 2.13 then gives new $a' = (a_1', \ldots, a_I')$ satisfying the inequality (SS). It follows that the inequality holds for a^t, $0 \leq t \leq 1$.

PROPOSITION 2.15

Theorem 1 *is true for* $G = GL(r)$.
(1) *If* $\rho_i \mid \pi_1(C_i)$ *are unitary representations for all* i, *then the vector bundle* F *on* Y *associated to* ρ *is* a-*semistable*.
(2) *If* $\rho_i \mid \pi_1(C_i)$ *are irreducible unitary for all* i, *then* F *is* a-*stable*.

Proof.

(1) As in Propositions 3.9 and 3.7(3) of [U2], it can be seen that the vector bundle F on Y corresponds to a QPG $\underline{E} = (E, F_j(E))$ on $\coprod Y_i$ and F is a-semistable (resp. a-stable) if and only if $\underline{E}$ is $(a, 1)$-semistable (resp. $(a, 1)$-stable). Note that $E = \coprod E_i$ is the pull-back of F to $\coprod Y_i$. By Theorem 2, [U3], the vector bundles E_i on Y_i associated to ρ_i are semistable for all i. Hence, for any subsheaf N_i of E_i, one has $\chi(N_i) \leq r_i(1-g_i)$, $r_i =$ rank N_i (note that degree $(E_i) = 0$). Thus

$$S_N = \left[\sum_i \chi(N_i) - \sum_j q^j(N)\right] \Bigg/ \sum_i a_i r_i \leq \sum_i r_i(1 - g_i) - \sum_j q^j(N) / \sum_i a_i r_i,$$

where the summation over j is taken for $1 \leq j \leq K$. For the choice of $\{a_i\}$ made in Proposition 2.13, we get

$$S_N \leq I - K - \sum g_i = \left(\sum_i \chi(E_i) - rK\right) \Bigg/ r.$$

Thus $(E, F_j(E))$ is $(a, 1)$-semistable and hence F is a-semistable.

(2) We need to consider two cases. With the notations in the proof of (1) if $r_i = 0$ for some i then by Proposition 2.13, we have $S_N < I - K - \sum g_i$. If $r_i \neq 0$ for all i, then there exists an i_0 such that $0 \neq r_{i_0} \neq r$. Since E_{i_0} is stable by Theorem 2 [U3], we have $\chi(N_{i_0}) < r_{i_0}(1 - g_{i_0})$. Therefore, $S_N < \sum_i r_i(1 - g_i) - \sum_j q^j(N)/\sum_i a_i r_i \leq I - K - \sum g_i$ (by Proposition 2.13). Thus $(E, F_j(E))$ is $(a, 1)$-stable and so F is a-stable.

Remark 2.16. The proof of Proposition 2.13 shows that there exist curves $Y^m, m = 0, \ldots, n+1$ such that (1)$Y^0 = Y$, (2) Y^{n+1} is a curve with ordinary nodes such that the dual graph of Y^{n+1} is a tree after omitting loops (3) Y^{m+1} is obtained from Y^m by blowing up a node which lies on two different components. For $m = 0, \ldots, n$, let $\varphi_m = Y^{m+1} \to Y^m$ be the natural surjective maps. Let F denote a unitary (resp. irreducible unitary) vector bundle on Y. The proofs of Propositions 2.13 and 2.15 together show how the 'polarization' $a = (a_1, \ldots, a_I)$ for which the vector bundles $\varphi_m^* F$ are a-semistable (resp. a-stable) varies as we go down the tower of curves $\{Y^m\}$.

Proof of Theorem 1.

(1) If ρ_{C_i} is unitary, so is $\mathrm{Ad} \circ \rho_{C_i} = (\mathrm{Ad} \circ \rho)_{C_i}$. Therefore there exist positive rational numbers $a_1, \ldots, a_I$ with $\sum a_i = 1$ (depending only on Γ and g_i) such that the vector bundle $\mathcal{F}_{\mathrm{Ad}\circ\rho} \approx \mathrm{Ad}\, \mathcal{F}$ associated to $\mathrm{Ad} \circ \rho$ is a-semistable (Proposition 2.15). Hence $\mathcal{F}$ is a-semistable.

(2) By Proposition 2.6, the principal G-bundles E_i on Y_i associated to ρ_i are stable for all i. We claim that for the choices of $\{a_i\}_i$ as in the proof of (1), the QPG $(E, (\sigma_j))$ corresponding to $\mathcal{F}$ is $(a, 1)$-stable. The result follows from the claim in view of Proposition 2.11. To prove the claim we check that the condition $(*2')$ of Lemma 1.9 is satisfied for any reduction s of the structure group to P. Let r_i be the rank of S_{s_i}. Since S_s is a proper subsheaf of $E(\mathbf{g})$, $\sum r_i \neq nI$. Since E_i are stable, by Lemma 2.3, $\chi(S_{s_i}) \leq r_i(1 - g_i)$ and the inequality is strict if $0 < r_i < n$. By Proposition 2.13, for the choices of $\{a_i\}$ as in (1), one has

$$\chi(S_s) - \sum_j q_j(S_s) \Big/ \sum_j a_i r_i < I - K - \sum_i g_i (= \chi(\mathrm{Ad}\, E)/n - K)$$

if $r_{i_0} = 0$ for some i_0 and $0 \neq \sum r_i$. If $r_i \neq 0$ for all i, since $\sum r_i \neq nI$, there exists an i_0 such that $0 < r_{i_0} < n$. Then $\chi(S_s) < \sum_i r_i(1 - g_i)$ and so

$$\begin{aligned}
\chi(S_s) - \sum_j q_j(S_s) &< \sum_i r_i(1 - g_i) - \sum_j q_j(S_s) \\
&\leq (I - K - \textstyle\sum g_i)\left(\displaystyle\sum_i a_i r_i\right), \text{ by Proposition 2.13,} \\
&= (\chi(\mathrm{Ad}\, E)/n - K)\sum_i a_i r_i.
\end{aligned}$$

This proves the claim.

