Generalized parabolic sheaves on an integral projective curve

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Abstract. We extend the notion of a parabolic vector bundle on a smooth curve to define generalized parabolic sheaves (GPS) on any integral projective curve X. We construct the moduli spaces \( M(X) \) of GPS of certain type on \( X \). If \( X \) is obtained by blowing up finitely many nodes in \( Y \) then we show that there is a surjective birational morphism from \( M(X) \) to \( M(Y) \). In particular, we get partial desingularisations of the moduli of torsion-free sheaves on a nodal curve \( Y \).

Keywords. Generalized parabolic sheaf; projective curve.

1. Introduction

In [1] we defined and studied GPBs (generalized parabolic bundles) on an irreducible nonsingular projective curve. The notion easily generalizes to a GPS (= generalized parabolic sheaf) on an integral projective curve \( X \). A GPS is a torsion-free sheaf \( E \) together with an additional structure called parabolic structure over disjoint effective Cartier divisors \( \{ D_j \}_{j \in J} \), \( J \) a finite set (see Definitions 1.3, 1.4). In [1] we constructed moduli spaces for GPBs with parabolic structure of certain type over a single divisor (i.e. \( J = \) singleton). Here we consider many divisors. Moreover, \( X \) being singular, the method used in [1] fails. Therefore we generalize the method of Simpson [4] for the construction of moduli spaces.

**Theorem 1.** There exists a (coarse) moduli space \( M_{X,J}(k, d) \) of semistable GPS \( F \) of rank \( k \), degree \( d \) with parabolic structure over \( D_j \) given by a flag \( \mathcal{F}_j: H^0(F \otimes \mathcal{O}_{D_j}) \supseteq F_j^1(F) \supseteq \cdots \supseteq F_j^{\chi}(F), \) and weights \((0, \varpi)\), where \( a_j = \dim F_j^1(F) \) and rational number \( \alpha \) are fixed with \( 0 < \alpha < 1 \). \( M_{X,J} = M_{X,J}(k, d) \) is a projective variety of dimension \( k^2(g-1) + 1 + \sum_j a_j(k \text{ degree } D_j - a_j), g = \text{arithmetic genus of } X \).

If \( X \) is nonsingular, then \( M(k, d) \) is normal. If further \((k, d) = 1, a_j = \text{multiple of } k \) and \( \alpha \) is close to 1 then \( M(k, d) \) is nonsingular and is a fine moduli space.

**Theorem 2.** Let \( X \) be the curve (proper transform) obtained by blowing up nodes \( \{ y_j \}_{j \in J} \) of an integral projective curve \( Y, \pi_{XY}: X \to Y \) surjection. For \( j \in J \), let \( D_j = \pi_{XY}^{-1}(y_j), a_j = k \). Then there exists a surjective birational morphism \( f_{XY}: M_{X,J,Y}: \to M_{Y,J'} \). In particular, if \( J' = \phi \), \( X \) = desingularization of \( Y \), \((k, d) = 1, \alpha \) close to 1, then \( M_{X,J} \) is the desingularization of the moduli space \( M_{Y,\phi} \) of semistable torsion-free sheaves on \( Y \). Further, if \( X' \) is a partial desingularization of \( Y \), obtained by blowing up \( y_j, j \in J' \), \( \pi_{X',X} \):
X → X', π_{X'}: X' → Y, then (with suitable D_j and parabolic structure as above) f_{XY} = f_{X'Y} \circ f_{XX}'. Thus M_{X', Y} is a partial desingularization of M_{X, Y}.

There is a close relationship between torsion-free sheaves on a singular curve Y and GPS on its desingularization. An analogue of Theorem 2 holds if \{y_j\} are ordinary cusps, and hopefully also in case each y_i is an ordinary n-tuple point with linearly independent tangents.

1. Preliminaries

Let X be an integral projective curve defined over an algebraically closed field k. Let \omega_X denote the dualising sheaf on X, it is a torsion-free sheaf. For a torsion-free sheaf E on X we denote by \( r(E) \) and \( d(E) \) respectively the rank and degree of E. Let \( \{D_j\} \) be finitely many effective divisors on X such that supports of \( D_j \) are mutually disjoint.

