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Schubert varieties are arithmetically Cohen-Macaulay

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Introduction

Let G be a semisimple algebraic group over an algebraically closed field of arbitrary characteristic. Let B be a Borel subgroup and $Q \supset B$ a parabolic subgroup of G. By a Schubert variety we mean the closure of a B-orbit in G/Q. The main result we prove here is that any Schubert variety X is Cohen-Macaulay, in fact has rational singularities in the sense of Kempf, and in any projective embedding given by an ample line bundle on G/Q it is arithmetically Cohen-Macaulay.

For Schubert varieties in Grassmannians such results were proved by Hochster, Kempf, Laksov and Musili (see [4, 6, 7, 10, 12]). When Q is of "classical" type it was proved by DeConcini and Lakshmibai [1]. Further results can be found in [5, 9]. For $SL(n)/B$ this result is due to Seshadri and Musili [15]. We are thankful to Seshadri for pointing out this reference.

The key point of the present method is the simple observation that (in characteristic p) there is a canonical Frobenius splitting of G/B which *simultaneously* splits *all* the Schubert varieties in G/B. This fact trivially implies that (in any characteristic) the scheme theoretic intersection of any set of unions of Schubert varieties is reduced. It also gives by the methods of [13] the following generalisation to unions of Schubert varieties of the vanishing theorem proved in [13]:

Let X be the union of a collection of Schubert varieties in G/Q taken with the reduced subscheme structure. Let L be a line bundle on G/Q such that $H^0(G/Q, L) \neq 0$. Then $H^q(X, L) = 0$ for $q > 0$ and the restriction map $H^0(G/Q, L) \to H^0(X, L)$ is surjective.

Having the vanishing theorem for (unions of) Schubert varieties and the pleasant intersection properties of Schubert varieties available a priori makes the inductive machinery of standard resolutions (or Bott-Samelson-Demazure desingularisations, [7, 2]) easier to handle. For we no longer require it to prove these results also as part of the induction.

Let $\psi: Z \to X$ be a standard resolution of the Schubert variety $X \subset G/B$

(constructed in [7, 2]). The canonical bundle K_Z is of the form $\mathcal{O}_Z(-\partial Z) \otimes \psi^* L^{-1}$ where ∂Z is a divisor with normal crossings and L is an ample line bundle on X. It is this fact, combined with duality for the Frobenius map $F: Z \to Z$, that enabled one to prove the Frobenius splitting of Schubert varieties and deduce the vanishing theorems in [11, 13]. In this paper too this very simple form of K_Z together with the inductive method of standard resolutions gives easily (for the reasons mentioned above) the vanishing of the higher direct images $R^q \psi_* K_Z$. It is then a result of duality theory that X is Cohen-Macaulay.

Once we have proved that X is Cohen-Macaulay its arithmetic Cohen-Macaulayness follows using the Frobenius splitting and projective normality proved in [11] and [13].

The result on the intersection of Schubert varieties stated above was conjectured by Seshadri in [9]. It was proved there for the "classical" case by the methods of standard monomial theory.

The application of these methods to standard monomial theory will be taken up in [14].

The results of this paper have been announced in [13].

1. Standard resolutions of Schubert varieties

Let G be a semisimple algebraic group over an algebraically closed field k of arbitrary characteristic. Let T be a maximal torus and B a Borel subgroup containing T. We take the set of roots in B to be positive. We denote by W the Weyl group and by s_α the reflection corresponding to the root α.

Let $Q \supset B$ be a parabolic subgroup of G. We mean by a *Schubert variety* in G/Q the closure of a B-orbit in G/Q. We will mostly be working with Schubert varieties in G/B. Properties of Schubert varieties in G/Q often follow easily from the corresponding properties of Schubert varieties in G/B by using the fibration $G/B \to G/Q$.

We have the Bruhat decomposition $G/B = \bigcup_{w \in W} BwB/B$ of G/B into B-orbits. Since the number of orbits is finite it follows that any B-invariant irreducible closed subvariety of G/B is a Schubert variety.

We recall the following useful fact from Kempf [7].

Lemma 1. *Let α be a simple root and P_α be the minimal parabolic subgroup containing B and the root $-\alpha$. Let $\pi: G/B \to G/P_\alpha$ be the natural map; it is a $P_\alpha/B = \mathbb{P}^1$-fibration. Let X be the Schubert variety $\overline{BwB/B}$ corresponding to $w \in W$.*

(a) *X is a $\mathbb{P}^1$-bundle over its image under π if and only if $l(ws_\alpha) = l(w) - 1$.*

(b) *X is mapped birationally onto its image $\pi(X)$ if and only if $l(ws_\alpha) = l(w) + 1$. In that case $\pi^{-1}\pi(X)$ is the Schubert variety corresponding to ws_α.*

For the proof we refer to [7].

Suppose X and α are such that Case (a) of the above lemma holds, i.e.

$l(ws_\alpha)=l(w)-1$. Let $Y=\overline{Bws_\alpha B/B}$. Then by Case (b) of the above lemma Y maps birationally onto its image $\pi(Y)$. Moreover $\pi(Y)=\pi(X)$ and $X=\pi^{-1}\pi(Y)$. Kempf then calls Y a *moving divisor in X moved by α.*

Definition 1. Let X be Schubert variety in G/B. We denote the union of all the Schubert varieties contained in X and of codimension 1 in X by ∂X. It is a closed subset of X and we give it the reduced subscheme structure. We sometimes think of ∂X as a Weil divisor in X.