3. The Picard group of the stack of QPGs

In this section Y denotes a reduced connected projective curve with ordinary nodes $\{y_j\}$, $j = 1, \ldots, J$ as only singularities. Let $\{Y_i\}$, $i = 1, \ldots, I$ be the irreducible components of Y and C_i the desingularization of Y_i. Let $C = \coprod_i C_i$ be the desingularization of Y. For convenience of notation, we fix an orientation of the dual graph of Y. For $1 \leq j \leq J$, let $i(j), t(j)$ denote the initial and terminal points of j in the dual graph. They correspond to curves $C_{i(j)}, C_{t(j)}$ intersecting at y_j. Let $x_j \in C_{i(j)}$ and $z_j \in C_{t(j)}$ be the two points of C mapping to $y_j \in Y$ and $D_j = x_j + z_j$, $j = 1, \ldots, J$. Let G denote an affine simply connected simple algebraic group over $\mathbf{C}$ (or an algebraically closed field of characteristic zero). For $i = 1, \ldots, I$, fix points $p_i \in C_i$, p_i not mapping to a singular point of Y. Let $C_i^* = C_i - \{p_i\}$, $C^* = C - \cup_i p_i$.

The results of this section were inspired by [LS]. If G is semisimple, then a principal G-bundle on a smooth curve C is trivial on the complement of a point in C. This no longer holds if C is replaced by a nodal curve Y. The results of [LS] cannot be generalized directly to G-bundles on Y. Hence we work with QPGs on C. Though we closely follow the ideas in [LS], the generalization to QPGs is not straightforward. All the functors involved have to be defined carefully to take care of the additional structure (generalized parabolic structure). Unlike the usual parabolic structure which is supported on isolated points, the generalized parabolic structure is supported on divisors, so one has the action of $G \times G$ rather than G.

3.1 *The stack* $Q^{\mathrm{gpar}}_{G,C}$ *and the stack* $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$

Let Aff/k be the flat affine site over the base field $k = \mathbf{C}$, i.e. the category of k-algebras equipped with $fppf$ topology. Let R denote a k-algebra, $C_{i,R} := C_i \times$ spec R and $C^*_R = C^* \times$ spec R. Let q_i be a local parameter at the point $p_i, i = 1, \ldots, I$. Let $L_{G,i}$ denote the k-group defined by associating to R the group $G(R(q_i))$. Let $L^+_{G,i}$ (resp. $L^{C_i}_G$) be the k-group defined by associating to R the group $G(R[[q_i]])$ (resp. $G(\Gamma(C^*_{i,R}, \mathcal{O}_{C^*_{i,R}})))$. Define $L_G = \prod_i L_{G,i}$, $L^+_G = \prod_i L^+_{G,i}$, $L^C_G = \prod_i L^{C_i}_G$. Let

$$Q_{G,C} = L_G/L^+_G = \prod_i L_{G,i}/L^+_{G,i}, \quad Q^{\mathrm{gpar}}_{G,C} = Q_{G,C} \times \prod_j G.$$

The indgroup L^C_G acts on $Q_{G,C}$. For each j, the evaluation at x_j and z_j gives an evaluation map $e_j : L^C_G \to G \times G$. $G \times G$ acts on G by $(g_1, g_2)g = g_1^{-1} g g_2$. Thus we have a natural action of L^C_G on $Q^{\mathrm{gpar}}_{G,C}$. Let $L^C_G \backslash Q^{\mathrm{gpar}}_{G,C}$ be the quotient stack.

To an object $R \in Aff/k$, associate the groupoid whose objects are families of QPGs $(E, (\sigma_j))$ on C parametrized by spec R and whose arrows are isomorphisms of the families of QPGs i.e. isomorphisms of E which preserve the parabolic structures (σ_j). For any morphism $R \to R'$ we have a natural functor between the associated groupoids. Thus we get a k-stack of (generalized) quasiparabolic G-bundles on C. We denote this stack by $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$.

Theorem 2. *There exists a canonical isomorphism of stacks*

$$\overline{\pi}_{\mathrm{par}} : L^C_G \backslash Q^{\mathrm{gpar}}_{G,C} \xrightarrow{\sim} \mathrm{Bun}^{\mathrm{gpar}}_{G,C}.$$

The projection $\pi_{\mathrm{par}} : Q^{\mathrm{gpar}}_{G,C} \to \mathrm{Bun}^{\mathrm{gpar}}_{G,C}$ *is locally trivial in etale topology.*

Proof. $Q_{G,C}$ represents the functor which associates to every k-algebra R the set of isomorphism classes of pairs (E, ρ) where E is a G-bundle over C_R and ρ is a trivialization of E over C_R^* ([LS], Proposition 3.10). Hence $Q_{G,C}^{\mathrm{gpar}}$ represents the functor P_G which associates to R the isomorphism classes of triples $(E, \rho, \mathbf{s})$ with (E, ρ) as above and $\mathbf{s} \in \prod_j G(R)$, $\mathbf{s} = (s_1, \ldots, s_J)$, $s_j \in G(R) = Mor$ (Spec R, G), G being the jth factor in $\prod_j G$. Such a triple gives a family of QPGs $(E, (\sigma_j))$ parametrized by $S =$ Spec R as follows. Let $\bar{s}_j : S \times x_j \times G \to S \times z_j \times G$ be given by $\bar{s}_j(s, x_j, g) = (s, z_j, g s_j(s))$ for $s \in S, g \in G$. Define $\sigma_j : E_{|S\times x_j} \overset{\approx}{\to} E_{|S\times z_j}$ by $\sigma_j = \rho^{-1}_{|S\times z_j} \circ \bar{s}_j \circ \rho_{|S\times x_j}$. Thus we get a universal QPG over $Q_{G,C}^{\mathrm{gpar}} \times C$, giving a map $\pi_{\mathrm{par}} : Q_{G,C}^{\mathrm{gpar}} \to \mathrm{Bun}_{G,C}^{\mathrm{gpar}}$. Being L_G^C-invariant, this map induces a morphism of stacks $\bar{\pi}_{\mathrm{par}} : L_G^C \backslash Q_{G,C}^{\mathrm{gpar}} \to \mathrm{Bun}_{G,C}^{\mathrm{gpar}}$.