DEFINITION 1.1

A quasi-parabolic structure on E over \( D_j \) is a flag \( F^j(E) \) of vector subspaces of \( H^0(E \otimes \mathcal{O}_{D_j}) \) viz.

\[ F^j(E): F^0_j(E) \supseteq H^0(E \otimes \mathcal{O}_{D_j}) \supseteq F^1_j(E) \supseteq \cdots \supseteq F^r_j(E) = 0. \]

DEFINITION 1.2

Let \( \mathcal{F} = \{F^j(E)\}_{j \in J} \). A QPS is a pair \( (E, \mathcal{F}(E)) \) where E is a torsion-free sheaf and \( \mathcal{F}(E) \) is a quasiparabolic structure on \( \{D_j\} \) as above.

DEFINITION 1.3

A parabolic structure on E over \( D_j \) is a quasiparabolic structure \( F^j(E) \) (Sec. 1.1) together with an \( r_j \)-tuple of real numbers \( \alpha^j = (\alpha^1_j(E), \ldots, \alpha^{r_j}_j(E)), 0 \leq \alpha^1_j(E) < \cdots < \alpha^{r_j}_j(E) < 1 \), called weights associated to \( F^j(E) \).

Let \( m^j_i = \dim F^i - \dim F^{i+1}, i = 1, \ldots, r_j \). Define \( \text{wt}_j(E) = \sum_{i=1}^{r_j} m^j_i \alpha^i_j(E), \text{wt}(E) = \sum_j \text{wt}_j(E) \). Let \( \text{par } d(E) = d(E) + \text{wt}(E) \), \( \text{par } \mu(E) = \text{par } d(E)/r(E) \).

DEFINITION 1.4

A GPS (generalized parabolic sheaf) is a triple \( (E, \mathcal{F}(E), \alpha) \) with \( \mathcal{F}, \alpha \) as in 1.1 and 1.3.

1.5

Let \( K \) be a subsheaf of E such that the quotient \( E/K \) is torsion-free in a neighbourhood of \( D \). Let \( h: K \rightarrow E \) be the inclusion map. Since D is a divisor and \( E/K \) is torsion-free, one has \( \text{Tor}_1^\ast(E/K, \mathcal{O}_E) = 0 \) and therefore \( h_{\mathcal{D}}: K_{\mathcal{D}} \rightarrow E_{\mathcal{D}} \) is an injection. Hence \( H^0(K \otimes \mathcal{O}_E) \) can be identified with a subspace \( F^j_1(K) \) of \( F^j_1(E) \). Define \( F^j_1(K) = F^j_1(K) \cap F^j_1(E) \). This gives (after omitting repetitions) a flag \( \mathcal{F}^j(K), j \in J \). The set \( \{\alpha^j_1(K)\} \) of weights for \( K \) is a subset of \( \{\alpha^j_1(E)\} \) defined as follows. One has \( F^j_1(K) = \)
$F^i(K) \cap F^j(E)$ for some $i$, let $i'_n$ be largest such $i$. Then $\alpha'^i(K) := \alpha'^i_n(E)$. Thus a subsheaf of a GPS with torsion-free quotient gets a natural structure of a GPS.

**DEFINITION 1.6**

A GPS $(E, \mathcal{F}(E), \alpha)$ is semistable (respectively stable) if for every (resp. proper) subsheaf $K$ of $E$ with torsion-free quotient, one has $\text{par } \mu(K) \leq (\text{resp. } <) \text{par } \mu(E)$.

**Remarks 1.7.**  (1) If $E/K$ is not torsion-free, then we may still define $F^i_n(K) = \text{image of } H^0(K \otimes \mathcal{O}_P) \text{ under } H^0(h_P)$ and define $\mathcal{F}^j(K)$ by intersecting $F^j_n(K)$ with the flag $\mathcal{F}^j(E)$. Thus we can talk of $wK$. If $M$ is the largest subsheaf of $E$ containing $K$, with $E/M$ torsion-free and $r(K) = r(M)$ then $\text{par } \mu(K) \leq \text{par } \mu(M)$. Then the condition of 1.6 is satisfied for every subsheaf $K$ of $E$ if $(E, \mathcal{F}(E), \alpha)$ is a semistable (resp. stable) GPS. (2) There exists a natural parabolic structure on a quotient sheaf also. Semistability and stability can also be defined equivalently using quotients instead of subsheaves. (See 3.4, 3.5 [1]).