Note that $X-\partial X$ is the Bruhat cell BwB/B where $X=\overline{BwB/B}$.

Lemma 2. *Let α be a simple root and $X\subset G/B$ the Schubert variety $\overline{BwB/B}$. Suppose $l(ws_\alpha)=l(w)-1$.*

(a) *$Y=\overline{Bws_\alpha B/B}$ is the unique moving divisor in X moved by α. Any other codimension 1 Schubert subvariety of X is saturated under the fibration $\pi\colon G/B\to G/P_\alpha$.*

(b) *$Y\not\subset\pi^{-1}\pi(\partial Y)$ and $\partial X=Y\cup\pi^{-1}\pi(\partial Y)$.*

Proof. (a) Let $Y_1=\overline{Bw_1B/B}$ be a Codimension 1 Schubert subvariety in X. Suppose Y_1 is not saturated under π. Since X is saturated we then have $X=\pi^{-1}\pi(Y_1)$. By Lemma 1, Case (b), $w_1s_\alpha=w$. Thus $w_1=ws_\alpha$.

(b) The saturation of Y under π is X. Therefore if Y is contained in the saturated set $\pi^{-1}\pi(\partial Y)$ then the later must be X. But that is not possible since $\dim\pi^{-1}\pi(\partial Y)<\dim X$. Thus $Y\not\subset\pi^{-1}\pi(\partial Y)$. The other assertion follows from (a).

We now recall the standard resolutions of Schubert varieties as done in Kempf [7].

Let $X=\overline{BwB/B}$ be a Schubert variety of dimension r. Let $w=s_{\alpha_1}\dots s_{\alpha_r}$ be a reduced expression for w. We also complete this into a reduced expression for the element w_N of maximum length: $w_N=s_{\alpha_1}\dots s_{\alpha_r}\dots s_{\alpha_N}$. Let $w_i=s_{\alpha_1}\dots s_{\alpha_i}$ and X_i the corresponding Schubert variety of dimension i. Note that $X_r=X$. Since $l(w_is_{\alpha_i})=l(w_i)-1$ it follows from Lemma 1 that X_i is saturated for the morphism $\pi_i\colon G/B\to G/P_{\alpha_i}$ and X_{i-1} maps birationally onto its image $\pi_i(X_{i-1})=\pi_i(X_i)$. Thus X_{i-1} is a moving divisor in X_i moved by α_i.

The standard modification φ_i: $M_i\to X_i$ is defined by the Cartesian square:

$$\begin{tikzcd} M_i \arrow[r, "\varphi_i"] \arrow[d] & X_i \arrow[d, "\pi_i"] \\ X_{i-1} \arrow[r, "\pi_i"'] & \pi_i(X_{i-1}) \end{tikzcd}$$

Thus M_i is a $\mathbb{P}^1$-bundle over the moving divisor X_{i-1} in X_i.

The standard resolutions ψ_i: $Z_i\to X_i$ are defined inductively. We start by taking $Z_0=X_0$, a point, and $\psi_0\colon Z_0\to X_0$ the identity morphism. Then $\psi_i\colon Z_i\to X_i$ is defined by the Cartesian square in the diagram:

$$\begin{tikzcd}
Z_i \arrow[rr, "\psi_i"] \arrow[dr] \arrow[dd, shift left, "f_i"] & & X_i \arrow[dd, "\varphi_i"] \\
& M_i \arrow[ur] \arrow[dd] & \\
Z_{i-1} \arrow[uu, shift left, "\sigma_i"] \arrow[rr] \arrow[dr, "\psi_{i-1}"'] & & \pi_i(X_{i-1}) \\
& X_{i-1} \arrow[ur, "\pi_i"'] &
\end{tikzcd}$$

Thus ψ_i is the pull back of $Z_{i-1}\to\pi_i(X_{i-1})$ by the $\mathbb{P}^1$-fibration $X_i\to\pi_i(X_{i-1})$. f_i: $Z_i\to Z_{i-1}$ being the pull back of the $\mathbb{P}^1$-bundle $X_i\to\pi_i(X_{i-1})$ is a $\mathbb{P}^1$-bundle. Hence Z_i is nonsingular by induction. The section σ_i: $Z_{i-1}\to Z_i$ of f_i is given by the inclusion $X_{i-1}\subset X_i$. Further ψ_i is birational since $\pi_i|X_{i-1}$ and ψ_{i-1} are birational.

In Z_i we have the divisors $Z_{ii}=\sigma_i(Z_{i-1})$ and $Z_{ij}=f_i^{-1}\dots f_{j+1}^{-1}(\sigma_j(Z_{j-1}))$, $j=1,\dots,i-1$, totalling i in number. They are nonsingular and intersect transversally and $\sigma_N\sigma_{N-1}\dots\sigma_i(Z_{i-1})=Z_{NN}\cap\dots\cap Z_{Ni}$.