To define a morphism $\mathrm{Bun}_{G,C}^{\mathrm{gpar}} \to L_G^C \backslash Q_{G,C}^{\mathrm{gpar}}$, for each R and $(E, (\sigma_j)) \in \mathrm{Bun}_{G,C}^{\mathrm{gpar}}(R)$ we have to give a L_G^C-bundle $T(R)$ on $\mathrm{Bun}_{G,C}^{\mathrm{par}}(R)$ together with an L_G^C-equivariant map $T(R) \to Q_{G,C}^{\mathrm{gpar}}(R)$. Take $(E, (\sigma_j)) \in \mathrm{Bun}_{G,C}^{\mathrm{gpar}}(R)$. For any R-algebra R', let Spec $R' = S'$ and $T(R') =$ the set of isomorphism classes of pairs $(\rho_{R'}, \sigma')$ where $\rho_{R'}$ is a trivialization of $E_{R'}$ over $C_{R'}^*$ and $\sigma' = (\sigma'_j)_j$, $\sigma'_j : E_{|S'\times x_j} \approx E_{|S'\times z_j}$ is the G-isomorphism which is the pull back of σ_j to R'. This defines an R-space T with the action of the group L_G^C (acting on $\rho_{R'}$). It is an L_G^C-bundle ([DS]; also [LS], Theorem 3.11). As $Q_{G,C}^{\mathrm{gpar}}$ represents the functor P_G, to every element $(\rho_{R'}, \sigma_{R'})$ of $T(R')$ corresponds an element of $Q_{G,C}^{\mathrm{gpar}}(R')$ giving a L_G^C-equivariant map $T \to Q_{G,C}^{\mathrm{par}}$. Hence we get a morphism of stacks $\mathrm{Bun}_{G,C}^{\mathrm{gpar}} \to L_G^C \backslash Q_{G,C}^{\mathrm{gpar}}$ which is clearly the inverse of $\bar{\pi}_{\mathrm{par}}$.

To check the local triviality of π_{par} in etale topology, we have to show that for any morphism f from a scheme S to $\mathrm{Bun}_{G,C}^{\mathrm{gpar}}$ the pull back of the fibration π_{par} to S is etale locally trivial i.e. admits local sections for the etale topology. Such a morphism corresponds to a QPG $(E, (\sigma_j))$ over $S \times C$. For $s \in S$, we can find an etale neighbourhood U of s and a trivialization ρ of $E_{|U\times C^*}$ ([DS]). Using ρ, the G-isomorphism σ_j gives a morphism $s_j : U \to G$. The triple $(E, \rho, (s_j))$ defines a morphism $f' : U \to Q_{G,C}^{\mathrm{gpar}}$ such that $\pi_{\mathrm{par}} \circ f' = f$; i.e. the section over U of the fibration π_{par}. This completes the proof of Theorem 2.

PROPOSITION 3.2

One has

(1) Pic $Q_{G,C} \approx \oplus_i \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1)$

(2) Pic $(Q_{G,C}^{\mathrm{gpar}}) \approx \oplus_i \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1)$.

Proof. (1) It is known that each Q_{G,C_i} is an ind-scheme which is an inductive limit of reduced projective Schubert varieties $X_{i,w}$, this ind-scheme structure coincides with the one by Kumar and Mathieu ([LS], Proposition 4.7). One has $H^1(X_{i,w}, \mathcal{O}) = 0$ ([KN, M]). It follows that Pic $Q_{G,C} \approx \oplus_i$ Pic Q_{G,C_i}. It is known that Pic $(Q_{G,C_i}) = \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1)$ for all i ([LS], 4.10; [M]; [NRS], 2.3) The first assertion follows.

(2) Since $\prod_j G$ is a simply connected affine algebraic group Pic $(\prod_j G)$ is trivial. The ind-scheme Q_{G,C_i} is the inductive limit of integral projective reduced (generalized) Schubert varieties X_{i,w_i} with $H^1(X_{i,w_i}, \mathcal{O}) = 0$. By III, Exer. 12.6 [H] it follows that Pic $(X_{1,w_1} \times \prod_j G) \approx \mathrm{Pic}(X_{1,w_1})$ ([H], III, Exer. 12.6) and therefore by induction on i one sees that

$$\mathrm{Pic}\left(\prod_i X_{i,w_i} \times \prod_j G\right) \approx \oplus_i \mathrm{Pic}(X_{i,w_i}) \approx \oplus_i \mathbf{Z}\mathcal{O}_{X_{i,w_i}}(1).$$

Since $(Q^{\mathrm{gpar}}_{G,C})$ is the inductive limit of $\prod_i X_{i,w_i} \times \prod_j G$ and the restriction $\mathcal{O}_{Q_{G,C_i}}(1) \mid_{X_{i,w_i}}$ $\approx \mathcal{O}_{X_{i,w_i}}(1)$ it follows that Pic $(Q^{\mathrm{gpar}}_{G,C}) \approx \oplus_i \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1)$.

The following result must be well-known, we are including a proof since we could not find a reference.

Lemma 3.3. *Let G be a connected semisimple algebraic group. Then any invertible regular function on G is constant.*

Proof. We remark first that the only regular invertible functions on SL_2 and the additive group G_a are constant functions. Let $f : G \to G_m$ be a regular function.

Claim. For any x in a 1-parameter unipotent subgroup U of G, one has $f(gx) = f(g)$ for all $g \in G$.

Proof of the claim. Consider the function $U \to G_m$ defined by $x \to f(gx)$. Since $U \approx G_a$, this function is constant i.e. $f(gx) = f(g)$ for all $g \in G$.

Since G is semisimple, G is generated by X_α, α varying over roots of G ([Sp], 9.4.1). Therefore, in view of the claim, one has $f(g) = f(h_{\alpha_1} \dots h_{\alpha_r})$ with $h_{\alpha_i} \in \mathrm{Im} SL_2$. The function $SL_2 \to G_m$ defined by $x \to f(h_{\alpha_1} \dots h_{\alpha_{r-1}} x)$ is constant. Hence $f(h_{\alpha_1} \dots h_{\alpha_r}) = f(h_{\alpha_1} \dots h_{\alpha_{r-1}})$. Repeating this process the result follows.