**Assumptions 1.8.** In this paper we want to study moduli spaces of GPS $(F, \mathcal{F}(F), \alpha)$ of the form $\mathcal{F}^j(F) : F^j(F) \Rightarrow F^j(F) \Rightarrow 0, \alpha^j = (0, \alpha), 0 < \alpha < 1$. We also assume that for all $j$ support of $D_j$ is contained in the set of nonsingular points of $X$. Henceforth we restrict ourselves to bundles of the above type. We also assume that the base field is that of complex numbers.

**DEFINITION 1.9**

A morphism of GPS is a morphism of torsion-free sheaves $f : F \to F'$ such that $(f|D_j)(F^j(F)) \subseteq F^j(F')$ for all $j$.

**Lemma 1.10.** Let $(F, \mathcal{F}(F), \alpha)$ be a semistable GPS. Then there exists an integer $n_1$, dependent only on $g (= \text{arithmetic genus of } X)$ and degree $D_j, j \in J$ such that if $\chi(F) = n > n_1$, then

1. $H^1(F) = 0, C^n \approx H^0(F)$,
2. $F$ is generated by global sections,
3. $H^0(F) \to H^0(F \otimes \mathcal{O}_D)$ is onto.

**Proof.** This follows from $H^i(F') \approx H^0(X, \text{Hom}(F', \omega_X))^*$ and the latter is zero if $p \mu(F')$ is sufficiently large (depending on $\chi(F'), g$). For (3) we need to take $F' = F(-D_j)$. (For details, see Lemma 3.7 [1]).

**Lemma 1.11.** A morphism $f$ of semistable GPS of same par $\mu$ is of constant rank. If the GPS have the same rank and one of them is stable, then either $f = 0$ or $f$ is an isomorphism.

**Proof.** This can be proved similarly as in Lemma 3.8 [1].

**COROLLARY 1.12.**

A stable GPS is simple i.e. its only endomorphisms are homotheties.
PROPOSITION 1.13.

The category $S$ of all semistable GPS on $X$ (of type described in 1.18) with a fixed par $\mu = m$ is an abelian category. Its simple objects are the stable GPS.

Proof. This follows from 1.11 and 1.12.

DEFINITION 1.14

In view of the above proposition, a semistable GPS $(E, \mathcal{F}, \alpha)$ in $S$ has a filtration with successive quotients stable GPS with par $\mu = m$. We denote by $\text{gr}(E, \mathcal{F}, \alpha)$ the associated graded object for the filtration. Up to isomorphism this object is independent of the choice of stable filtration. Define an equivalence relation on $S$ by $(E, \mathcal{F}, \alpha)$ is equivalent to $(E', \mathcal{F}', \alpha)$ iff $\text{gr}(E, \mathcal{F}, \alpha) \approx \text{gr}(E', \mathcal{F}', \alpha)$.

Remark 1.5. We may (for convenience) use the terminology ‘a GPS $E$’ when there is no confusion about parabolic structure possible.

2. Construction of the moduli space

2.1 Consider semistable GPS $F$ of type described in 1.8 with rank $k$, Euler characteristic $n > n_1$ fixed. Let $P(m)$ be the Hilbert polynomial of $F$. Let $Q = Q(\mathcal{O}^n, P(m))$ be the quotient scheme of coherent sheaves over $X$ which are quotients of $\mathcal{O}^n$ and have Hilbert polynomial equal to $P$. Let $\mathbb{F}$ denote the universal quotient sheaf on $Q \times X$. Let $R$ be the open subscheme of $Q$ consisting of points $q \in Q$ such that $\mathcal{F}_q = \mathcal{F} | q \times X$ is torsion-free and the map $H^0(\mathcal{O}^n) \to H^0(\mathcal{F}_q)$ is an isomorphism. It follows that $H^1(\mathcal{F}_q) = 0$ for $q \in R$. For every $j$, let $p_j: R \times D_j \to R$ be the canonical map and define $V_j = (p_j)_*(\mathcal{F}(\mathcal{O}))$. Let $G(V_j)$ be the flag bundle over $R$ of the type determined by the parabolic structure over $D_j$. It is a relative Grassmannian bundle of quotients of rank $q_j$. Let $\bar{R}$ denote the fibre product of $\{G(V_j)\}_{j}$ over $R$. Let $\bar{R}^s$ (resp. $\bar{R}^{ss}$) denote the subset of $\bar{R}$ corresponding to stable (resp. semistable) GPS. Similarly we can define $\bar{Q}$, $\bar{Q}^s$ and $\bar{Q}^{ss}$.