Since $X_r=X$ we get by this process a standard modification and a standard resolution for the Schubert variety X we started with. This of course depends on the reduced expression chosen for w. In the induction process we could have stopped at the r^{th} stage if we are interested only in constructing a resolution for X; but we will find it useful later to have gone up all the way to $X_N=G/B$.

Definition 2. For Z_i we define ∂Z_i to be the union $\bigcup_{j=1}^{i} Z_{ij}$. We give it the reduced subscheme structure. It is a Cartier divisor in Z_i.

The following proposition shows that the standard resolution ψ_i: $Z_i\to X_i$ is a B-equivariant resolution which converts the Weil divisor ∂X_i into the divisor ∂Z_i with normal crossings.

Proposition 1. (a) *There is a natural action of B on Z_i for every i, which makes the maps ψ_i: $Z_i\to X_i$, f_i: $Z_i\to Z_{i-1}$ and σ_i: $Z_{i-1}\to Z_i$ B-equivariant.*

(b) *$\psi_i(\partial Z_i)=\partial X_i$ and ψ_i maps $Z_i-\partial Z_i$ isomorphically onto $X_i-\partial X_i$.*

Proof. We assume by induction that the results hold for $i-1$. Since ψ_i is the pull back of the B-equivariant map $\pi_i\psi_{i-1}$: $Z_{i-1}\to\pi_i(X_{i-1})$ by the B-equivariant map $X_i\to\pi_i(X_{i-1})$ it follows that B acts on Z_i and ψ_i is B-equivariant. The same Cartesian square

$$\begin{tikzcd}
Z_i \arrow[r] \arrow[d, shift left, "f_i"] & X_i \arrow[d] \\
Z_{i-1} \arrow[u, shift left, "\sigma_i"] \arrow[r, "\pi_i\psi_{i-1}"'] & \pi_i(X_{i-1})
\end{tikzcd}$$

gives the B-equivariance of f_i. The B-equivariance of σ_i follows from that of $X_{i-1}\hookrightarrow X_i$.

Now π_i is an isomorphism when restricted to the Bruhat cell $Bw_{i-1}B/B$. By induction ψ_{i-1} is an isomorphism of $Z_{i-1}-\partial Z_{i-1}$ onto this cell. It then follows from the above diagram that ψ_i is an isomorphism of $Z_i - f_i^{-1}(\partial Z_{i-1})$ onto its image. Since $\psi_i(Z_{ii}) = X_{i-1}$ the proposition follows from Lemma 2 Part (b).

We now proceed to compute the canonical bundle K_{Z_i} of Z_i. This has already been done in [11, Section 3] and is a crucial fact in proving the Frobenius splitting of Schubert varieties. We give here a more geometric derivation of the result keeping the group theory to a minimum. We will need the following simple lemma to provide the key inductive step.

Lemma 3. *Let $f\colon X \to Y$ be a $\mathbb{P}^1$-bundle with X and Y smooth varieties. Let $\sigma\colon Y \to X$ be a section, D the divisor $\sigma(Y)$ in X and L_D the line bundle $\mathcal{O}_X(D)$ corresponding to the divisor D.*

(a) *The relative canonical bundle $K_{X/Y} = K_X \otimes f^* K_Y^{-1}$ is isomorphic to $L_D^{-2} \otimes f^* \sigma^* L_D$.*

(b) *If L is any line bundle on X whose degree along the fibres of f is 1 then $K_{X/Y} = L_D^{-1} \otimes (L^{-1} \otimes f^* \sigma^* L)$.*

Proof. From the normal bundle exact sequence $0 \to T_D \to \sigma^* T_X \to N_{X/D} \to 0$, where T_D and T_X are the tangent bundles of D and X and $N_{X/D}$ is the normal bundle of D in X, we get that the restriction of $K_{X/Y}$ to D is the conormal bundle of D in X. On the other hand since L_D^{-1} is the ideal sheaf of D in X the conormal bundle of D in X is also L_D^{-1} restricted to D. Thus we get $\sigma^* K_{X/Y} = \sigma^* L_D^{-1}$. Tensoring this with $\sigma^* L_D^2$ we get $\sigma^*(K_{X/Y} \otimes L_D^2) = \sigma^* L_D$. Now $K_{X/Y} \otimes L_D^2$ has degree zero along the fibres of f and hence comes from Y so that $K_{X/Y} \otimes L_D^2 = f^* \sigma^*(K_{X/Y} \otimes L_D^2)$. Comparing these two equalities we get $K_{X/Y} \otimes L_D^2 = f^* \sigma^* L_D$ proving (a).

Part (b) follows from (a) and the remark that if L_1, L_2 are two line bundles on X with the same degree along the fibres of f then $L_1^{-1} \otimes f^* \sigma^* L_1 = L_2^{-1} \otimes f^* \sigma^* L_2$. To prove this observe that $L_1^{-1} \otimes L_2$ comes from Y so that $L_1^{-1} \otimes L_2 = f^* \sigma^* (L_1^{-1} \otimes L_2)$. Rearranging this gives $L_1^{-1} \otimes f^* \sigma^* L_1 = L_2^{-1} \otimes f^* \sigma^* L_2$.