3.4. For each i, there are morphisms of stacks $\pi_i : Q_{G,C_i} \to \mathrm{Bun}_{G,C_i}$ inducing isomorphisms $\pi_i^* : \mathrm{Pic}(\mathrm{Bun}_{G,C_i}) \to \mathrm{Pic}(Q_{G,C_i})$. If L'_i denotes the generator of Pic (Bun_{G,C_i}) as well as its pull back to $\mathrm{Bun}_{G,C}$, then $\pi_i^* L'_i = \mathcal{O}_{Q_{G,C_i}}(1)$ ([LS, So, T]). Hence if we denote the pull back of $\mathcal{O}_{Q_{G,C_i}}(1)$ to $Q_{G,C}$ by $\mathcal{O}_{Q_{G,C_i}}(1)$ again, we have $\pi^*(L'_i) = \mathcal{O}_{Q_{G,C_i}}(1)$. Since Pic $Q_{G,C} = \oplus_i \mathbf{Z}\mathcal{O}_{Q_{G,C_i}}(1)$ it follows that π^* is surjective. Similar argument using Proposition 3.2 shows that π^*_{par} is surjective. One has $\pi^*_{\mathrm{par}}(L_i) = \mathcal{O}_{Q_{G,C_i}}(1)$, where L_i denotes the pull back of L'_i under the forgetful morphism $\mathrm{Bun}^{\mathrm{gpar}}_{G,C} \to \mathrm{Bun}_{G,C}$. Note that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(\mathrm{Bun}_{G,C}) & \overset{\pi^*}{\to} & \mathrm{Pic}(Q_{G,C}) \\ \Phi^* \downarrow & & \downarrow\approx \\ \mathrm{Pic}(\mathrm{Bun}^{\mathrm{gpar}}_{G,C}) & \overset{\pi^*_{\mathrm{par}}}{\to} & \mathrm{Pic}(Q^{\mathrm{gpar}}_{G,C}). \end{array}$$

We now check that π^*_{par} is injective, the injectivity of π^* follows similarly. Denote by $\mathrm{Pic}^{L^C_G}(Q^{\mathrm{gpar}}_{G,C})$ the group of L^C_G-linearized line bundles on $Q^{\mathrm{gpar}}_{G,C}$. Since π_{par} is locally trivial (Theorem 2), for any line bundle L on $\mathrm{Bun}^{\mathrm{gpar}}_{G,C}$, π_{par} induces an isomorphism between the sections of L and L^C_G-invariant sections of $\pi^*_{\mathrm{par}}L$. Therefore we have an injection $\mathrm{Pic}(\mathrm{Bun}^{\mathrm{gpar}}_{G,C}) \to \mathrm{Pic}^{L^C_G}(Q^{\mathrm{gpar}}_{G,C})$ induced by π^*_{par}. The kernel of the forgetful morphism

$\mathrm{Pic}^{L_G^C}(Q_{G,C}^{\mathrm{gpar}}) \to \mathrm{Pic}(Q_{G,C}^{\mathrm{gpar}})$ is the set of L_G^C-linearizations of the trivial line bundle. Any such linearization is given by an invertible (regular) function h on $L_G^C \times Q_{G,C}^{\mathrm{gpar}}$ satisfying a cocycle condition. $Q_{G,C}$ being an inductive limit of integral projective schemes ([LS], 4.6) has no non constant regular functions. Since G is simple, $\prod_j G$ has no invertible nonconstant regular functions (Lemma 3.3). Hence h is the pull back of an invertible function on L_G^C. Since it satisfies a cocycle condition, it is in fact a character on L_G^C. By [LS], Lemma 5.2, h is trivial. Thus the forgetful morphism is injective. Hence the composite $\pi_{\mathrm{gpar}}^* : \mathrm{Pic}(\mathrm{Bun}_{G,C}^{\mathrm{gpar}}) \to \mathrm{Pic}^{L_G^C}(Q_{G,C}^{\mathrm{par}}) \to \mathrm{Pic}(Q_{G,C}^{\mathrm{gpar}})$ is injective. Thus π_{par}^* is an isomorphism. Similarly π^* is an isomorphism and hence Φ^* is also an isomorphism. We have proved the following theorem.

Theorem 3. *Let G be a simple simply connected affine algebraic group over* **C**. *Then we have the following isomorphisms*.

(1) $\mathrm{Pic}\,(\mathrm{Bun}_{G,C}) \approx \oplus_{i=1}^{I} \mathbf{Z} L_i'$,

(2) $\mathrm{Pic}\,(\mathrm{Bun}_{G,C}^{\mathrm{gpar}}) \approx \oplus_{i=1}^{I} \mathbf{Z} L_i$,

where L_i' and L_i are the pullbacks of the generator of $\mathrm{Pic}\,(\mathrm{Bun}_{G,C_i})$ *to* $\mathrm{Bun}_{G,C}$ *and* $\mathrm{Bun}_{G,C}^{\mathrm{gpar}}$ *respectively*.

Remark 3.5. For $G = GL(n), SL(n), Sp(2n)$, the moduli stack (resp. moduli space) of bundles on Y is isomorphic to the moduli stack (resp. moduli space) of QPGs on C ([U1, U2, U4]). Hence we have

$$\mathrm{Pic}(\mathrm{Bun}_{G,Y}) \approx \oplus_i \mathbf{Z}$$

for $G = GL(n), SL(n)$ or $Sp(2n)$.