The quotient scheme $Q$ has a natural embedding in a Grassmannian. For $m \geq M_1(n)$, the natural map $H^0(\mathcal{O}^n)(m) \to H^0(\mathcal{F}_q(m))$ is surjective for all $q \in Q$. Let $W = H^0(\mathcal{O}(m))$, then $H^0(\mathcal{F}_q(m))$ is a quotient of $W$ of dimension $P(m)$ for $m \geq M_1$. This gives a closed embedding $Q \to \text{Grass}_{pm}(\mathbb{C}^n \otimes W)$. A point $q$ of $\bar{Q}$ gives for each $j$, a $q_j$-dimensional quotient of $\mathbb{C}^n$. Hence we get an embedding $\bar{Q} \to Z = \text{Grass}_{pm}(\mathbb{C}^n \otimes W) \times (\times_j \text{Grass}_{q_j}(\mathbb{C}^n))$. This embedding is equivariant under the action of $\text{PGL}(n)$. The $\text{PGL}(n)$ action on $\bar{Q}$ and $\mathbb{C}^n$ is the natural one, while on $W$ it acts trivially. On $Z$ we take the polarization

$$a(n - \sum_j q_j)/km \times a\alpha \times \cdots \times a\alpha,$$

where $1$ denotes $\mathcal{O}(1)$ and $a$ is a sufficiently big integer to make all the numbers above integers, $n > \sum_j q_j$.

We denote a point of $Z$ by $(P_j, (p_j))$ where $P: \mathbb{C}^n \otimes W \to U$, $P_j: \mathbb{C}^n \to U_j$ are surjective maps, dim $U = P(m)$, dim $U_j = q_j$ for all $j$. Similarly a point of $\bar{Q}$ is denoted by $(p, (p_j))$. 
where \( p: \mathcal{O}^n \to F, p_j: H^0(F|D_j) \to Q^j \) are surjections, \( \dim Q^j = q_j \forall j \). For a subsheaf \( E \) of \( F \), we define \( Q^j_E = p_j(H^0(E|D_j)) \). For a quotient \( q:F \to G \), define \( Q^j_q = H^0(G|D_j)/q(\text{Ker} \ p_j) \). For simplicity of notation, we denote \( \dim(Q^j) \) (\( \dim(Q^j_E) \)) by \( q_j(E) \) (by \( q_j(G) \)). In particular, \( q_j = q_j(F) \).

**PROPOSITION 2.2**

For a nontrivial proper subspace \( H \subset C^n \) of dimension \( h \) define \( \sigma_H \) by

\[
\sigma_H = \left( -\alpha \sum_j q_j \right) \left( n \text{dim} P(H \otimes W) \right)
\]

\[
+ \alpha \sum_j (q_j h - \text{dim} P_j(H)).
\]

Then a point \( (P_j, p_j) \) of \( Z \) is semistable (resp. stable) for \( PGL(n) \) action (with the above polarization) if and only if \( \sigma_H \leq 0 \) (respectively < 0).

**Proof.** See [3, Proposition 5.1.1] and [2, Proposition 4.3].

**DEFINITION 2.3**

Let \( F \) be a torsion-free sheaf of rank \( k \) on \( X \). For every subsheaf \( E \) of \( F \) and \( m \geq 0 \) integer define

\[
\chi_E(m) = \left( -\alpha \sum_j q_j \right) \left( n \chi(E) P(m) - n \chi(E(m)) \right)
\]

\[
+ \alpha \sum_j (q_j \chi(E) - n q_j(E)),
\]

\[
\sigma_E(m) = \left( n \text{dim} h^0(E) \right) \left( n \chi(E) P(m) - n \chi(E(m)) \right)
\]

\[
+ \alpha \sum_j (q_j h^0(E) - n q_j(E)).
\]

**Lemma 2.4.** Let \( F \) be a torsion-free sheaf corresponding to a point \( (p_j, (p_j)_j) \) of \( \mathcal{O} \). Then \( F \) is semistable (respectively stable) if and only if for every subsheaf \( E \) of \( F \) we have \( \chi_E = \chi_E(m) \leq 0 \) (resp. < 0) for any integer \( m \).