Remark 1. The canonical bundle $K_{G/B}$ of G/B is given by $K_{G/B} = L_\rho^{-2}$ where L_ρ is the line bundle on G/B associated to the B-bundle $G \to G/B$ by the character ρ, the half sum of positive roots. L_ρ is also the line bundle $\mathcal{O}_{G/B}(D)$ corresponding to the divisor $\partial(G/B) = \sum \overline{B w_N s_\beta B/B}$ the summation running through the simple roots β. Let $\tilde{B} = w_N B w_N$ be the opposite Borel subgroup. Then the $\tilde{B}$-orbit closures in G/B are called the *opposite Schubert varieties.* They are translates by w_N of the usual Schubert varieties. G being a rational variety translation preserves linear equivalence of divisors and hence the divisor D is linearly equivalent to $\tilde{D} = \sum \overline{\tilde{B} s_\beta B/B}$, the sum of codimension 1 opposite Schubert varieties. Thus $K_{G/B}^{-1} = \mathcal{O}_{G/B}(D + \tilde{D})$ so that $K_{G/B}^{-1}$ has a section whose divisor of zeroes is $D + \tilde{D}$.

Proposition 2. *The canonical bundle K_{Z_i} of the standard resolution is given by $K_{Z_i}^{-1} = \mathcal{O}_{Z_i}(\partial Z_i) \otimes \psi_i^* L_\rho$.*

Proof. Observe that $\psi_i^* L_\rho$ has degree 1 along the fibres of f_i for all i. For, by construction the general fibre of f_i is a fibre of $G/B \to G/P_{\alpha_i}$ and the degree of L_ρ

along the latter fibre is $\langle \alpha_i^\vee, \rho \rangle$ which is 1 (cf. [2]). To find K_{Z_i} we only have to add $K_{Z_i/Z_{i-1}}$ to $f_i^* K_{Z_{i-1}}$. The proposition follows easily using induction on i and Lemma 3, Part (b).

Remark 2. The above proposition shows that $K_{Z_N}^{-1}$ has a section with zeroes $\partial Z_N + \psi_N^{-1}(\tilde{D})$ where $\tilde{D}$ is as in Remark 1. Composing with the differential $d\psi_N$: $K_{Z_N}^{-1} \to \psi_N^* K_{G/B}^{-1}$ we get a section of $\psi_N^* K_{G/B}^{-1}$. Since G/B is smooth $\psi_{N*}\mathcal{O}_{Z_N} = \mathcal{O}_{G/B}$ and hence by the projection formula the global sections of $\psi_N^* K_{G/B}^{-1}$ are the same as those of $K_{G/B}^{-1}$. Thus we get a section σ of $K_{G/B}^{-1} = L_\rho^2$. Further ψ_N being an isomorphism on $Z_N - \partial Z_N$ (Proposition 1) σ has zeroes of multiplicity 1 on $\tilde{D}$ and it can possibly have other zeroes only along the components of D. But $K_{G/B}^{-1} = \mathcal{O}_{G/B}(D + \tilde{D})$ as we have seen in Remark 1. Thus $\sigma = 0$ must be the divisor $D + \tilde{D}$. Thus by this process whatever be the reduced expression for w_N we start with we arrive (upto scalar) at the same section σ of $K_{G/B}^{-1}$.

2. Simultaneous splitting of Schubert varieties

In this section the base field k will be of characteristic $p > 0$ unless otherwise mentioned. By a variety we mean a reduced scheme, not necessarily irreducible.

For convenience we recall some definitions and results from [11] and [13]. For more details see these references.

2.1. For any k-algebra A the map sending $a \in A$ to a^p is a ring homomorphism. For any scheme X this gives rise to the absolute Frobenius morphism $F: X \to X$. We call X *Frobenius split* if the p^{th} power map $\mathcal{O}_X \to F_* \mathcal{O}_X$ admits a splitting $F_* \mathcal{O}_X \to \mathcal{O}_X$. Note that Frobenius split schemes are reduced schemes.

2.2. For a line bundle L on X, $H^i(X, L \otimes F_* \mathcal{O}_X)$ is (semilinearly) isomorphic to $H^i(X, L^p)$. It follows that if X is Frobenius split then the Frobenius map $H^i(X, L) \to H^i(X, L^p)$ is injective. Therefore if L is ample and X is Frobenius split $H^i(X, L) = 0$ since $H^i(X, L^{p^\nu}) = 0$ for ν large.

2.3. If Y is a subvariety of X we say Y is compatibly split in X if there is a splitting $\varphi: F_* \mathcal{O}_X \to \mathcal{O}_X$ of X such that $\varphi(F_* I) \subset I$ where $I \subset \mathcal{O}_X$ is the ideal sheaf of Y in X. It then follows that Y is Frobenius split and for an ample line bundle L on X the restriction $H^0(X, L) \to H^0(Y, L)$ is surjective.

2.4. If the splitting $\varphi: F_* \mathcal{O}_X \to \mathcal{O}_X$ factors through $F_* \mathcal{O}_X \to F_*(\mathcal{O}(D))$ for some effective divisor D then for any bundle L on X we have $H^i(X, L) = 0$ if $H^i(X, L^p \otimes \mathcal{O}(D)) = 0$. Suppose further φ splits Y compatibly and that no irreducible component of Y is contained in the support of D. Then if $D = (p-1) D_0$ and L is a line bundle on X such that $L \otimes \mathcal{O}_X(D_0)$ is ample then $H^q(Y, L) = 0$ for $q > 0$ and $H^0(X, L) \to H^0(Y, L)$ is surjective. The proof of Theorem 2, Section 3 of [13] gives this result.