PROPOSITION 3.6

Assume that C is irreducible and G as in Theorem 3. *Let* $(\mathrm{Bun}_{G,C}^{\mathrm{gpar}})^{\mathrm{ss}}$ *denote the substack corresponding to α-semistable QPGs. Then*

$$\mathrm{Pic}(\mathrm{Bun}_{G,C}^{\mathrm{gpar}})^{\mathrm{ss}} \approx \mathbf{Z}.$$

Proof. We claim that a QPG (E, σ) is α-semistable (resp. stable) for any $\alpha, 0 \le \alpha \le 1$ if the underlying bundle E is semistable (resp. stable). The semistability (resp. stability) of E implies that $\deg E(\mathbf{p}) \le$ (resp. $<$)0. Since σ_j is an isomorphism, the subspace σ_j of $E(\mathbf{g})_{x_j} \oplus E(\mathbf{g})_{z_j}$ maps isomorphically onto $E(\mathbf{g})_{x_j}$ under the projection map. Hence $\sigma_j \cap (E(\mathbf{p})_{x_j} \oplus E(\mathbf{p})_{z_j})$ maps injectively into $E(\mathbf{p})_{x_j}$ and hence has dim.$\le$ rank $E(\mathbf{p})$. It follows that pardeg $E(\mathbf{p}) \le$ (resp. $<$)αJ rank$E(\mathbf{p})$. The claim now follows from Lemma 1.6.

The morphism $\phi : \mathrm{Bun}_{G,C}^{\mathrm{gpar}} \to \mathrm{Bun}_{G,C}$ (forgetting the quasiparabolic structure) is a surjective morphism with isomorphic fibres. It follows from the claim that $\phi^{-1}(\mathrm{Bun}_{G,C} - \mathrm{Bun}_{G,C}^{\mathrm{ss}}) \supseteq \mathrm{Bun}_{G,C}^{\mathrm{gpar}} - (\mathrm{Bun}_{G,C}^{\mathrm{gpar}})^{\mathrm{ss}}$. Hence $\mathrm{codim.}_{\mathrm{Bun}_{G,C}^{\mathrm{gpar}}}(\mathrm{Bun}_{G,C}^{\mathrm{gpar}} - (\mathrm{Bun}_{G,C}^{\mathrm{gpar}})^{\mathrm{ss}}) \ge \mathrm{codim.}_{\mathrm{Bun}_{G,C}}(\mathrm{Bun}_{G,C} - \mathrm{Bun}_{G,C}^{\mathrm{ss}})$. Since the latter is ≥ 2 for $g \ge 2$ ([L-S], 9.3) the same is true for the former. The result now follows from Theorem 3.

3.7. *Results in case* $G = GL(n), SL(n)$

In case of vector bundles we have the following results on Picard groups of moduli spaces ([U5, U6]). Let Y denote an irreducible reduced curve over $\mathbf{C}$ with at most ordinary nodes as singularities. Let $\mathcal{L}$ be a line bundle on Y. Let $U'_Y(n,d)$ (resp. $U'_{\mathcal{L}}(n,d)$) denote the moduli space of semistable vector bundles of rank n and degree d (resp. with fixed determinant $\mathcal{L}$) on Y. Let $U'^s_Y(n,d)$ (resp. $U'^s_{\mathcal{L}}(n,d)$) denote the open subset of $U'_Y(n,d)$ (resp. $U'_{\mathcal{L}}(n,d)$) consisting of stable vector bundles. Let g_C (resp. g_Y) denote the geometric (resp. arithmetic) genus of Y.

I. Assume that $g_C \geq 2$. Then, except possibly for $g_C = 2, n = 2, d$ even, one has

1. Pic $U'^s_{\mathcal{L}}(n,d) \approx$ Pic $U'_{\mathcal{L}}(n,d) \approx \mathbf{Z}$.

2. Pic $U'^s(n,d) \approx$ Pic $U'(n,d) \approx$ Pic $J \oplus \mathbf{Z}$, where J denotes the Jacobian of Y.

II. Assume that $g_Y = 2, n = 2$. Then

$$\text{Pic } U'_{\mathcal{L}}(2,d) \approx \mathbf{Z}.$$

4. Compactifications

In general, the moduli spaces M_G of principal G-bundles on a nodal curve Y are not complete. In case $G = GL(n)$ a compactification of M_G is given by the moduli space of torsionfree sheaves of rank n (and fixed degree) on Y, this compactification is not normal. A normal compactification of M_G is obtained as the moduli space of (generalized) parabolic bundles on the desingularization C of Y ([U1, U2]). This can be done for other classical groups $G = O(n), SO(n), Sp(2n)$ also, we briefly describe the main result (Theorem 5). The details will appear elsewhere [U4]. To construct a normal compactification of M_G, one needs a good compactification of G and hence a good representation of G. In case of classical groups we use their natural representations. For a general group G, a natural choice is the adjoint representation. Unfortunately it gives a compactification of G only if G is of adjoint type ([DP], §6; [S]; [D]). Using this compactification we give a more general definition of QPGs in case G has *trivial centre*. For classical groups and adjoint groups we 'compactify' the stack $\text{Bun}^{\text{gpar}}_{G,C}$ and also compute the Picard group of the compactification. In case of classical groups, the compactifications of moduli spaces obtained are complete normal varieties (see Theorem 5). We do not prove that the 'compactification' is a proper stack in case of adjoint groups. It will be useful to know a natural (canonically defined) compactification of G in the general case.

4.1. Let the notations be as in §3. We further assume that G is a semisimple algebraic group with trivial centre. Let $\mathbf{g}$ denote the Lie algebra of G, $n = \dim \mathbf{g}$. $G \times G$ acts on $\mathbf{g} \oplus \mathbf{g}$ (via adjoint representation) and hence on the Grassmannian $\text{Gr}(n, \mathbf{g} \oplus \mathbf{g})$ of n-dimensional subspaces of $\mathbf{g} \oplus \mathbf{g}$. Let ΔG denote G embedded in $G \times G$ diagonally. Since G has trivial centre the adjoint representation is faithful. Hence $G \approx G \times G/\Delta G$ gets embedded in $\text{Gr}(n, \mathbf{g} \oplus \mathbf{g})$ as $G \times G$-orbit of $\Delta \mathbf{g} \in \text{Gr}(n, \mathbf{g} \oplus \mathbf{g})$. Let F be the closure of the $G \times G$ orbit of $\Delta \mathbf{g}$ in $\text{Gr}(n, \mathbf{g} \oplus \mathbf{g})$.