2.5. If $f: X \to Z$ is a morphism of projective varieties such that $f_* \mathcal{O}_X = \mathcal{O}_Z$ and $Y \subset X$ is compatibly split then the direct image of the splitting of Y in X gives a compatible splitting of $f(Y)$ in Z.

If X is nonsingular the duality for the finite flat map $F\colon X \to X$ (the absolute Frobenius) identifies $\mathcal{H}om(F_* \mathcal{O}_X, \mathcal{O}_X)$ with K_X^{1-p} (see [3], Exercises III 6.10 and 7.2). Thus a splitting $F_* \mathcal{O}_X \to \mathcal{O}_X$ can be identified with a section of K_X^{1-p}. Using this and Proposition 8 from [11] one can prove that the section $\tilde{\sigma}^{p-1}$ of $K_{Z_N}^{1-p}$, where $\tilde{\sigma}$ is the section of $K_{Z_N}^{-1} = \mathcal{O}_{Z_N}(\partial Z_N) \otimes \psi_N^*(\mathcal{O}_{G/B}(\tilde{D}))$ with its divisor of zeroes $\partial Z_N + \psi_N^{-1}(\tilde{D})$ (see Remark 2, Section 1), gives a splitting of Z_N which compatibly splits all the Z_i (considered as a subvariety of Z_N through the embedding $\sigma_N \ldots \sigma_{i+1}$). By considering the morphism $\psi_N\colon Z_N \to G/B$ the result stated in the previous paragraph then shows that the section σ^{p-1} of $K_{G/B}^{1-p}$ (see Remark 2, Section 1) which is the image of $\tilde{\sigma}^{p-1}$ gives a splitting of G/B which compatibly splits the N Schubert varieties X_i, $i = 0, \ldots, N-1$. But as observed in Remark 2, Section 1, the section σ is independent of the reduced expression chosen for w_N. Since any Schubert variety X occurs as a X_i for a suitable choice of reduced expression for w_N we have proved the following theorem for G/B. To prove it for G/Q we just take the direct image of this splitting of G/B under $G/B \to G/Q$.

Theorem 1. *Let the base field be of characteristic $p > 0$ and Q a parabolic subgroup of G. Then the homogeneous space G/Q has a natural Frobenius splitting which compatibly splits all the Schubert varieties in G/Q.*

This theorem and the result from [13] stated in §2.4 above lead to the following generalisation of the vanishing theorem proved in [13].

Theorem 2. *For this theorem the base field can be of arbitrary characteristic. Let L be a line bundle on G/Q, where Q is a parabolic subgroup of G, such that $H^0(G/Q, L) \neq 0$. Let $X_1, \ldots, X_r$ be a collection of Schubert varieties in G/Q and $X = X_1 \cup \ldots \cup X_r$ their union taken with the reduced subscheme structure. Then:*

(a) $H^q(X, L) = 0$ *for* $q > 0$

(b) *The restriction map $H^0(G/Q, L) \to H^0(X, L)$ is surjective.*

Proof. First we show that we can reduce to characteristic $p > 0$ base fields. This is shown in [11, Lemma 3, Section 3] and we repeat part of it here for convenience.

Let $\mathbf{G} \to \operatorname{Spec} \mathbb{Z}$ be the Chevalley group scheme of type G and $\mathbf{Q} \subset \mathbf{G}$ the corresponding parabolic subgroup scheme over $\mathbb{Z}$. Then $\mathbf{G}/\mathbf{Q} \to \operatorname{Spec} \mathbb{Z}$ is flat and the line bundle L on G/Q also comes from a line bundle $\mathbf{L}$ on $\mathbf{G}/\mathbf{Q}$.

We can also show that the Schubert variety $X \subset G/Q$ can be extended to a scheme $\mathbf{X} \to U$ where U is a nonempty open subscheme of $\operatorname{Spec} \mathbb{Z}$ such that for any prime $p \in U$ the reduction $\bmod p$ gives the Schubert variety in characteristic p. This can be seen by taking the scheme theoretic image of the orbit map $\mathbf{B} \to \mathbf{G}/\mathbf{Q}$ of the section $\operatorname{Spec} \mathbb{Z} \to \mathbf{G}/\mathbf{Q}$ given by the Weyl group element corresponding to X and using the lemma on generic flatness. (See [11, Lemma 3, Section 3].)

Then a simple application of semicontinuity to $\mathbf{L}$ on $\mathbf{G}/\mathbf{Q}$ and $\mathbf{X}$ over $U \subset \operatorname{Spec} \mathbb{Z}$ shows that it is enough to prove the theorem in positive characteristic.

We can also assume $Q = B$ by considering the fibration $\pi\colon G/B \to G/Q$ and the inverse images $\pi^{-1}(X_i)$.

Now that we are in characteristic $p>0$ we can make use of Theorem 1.

We use the notation of Remarks 1 and 2 of Section 1. The section σ of $K_{G/B}^{-1}$ has $D+\tilde{D}$ as its divisor of zeroes. Since the point Schubert variety $X_0 = B/B$ is contained in any Schubert variety and is not contained in $\operatorname{supp}\tilde{D}$ we see that none of the X_i can be contained in $\operatorname{supp}\tilde{D}$. Moreover we know that for L with nontrivial sections $L\otimes\mathcal{O}_{G/B}(\tilde{D})$ is ample. Now the theorem follows from the result of [13] stated in §2.4.

We next note below a trivial consequence of Theorem 1. It will be of crucial importance later in the proof of the Cohen-Macaulay property of Schubert varieties (cf. proof of Theorem 4 below).

Theorem 3. *For this theorem the base field can be of arbitrary characteristic. Let $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ be two collections of Schubert varieties in G/Q. Let $X = X_1 \cup \ldots \cup X_r$ and $Y = Y_1 \cup \ldots \cup Y_s$ be their unions taken with the reduced subscheme structures. Then their scheme theoretic intersection $X\cap Y$ is reduced.*

Proof. As in the proof of the previous theorem we first remark that we can assume that the base field is of positive characteristic. For, we can extend X and Y to subschemes $\mathbf{X}, \mathbf{Y} \subset \mathbf{G}/\mathbf{Q}$ flat over a nonempty open $U \subset \operatorname{Spec}\mathbb{Z}$ and taking intersection and reducing $\operatorname{mod} p$, for p in a suitable nonempty open subset of $\mathbb{Z}$ commute for $\mathbf{X}$ and $\mathbf{Y}$ because they commute for the generic point $0\in U$, $\mathbb{Z}\hookrightarrow\mathbb{Q}$ being flat. If there are nilpotents in the generic fibre they will show up in most special fibres also.

So we assume the base field is of positive characteristic. Then the theorem is a general fact about intersection of simultaneously compatibly split subvarieties. Let $I_l \subset \mathcal{O}_{G/Q}$ be the ideal sheaf of $X_l \subset G/Q$ and J_m that of Y_m. By Theorem 1 there is a splitting $\varphi\colon F_*\mathcal{O}_{G/Q} \to \mathcal{O}_{G/Q}$ of G/Q with the property $\varphi(F_* I_l) \subset I_l$ and $\varphi(F_* J_m) \subset J_m$. Since the ideal I of $X\cap Y$ is $(I_1 \cap \ldots \cap I_r) + (J_1 \cap \ldots \cap J_m)$ it follows that $\varphi(F_* I) \subset I$. This shows that $X\cap Y$ is compatibly split in G/Q. In particular it is Frobenius split which implies that it is reduced, for otherwise the map $\mathcal{O}_{X\cap Y} \to F_*\mathcal{O}_{X\cap Y}$ will not be injective and has no chance of having a splitting.

This result was conjectured in [9] and proved there for the "classical" case. Kempf had proved it for special Schubert varieties [7, Cor. 3, Section 5]. For the other conjectures stated in [9] see [14].

Remark 3. Actually the splitting given by σ^{p-1} compatibly splits the opposite Schubert varieties as well. This follows from symmetry or from the remark preceding Proposition 9 in [11]. So in Theorem 3 we can allow some of the X_l and Y_m to be opposite Schubert varieties also.

3. Singularities of Schubert varieties

In this section the base field can have arbitrary characteristic. We first recall the notions of trivial morphism and rational resolution introduced by Kempf.

Definition 3 (Kempf [7], p. 567). Let $f\colon X\to Y$ be a morphism of schemes. We call f to be *trivial* if the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective and the higher direct images $R^q f_* \mathcal{O}_X$ vanish for $q>0$. The *image* of a trivial morphism f is the closed subscheme of Y having $f_* \mathcal{O}_X$ as structure sheaf. We denote the image scheme by $f(X)$. It is the scheme theoretic image of f.

Remarks 4. (a) If $f\colon X \to Y$ is trivial with image Y and L is a line bundle on Y then $H^i(Y, L) = H^i(X, f^* L)$. This follows from Leray spectral sequence and the projection formula.

(b) Any closed immersion is a trivial map.

(c) Base change of a trivial morphism by a flat morphism is trivial, the image of the base change being the base change of the image.

(d) If $f\colon X \to Y$ is a trivial morphism and $g\colon Y \to Z$ is such that $g|f(X)\colon f(X) \to Z$ is trivial then the composite $gf\colon X \to Z$ is trivial. This follows from the Grothendieck spectral sequence for the composite.

We have the following ingenious and very useful proposition due to Kempf.

Proposition 3. *Let $f\colon X \to Y$ be a proper morphism of algebraic schemes and $X' \subset X$ a closed subscheme. Let $Y' \subset Y$ be the scheme theoretic image of X' and L an ample line bundle on Y. Suppose that the following conditions hold:* (a) $f_* \mathcal{O}_X = \mathcal{O}_Y$, (b) $H^q(X, f^* L^n) = 0$ *and* $H^q(X', f^* L^n) = 0$ *for* $q>0$ *and n large and* (c) $H^0(X, f^* L^n) \to H^0(X', L^n)$ *is surjective for large n. Then f is trivial with image Y and the restriction $f'\colon X' \to Y$ of f to X' is also trivial with image Y'.*

This follows from a Leray spectral sequence argument, (see [2], Proposition 2 § 5). We remark that if X is reduced then its scheme theoretic image is also reduced so that in this case if Conditions (a), (b), (c) hold the image of the trivial map f' is reduced.