Given a principal G-bundle E and disjoint divisors $D_j = x_j + z_j$ on C, define

$$E^j = E_{x_j} \times E_{z_j} \cong G \times G, E^j(F) = E^j \times_{(G \times G)} (F), j = 1, \ldots, J.$$

A QPG (quasiparabolic G-bundle) is a pair $(E, (\sigma_j))$ where E is a principal G-bundle and $\sigma_j \in E^j(F)$, $j = 1, \ldots, J$.

DEFINITION 4.2

QPGs $(E, (\sigma_j))$ and $(E', (\sigma'_j))$ are isomorphic if there is an isomorphism $f : E \to E'$ of principal G-bundles which maps σ_j to (σ'_j) i.e. for the isomorphism $f_F^j : E^j(F) \to E'^j(F)$ one has $f_F^j(\sigma_j) = \sigma'_j$.

4.3. A family of QPGs $(\mathcal{E}, (\sigma_j)) \to C \times T$ is a family of G-bundles $\mathcal{E} \to C \times T$ together with a section $\sigma_j : T \to \mathcal{E}^j(F)$.

Remark 4.4. (1) The following diagram commutes

$$\begin{array}{ccc} G \times G & \xrightarrow{h_1} & F = \overline{G \times G/\Delta G} \\ t_2 \downarrow & & \downarrow t_1 \\ GL(\mathbf{g}) \times GL(\mathbf{g}) & \overset{h_2}{\hookrightarrow} & \mathrm{Gr}(n, \mathbf{g} \oplus \mathbf{g}). \end{array}$$

Here t_1 is inclusion, t_2 = product of adjoint representations of G in $\mathbf{g}$, $h_2(f_1, f_2)$ = subspace of $\mathbf{g} \oplus \mathbf{g}$ generated by $\{(f_1 v, f_2 v), v \in \mathbf{g}\}$ and $t_1 \circ h_1$ is the map inducing the Demazure embedding of G (by identifying G with $G \times G/\Delta G$).

(2) Recall that a (generalized) quasiparabolic structure (over $D_j = x_j + z_j$, $j = 1, \ldots, J$) on a vector bundle N of rank n is given by an n-dimensional subspace of $N_{x_j} \oplus N_{z_j}$, $j \in J$ i.e. by an element of $\Pi_j \mathrm{Gr}(n, N_{x_j} \oplus N_{z_j})$ [U1]. Given a family of QPGs $\mathcal{E} \to C \times T$, let $\mathcal{E}(\mathbf{g})$ be the family of vector bundles of rank n associated to $\mathcal{E}$ via the adjoint representation of G in $\mathbf{g}$. It follows from the above commutative diagram that σ composed with the injection $\Pi_j \mathcal{E}^j(F) \to (\Pi_j \mathcal{E}^j(\mathrm{Gr}(n, \mathbf{g} \oplus \mathbf{g})))$ gives a quasi parabolic structure on $\mathcal{E}(\mathbf{g})$.

4.5 *The stack $\overline{Q}_{G,C}^{\mathrm{gpar}}$ and the stack* $\overline{\mathrm{Bun}}_{G,C}^{\mathrm{gpar}}$

Let the notations be as in 3.1. Let $\overline{Q}_{G,C}^{\mathrm{gpar}} = Q_{G,C} \times \prod_j F$. The ind-scheme $Q_{G,C}$ is ind-proper, so is $\overline{Q}_{G,C}^{\mathrm{gpar}}$. The indgroup L_G^C acts on $Q_{G,C}$. For each j, the evaluation at x_j and z_j gives an evaluation map $e_j : L_G^C \to G \times G$. $G \times G$ acts on F naturally. Thus we have a natural action of L_G^C on $\overline{Q}_{G,C}^{\mathrm{gpar}}$. Let $L_G^C \backslash \overline{Q}_{G,C}^{\mathrm{gpar}}$ be the quotient stack.

As in 3.1, we define the k-stack of (generalized) quasiparabolic G-bundles on C (with extended definition of the parabolic structure using F). We denote this stack by $\overline{\mathrm{Bun}}_{G,C}^{\mathrm{gpar}}$. It contains $\mathrm{Bun}_{G,C}^{\mathrm{gpar}}$ as an open substack.

Theorem 4. (1) *There exists a canonical isomorphism of stacks*

$$\overline{\pi}_{\mathrm{par}} : L_G^C \backslash \overline{Q}_{G,C}^{\mathrm{gpar}} \tilde{\to} \overline{\mathrm{Bun}}_{G,C}^{\mathrm{gpar}}.$$

Moreover the projection map $\overline{Q}_{G,C}^{\mathrm{gpar}} \to \overline{\mathrm{Bun}}_{G,C}^{\mathrm{gpar}}$ *is locally trivial for étale topology.*
(2) *Let G be a simple, simply connected affine algebraic group over* $\mathbf{C}$. *Then there exists an isomorphism*

$$\mathrm{Pic}\ (\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}) \approx \oplus_i \mathbf{Z} L_i \oplus \oplus_j \mathrm{Pic} F,$$

where L_i are line bundles coming from Bun_{G,C_i}.

Proof. The proof is on similar lines as that of Theorem 2 and Theorem 3, we omit some details to avoid repetition.

(1) $\overline{Q}^{\mathrm{gpar}}_{G,C}$ represents the functor $\overline{P}_G$ which associates to every k-algebra R the set of isomorphism classes of triples $(E, \rho, \mathbf{s})$ where E is a principal G-bundle on C_R, ρ is a trivialization of E over C_R^* and $\mathbf{s} \in Mor$ (Spec $R, \prod_j F$). Then $\mathbf{s} = (s_1, \dots, s_J), s_j \in Mor$ (Spec R, F) for all j. We can associate to such a triple a QPG $(E, (\sigma_j))$ on C_R. We only need to define for each j, morphism $\sigma_j : S \to E^j(F), S =$ Spec R. The restriction of ρ^{-1} gives isomorphisms $S \times x_j \times G \approx E_{|S\times x_j}, S \times z_j \times G \approx E_{|S \times z_j}$ and hence an isomorphism of $G \times G$-bundles $S \times G \times G = (S \times x_j \times G) \times_S (S \times z_j \times G) \approx E_{|S\times x_j} \times_S E_{|S\times z_j} = E^j$. Therefore we have an isomorphism of associated fibre bundles $\rho_j(F) : S \times F \overset{\approx}{\to} E^j(F)$. Define σ_j by $\sigma_j(s) = \rho_j(F)(s, s_j(s))$. It follows that $\overline{Q}^{\mathrm{gpar}}_{G,C} \times C$ has a universal QPG and we have an L^C_G-equivariant morphism of stacks $\pi_{\mathrm{par}} : \overline{Q}^{\mathrm{gpar}}_{G,C} \to \overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$. This induces the morphism $\overline{\pi}_{\mathrm{par}}$ on the quotient stack.

To define the inverse of $\overline{\pi}_{\mathrm{par}}$, let $(E, (\sigma_j)) \in \overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}(R)$. Let R' be an R-algebra, $S' =$ Spec R'. Let $T(R')$ be the set of pairs $(\rho_{R'}, \sigma')$ where $\rho_{R'}$ is a trivialization of $E_{R'}, \sigma' = (\sigma'_1, \dots, \sigma'_J)$ where σ'_j is a pull back of $\sigma_j \forall j$. This defines a T-space with an action of L^C_G (via $\rho_{R'}$), it is an L^C_G-bundle [DS]. We now define a L^C_G-equivariant map $T \to \overline{Q}^{\mathrm{gpar}}_{G,C}$. Given $(\rho_{R'}, \sigma') \in T(R')$, we define $s'_j : S' \to F$ by $s'_j = pr_F \circ ((\rho_{R'})_j(F))^{-1} \circ \sigma'_j$. Then $(E_{R'}, \rho_{R'}, (s'_j)) \in \overline{P}_G(R')$. Since $\overline{Q}^{\mathrm{gpar}}_{G,C}$ represents the functor $\overline{P}_G$, this defines a map $\alpha : T \to \overline{Q}^{\mathrm{gpar}}_{G,C}$, it is L^C_G-equivariant. The L^C_G-bundle T together with α give a morphism of stacks from $\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$ to the quotient stack $L^C_G \backslash \overline{Q}^{\mathrm{gpar}}_{G,C}$ which is easily seen to be the inverse of $\overline{\pi}_{\mathrm{par}}$.

The assertions about local triviality of $\overline{\pi}_{\mathrm{par}}$ follow similarly as in Theorem 2.

(2) Using the facts that each Q_{G,C_i} is an inductive limit of reduced projective varieties $X_{i,w}$ with $H^1(X_{i,w}, \mathcal{O}) = 0$ and F is a projective variety with $H^1(F, \mathcal{O}) = 0$, it can be proved that Pic $\overline{Q}^{\mathrm{gpar}}_{G,C} \approx \oplus_i \mathbf{Z} \mathcal{O}_{Q_{G,C_i}}(1) \oplus \oplus_j \mathrm{Pic} F$ (similarly as Proposition 3.2). The injectivity of π^*_{par} follows exactly as in Theorem 3. Note that F being a projective variety $\overline{Q}^{\mathrm{gpar}}_{G,C}$ is an inductive limit of integral projective schemes and hence has no nonconstant regular functions.

We now check the surjectivity of π^*_{par}. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(\mathrm{Bun}_{G,C}) & \overset{\pi^*}{\to} & \mathrm{Pic}(Q_{G,C}) = \oplus \mathbf{Z} \mathcal{O}_{Q_{G,C_i}}(1) \\ \varphi^* \downarrow & & \downarrow \\ \mathrm{Pic}(\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}) & \overset{\pi^*_{\mathrm{par}}}{\to} & \oplus_j \mathrm{Pic} F \oplus \mathbf{Z} \mathcal{O}_{Q_{G,C_i}}(1) \end{array}$$

with φ the forgetful morphism and the right vertical arrow is the inclusion as direct summand. Hence one has $\pi^*_{\mathrm{par}}(\varphi^* L'_i) = \mathcal{O}_{Q_{G,C_i}}(1), L'_i$ being the pull back of the generator of Pic (Bun_{G,C_i}) to Pic $(\mathrm{Bun}_{G,C})$. Thus for the surjectivity of π^*_{par} it suffices to show that there exist line bundles $\{L'_{i,j}\}$ on $\overline{\mathrm{Bun}}^{\mathrm{gpar}}_{G,C}$ which pullback to the generators of $\oplus_j$ Pic F.

From the construction and results in [S], it follows that Pic F is a lattice of rank r generated by $L'_i, i = 1, \ldots, r, r =$ rank of G. For each i, there exists a $G \times G$ module W_i and a $G \times G$ equivariant embedding $F \to P(W_i)$ such that $\mathcal{O}_{P(W_i)}(1)$ restricts to L'_i on F. Given a family of QPGs $(E, (\sigma_j))$ on $C \times \text{Spec } R$ one has $E^j(F) \subset E^j(P(W_i))$. Let L'_{ij} denote the line bundle on $E^j(P(W_i))$ (and also its restriction to $E^j(F)$) which restricts to $\mathcal{O}_{P(W_i)}(1)$ on each fibre. The pull-back of L'_{ij} by σ_j: Spec $R \to E^j(F)$ is a line bundle $L'_{ij,R}$ on Spec R. This construction can be done for any R. Hence $\{L'_{ij,R}\}$ define a line bundle L'_{ij} on the stack $\overline{\text{Bun}}^{\text{gpar}}_{G,C}$. By construction, $\overline{\pi}^*_{\text{par}}(L'_{ij})$ is the generator of the jth factor Pic F in Pic $(\overline{Q}^{\text{gpar}}_{G,C})$.