Lemma 4. *Let $f\colon X \to Y$ be a proper morphism of schemes and $X' \subset X$ a closed subscheme whose ideal sheaf is $I \subset \mathcal{O}_X$. Suppose f is trivial with image Y. Then the following two conditions are equivalent:*

(a) $R^q f_* I = 0$ *for* $q>0$

(b) $f|X'\colon X' \to Y$ *is trivial.*

When either of these conditions holds $f_* I \subset \mathcal{O}_Y$ is the ideal sheaf of the image of $f|X'$.

Proof. Everything follows easily from the long exact sequence

$$0 \to f_* I \to f_* \mathcal{O}_X \to f_* \mathcal{O}_{X'} \to R^1 f_* I \ldots$$

obtained from $0 \to I \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$ by applying the functor f_*.

We record the following trivial lemma for later reference.

Lemma 5. *Let X, Y be reduced closed subschemes of Z with ideal sheaves I_X and I_Y in $\mathcal{O}_Z$. Consider the set theoretic intersection $(X \cap Y)$ as a reduced closed subscheme of X and $J \subset \mathcal{O}_X$ be the corresponding ideal sheaf. Then if the scheme theoretic intersection $X \cap Y$ in Z is reduced then the restriction map $\mathcal{O}_Z \to \mathcal{O}_X$ maps I_Y surjectively onto J.*

Proof. Trivial.

Definition 4 (Kempf [8], p. 50). A proper birational morphism $f\colon X \to Y$ with X smooth is called a *rational resolution* of Y if (a) f is trivial with image Y, i.e.

$f_* \mathcal{O}_X = \mathcal{O}_Y$ and $R^q f_* \mathcal{O}_X = 0$ for $q > 0$ and (b) $R^q f_* K_X = 0$ for $q > 0$ where K_X is the canonical line bundle of X.

Remark. Condition (b) is always satisfied if the base field is of characteristic zero by a generalisation of Kodaira vanishing theorem (due to Grauert-Riemannschneider).

Proposition 4. *Suppose* (for simplicity) *that X and Y are projective varieties. Then if $f\colon X \to Y$ is a rational resolution then Y is Cohen-Macaulay.*

Proof. The proof is by duality. We know [3, Chapter III, Theorem 7.6] that the projective variety Y is Cohen-Macaulay if and only if for an ample line bundle L on Y we have $H^i(Y, L^{-n}) = 0$ for $i < \dim Y$ and n large. Since f is trivial by Leray spectral sequence $H^i(Y, L^{-n}) = H^i(X, f^* L^{-n})$. Since X is smooth projective by Serre duality $H^i(X, f^* L^{-n})$ is dual to $H^{N-i}(X, K_X \otimes f^* L^n)$ where N is the dimension of X. Again by Leray spectral sequence and the fact $R^q f_* K_X = 0$ for $q > 0$ it follows that $H^{N-i}(X, K_X \otimes f^* L^n) = 0$ which completes the proof.

We now have everything we need to prove the main the theorem.

Theorem 4. *Any standard resolution $\psi\colon Z \to X$ of a Schubert variety X in G/B is a rational resolution.*

Proof. We choose a reduced expression for the maximal element w_N such that the resolution $\psi\colon Z \to X$ occurs at some step in the ladder of resolutions $\psi_i\colon Z_i \to X_i$ constructed in Section 1. We keep the notation from Section 1. We use induction on i to prove the theorem. For convenience we temporarily drop the subscript $i+1$ from $Z_{i+1}, X_{i+1}, \sigma_{i+1}, f_{i+1}$ etc. and denote them simply by Z etc.

We have the basic Cartesian diagram:

$$\begin{tikzcd}
Z \arrow[rr, "\psi"] \arrow[d, shift left, "f"] & & X \arrow[d, "\pi"] & \subset G/B \arrow[d, "\pi"] \\
Z_i \arrow[u, shift left, "\sigma"] \arrow[r, "\psi_i"'] & X_i \arrow[r, "\pi"'] & \pi(X) & \subset G/P_\alpha
\end{tikzcd}$$

We first prove the following claim:

A_{i+1}: $\psi\colon Z \to X$ is a trivial morphism with image X. The restriction $\psi | f^{-1}(\partial Z_i)$: $f^{-1}(\partial Z_i) \to X$ is also trivial and its image is the reduced subvariety $\pi^{-1}(\pi \partial X_i) = \overline{\partial X - X_i}$.

We prove this by induction. Thus we assume that the corresponding statement A_i at the i^{th} step is true.

Let L be an ample bundle on G/P_α. Consider the line bundle $\pi^* L$ on G/B and the Schubert variety $X_i \subset G/B$ and the union of Schubert varieties $\partial X_i \subset G/B$. Then by the vanishing theorem (Theorem 2, Section 2) it follows that the Conditions (a), (b) and (c) of Kempf's proposition (Proposition 3 of this section) are satisfied for both the subvarieties X_i and ∂X_i of G/B for the line bundle $\pi^* L$. Therefore we conclude that $\pi | X_i$: $X_i \to \pi(X)$ is trivial with image $\pi(X)$ and $\pi | \partial X_i$: $\partial X_i \to \pi(X)$ is trivial with the reduced subscheme $\pi(\partial X_i)$

as its image. Since ψ and $\psi|f^{-1}(\partial Z_i)$ are the base change of $(\pi|X_i)\circ\psi_i$ and $(\pi|\partial X_i)\circ\psi_i$ by the flat map $\pi\colon X\to\pi(X)$ the claim follows. In particular we have proved that standard resolutions are trivial maps.

We next turn to proving that $R^q\psi_* K_Z$ vanishes for $q>0$. By Proposition 2, Section 1 we know that $K_Z=\mathcal{O}_Z(-\partial Z)\otimes\psi^* L_\rho^{-1}$. Using the projection formula this gives $R^q\psi_* K_Z=L_\rho^{-1}\otimes R^q\psi_*\mathcal{O}_Z(-\partial Z)$. So we only have to prove that $R^q\psi_*\mathcal{O}_Z(-\partial Z)=0$ for $q>0$. Since $\mathcal{O}_Z(-\partial Z)\subset\mathcal{O}_Z$ is the ideal sheaf of ∂Z in Z by Lemma 4 above it is enough to prove the following claim

B_{i+1}: $\psi|\partial Z\colon \partial Z\to X$ is trivial with image the reduced subvariety ∂X.

This is done by induction on i. So we assume that the statement B_i holds. Consider the exact sequence

$$0\to\mathcal{O}_Z(-\sigma(Z_i))\to\mathcal{O}_Z\to\sigma_*\mathcal{O}_{Z_i}\to 0.$$

Tensor this with $f^*\mathcal{O}_{Z_i}(-\partial Z_i)$ to get:

$$0\to\mathcal{O}_Z(-\partial Z)\to f^*\mathcal{O}_{Z_i}(-\partial Z_i)\to\sigma_*\mathcal{O}_{Z_i}(-\partial Z_i)\to 0.$$

The middle sheaf has vanishing higher direct images under ψ by the claim A_{i+1} proved above and Lemma 4. The last sheaf has the same property by induction assumption. Hence it follows that $R^q\psi_*\mathcal{O}_Z(-\partial Z)=0$ for $q\geqq 2$. To prove it for $q=1$ we should prove that $\psi_* f^*\mathcal{O}_{Z_i}(-\partial Z_i)\to\psi_*\sigma_*\mathcal{O}_{Z_i}(-\partial Z_i)$ is surjective. Since ψ is trivial and $\psi f^{-1}(\partial Z_i)=\overline{\partial X-X_i}$ (Lemma 2) it follows from Lemma 4 that $\psi_* f^*\mathcal{O}_{Z_i}(-\partial Z_i)$ is the ideal sheaf of the reduced subvariety $\overline{\partial X-X_i}$ in X. For similar reasons $\psi_*\sigma_*\mathcal{O}_{Z_i}(-\partial Z_i)$ is the ideal sheaf of the reduced subvariety ∂X_i in X_i extended by zero to the whole of X. Since by Theorem 3, $\overline{\partial X-X_i}\cap X_i$ is *reduced* it follows that the required restriction map is surjective by Lemma 5 above. This completes the proof of the theorem.

Theorem 5. *Let Q be a parabolic subgroup of G and X a Schubert variety in G/Q. Then*

(a) *X is Cohen-Macaulay*

(b) *X is arithmetically Cohen-Macaulay in any projective embedding by an ample line bundle on G/Q.*

Proof. By Theorem 4 and Proposition 4 it follows that Schubert varieties in G/B are Cohen-Macaulay. For $X\subset G/Q$ the inverse image $\pi^{-1}(X)$ under $\pi\colon G/B\to G/Q$ is a Schubert variety in G/B and hence Cohen-Macaulay. Since $\pi^{-1}(X)\to X$ is a Q/B-fibration it follows that X is Cohen-Macaulay.

Let L be an ample line bundle on G/Q. Since X is Cohen-Macaulay $H^q(X,L^{-n})=0$ for $q<d$ and large n, [3, III Theorem 7.6]. Since X is Frobenius split in characteristic p this implies $H^q(K,L^{-n})=0$ for $q<d$ and all $n\geqq 1$ in characteristic p and hence in all characteristics by semicontinuity. For the same reason $H^q(X,L^n)=0$ for $q>0$ and $n\geqq 1$. Now, the standard resolution $\psi\colon Z\to\pi^{-1}(X)$ is trivial and the Q/B fibration $\pi^{-1}(X)\to X$ is also trivial. Therefore $H^q(X,\mathcal{O}_X)=H^q(Z,\mathcal{O}_Z)$ and the latter vanishes for $q>0$ since Z is built as a sequence of $\mathbb{P}^1$-fibrations starting from a point.

So the upshot is that $H^q(X, L^n)=0$ for $d>q>0$ and all $n\in\mathbb{Z}$. Moreover the projective normality of X, proved in [13], implies that the natural map from the m^{th} symmetric power $S^m H^0(X,L)$ to $H^0(X,L^m)$ is surjective for all $m\geqq 1$. These facts give the arithmetic Cohen-Macaulay property.

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