Case of classical groups

For the simple and simply connected classical groups $SL(n)$ and $Sp(2n)$ the compactifications F of G are defined using natural representations (described below). We claim that Theorem 4 holds in these cases also. The existence of the isomorphism $\overline{\pi}_{\text{par}}$ and injectivity of $\overline{\pi}^*_{\text{par}}$ can be seen exactly as in the proof of the Theorem 4. We only need to check the surjectivity of $\overline{\pi}^*_{\text{par}}$, this is done below.

4.6. *Case* $G = SL(n)$. For $G = SL(n)$, the compactification F of G using natural representation of G ([U1, U4]) can be described as follows. $SL(n) \times SL(n)$ is embedded diagonally in $SL(2n) \subset GL(2n)$. Let $G \to GL(V)$ be the natural representation. Let $P \subset SL(2n)$ be the stabilizer of the diagonal in $V \oplus V$; P is a maximum parabolic subgroup. The Grassmannian Gr $= SL(2n)/P$ is embedded in $\mathbf{P}(\overset{n}{\wedge} (V \oplus V))$ by Plücker embedding. Let $\{P_{i_1,\ldots,i_n}\}$ denote the Plücker coordinates. Let F be the hyperplane section of Gr defined by $P_{1,\ldots,n} = P_{n+1,\ldots,2n}$. Then F can be regarded as a compactification of $SL(n)$ with $SL(n)$ identified with the subset of F defined by $P_{1,\ldots,n} \neq 0$. The generator of Pic Gr $\approx \mathbf{Z}$ is the line bundle associated to the character w_n on P and its restriction to F is the generator L' of Pic $F \approx \mathbf{Z}$. $F - SL(n)$ is a divisor D' in F to which L' is associated. Given a family of QPGs $(E, (\sigma_j))$ on $C \times$ Spec R, one has $E^j(F) \subset E^j(\text{Gr})$. Let L'_j be the line bundle on $E^j(\text{Gr})$ associated to the P-bundle $E^j(SL(2n)) \to E^j(\text{Gr})$ via the character w_n. The pull back of L'_j by σ_j : Spec $R \to E^j(F) \subset E^j(\text{Gr})$ is a line bundle $L'_{j,R}$ on Spec R, $L'_{j,R}$ define a line bundle L'_j on the stalk $\overline{\text{Bun}}^{\text{gpar}}_{G,C}$. By construction, $\pi^*_{\text{par}}(L'_j)$ is the generator of the jth factor Pic F in Pic $\overline{Q}^{\text{gpar}}_{G,C}$. Hence the morphism in Theorem 4(2) is a surjection and thus an isomorphism for $G = SL(n)$ and F as above.

4.7. *Case* $G = Sp(2n)$. In case $G = Sp(2n)$ also one can use the natural representation of G to define F (§5, [U4]). Let $G \to GL(V)$ be the natural representation. We regard $Sp(2n)$ as the group $Sp(q, V)$ of automorphisms of V preserving a symplectic form (nondegenerate alternating form) q on V. Then F is the variety of maximum isotropic subspaces for $q \oplus (-q)$ on $V \oplus V$. The group $Sp(2n) \times Sp(2n) = Sp(q, V) \times Sp(-q, V)$ is embedded in $Sp(q \oplus (-q), V \oplus V) = \text{Sp}(4n)$ diagonally. Then $F \approx Sp(4n)/P$, P being the maximum parabolic subgroup of $Sp(4n)$ which is the stabilizer of the maximum isotropic subspace Δ_V of $V \oplus V$, Pic $F = \mathbf{Z}L'$, L' being the line bundle associated to the fundamental weight w_{2n}. Given a family of QPGs $(E, (\sigma_j))$ on C parametrized by $S =$ Spec R, $\sigma_j : S \to E^j(F)$, one has $E^j(F) = E^j(Sp(4n)/P)$ and $E^j(Sp(4n)) \to E^j(F)$ is a P-bundle. Let L'_j denote the line bundle on $E^j(F)$ associated to this P-bundle via the character w_{2n}. Let $L'_{j,R}$ denote the line bundle on S which is the pullback of this line bundle by σ_j. This construction being valid for any R, it defines a line bundle L'_j on the stack $\overline{\mathrm{B}}\mathrm{un}^{\mathrm{gpar}}_{G,C}$. Clearly, $\pi^*_{\mathrm{par}}(L'_j)$ is the generator of Pic F, the jth factor. It follows that the injection in Theorem 4(2) is an isomorphism for $G = Sp(2n)$ with F defined as above.

The following definitions and results are stated for $O(n)$-bundles, they hold for $Sp(2n)$-bundles also with orthogonal replaced by symplectic and n replaced by $2n$.

DEFINITION 4.8

An orthogonal bundle (E, q) on C is an I-tuple of vector bundles $E = (E_1, \ldots, E_I)$, $E_i =$ a vector bundle on Y_i with a nondegenerate quadratic form q_i and $q = (q_1, \ldots, q_I)$. We assume that rank $E_i = n$ for all i, we call n the rank of E. For a closed point $x \in C$, let q_x denote the induced quadratic form on the fibre E_x.

DEFINITION 4.9

A generalized quasiparabolic orthogonal bundle (*orthogonal QPB* in short) on C is an orthogonal bundle (E, q) of rank n together with n-dimensional vector subspaces $F^j_1(E)$ of $E_{x_j} \oplus E_{z_j}$ which are totally isotropic for $q_{x_j} \oplus (-q_{z_j})$.

Theorem 5. *Assume further that Y is irreducible. Then there is a coarse moduli space M for α-semistable orthogonal QPBs of rank n, $\alpha \in (0, 1)$ being rational. M is normal and complete*.

Let $\mathcal{U}$ be the moduli space of orthogonal sheaves of rank n on Y. Assume that $0 < \alpha < 1$, α is close to 1. Then

(1) there exists a morphism $f : M \to \mathcal{U}$.
(2) Let $\mathcal{U}^s_n$ be the subset of $\mathcal{U}$ corresponding to stable orthogonal bundles. Then the restriction of f to $f^{-1}(\mathcal{U}^s_n)$ is an isomorphism onto $\mathcal{U}^s_n$.

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