**Proof.** Let \( E \) be a subsheaf of \( F \) with \( F/E \) torsion-free. Substituting \( P(m) = km + n \), \( \chi(E(m)) = \chi(E) + m r(E)(r(E) = \text{rank of } E) \) in the expression for \( \chi_E \) and simplifying one gets

\[
\chi_E(m) = nr(E) \left[ \frac{\chi(E)}{r(E)} - \frac{n}{k} + \alpha \sum q_j/k - \alpha \sum q_j(E)/r(E) \right].
\]

By definition \( F \) is semistable (respectively stable) if and only if the expression in the square bracket is \( \leq 0 \) (resp. < 0).
Suppose now that $F/E$ is not torsion-free. Then there exists $\tilde{E}, E \subset \tilde{E}$ such that $F/\tilde{E}$ is torsion-free, rank $E = \text{rank} \ E$. Let $\tilde{E}/E = \tau = \tau + \Sigma j \tau j$, where $\tau j = \tau j d j$. By the above argument, $\chi_{\tilde{E}}(m) \leq 0$. We claim that $\chi_{\tilde{E}}(m) < \chi_{E}(m)$. Using $\chi(\tilde{E}(m)) - \chi(E(m)) = h^0(\tau)$ for $m \geq 0$, $q_j(\tilde{E}) - q_j(E) \leq h^0(\tau j)$ we get $\chi_{\tilde{E}}(m) - \chi_{E}(m) = n(\alpha \Sigma j h^0(\tau j) - h^0(\tau)) < 0$ since $\alpha < 1$.

Lemma 2.5. There exists an integer $M_2(n) \geq M_1(n)$ such that if $(p,(p_j)) \in \tilde{Q}$ is a point satisfying the following two conditions, then the image of the point in $Z$ is semistable (resp. stable).

(1) The canonical map $\mathbb{C}^n \rightarrow H^0(F)$ is an isomorphism.
(2) For every subsheaf $E$ of $F$, generated by global sections, $\sigma_E(m) \leq 0$ (resp. $< 0$) for $m \geq M_2(n)$.

Proof. Let $H \subset \mathbb{C}^n$ be a subspace. Let $E$ be the subsheaf of $F$ generated by $H$ and let $K$ be the kernel of the surjection $H \otimes \mathcal{O}_X \rightarrow E$. As $H$ varies over subspaces of $\mathbb{C}^n$ and $F$ varies over $\tilde{Q}$, the sheaves $E$ and hence $K$ form a bounded family. Hence there exists $M_2(n)$ such that for $m \geq M_2(n)$, $h^1(E(\tilde{E})) = 0$, $h^1(E(m)) = 0$ and $h^1(K(m)) = 0$ for all such $E$ and $K$. It follows that $\dim P(H \otimes W) = \chi(E(m))$. Clearly $\dim H \leq h^0(E)$, $\dim P_j(H) = q_j(E)$. Therefore $\sigma_H \leq \sigma_E(m) \leq 0$ (resp. $< 0$). Thus the image $(P, (P_j))$ of $(p,(p_j))$ is semistable (resp. stable).

Lemma 2.6. One can find $M_3(n) \geq M_2(n)$ such that for $m \geq M_3(n)$ the following holds. If $(p,(p_j)) \in \tilde{Q}$ is a point whose image in $Z$ is semistable then (i) $\mathbb{C}^n \rightarrow H^0(F)$ is injective and (ii) for all torsion-free quotients $F \rightarrow G \rightarrow 0$, one has $\tau_G \leq 0$. Here $\tau_G$ is defined by

$$\tau_G = \left( \frac{n - \alpha \Sigma j q_j}{k} \right) \left( -kh^0(G) + nr(G) \right) + \alpha \Sigma j (nq_j(G) - q_jh^0(G)).$$

Proof. Note that if $H_0$ is the kernel of the map $\mathbb{C}^n \rightarrow H^0(F)$, then $\sigma_{H_0} > 0$, contradicting the semistability of the image point in $Z$ (Proposition 2.2). Hence (i) follows. For (ii), suppose that there exists a torsion-free quotient $G$ with $\tau_G > 0$. Then $h^0(G) < n$, for $h^0(G) \geq n$ implies $\tau_G \leq 0$. Let $H$ be the kernel of the composite $\mathbb{C}^n \rightarrow H^0(F) \rightarrow H^0(G)$. Let $E$ denote the subsheaf of $F$ generated by $H$. Clearly we have $r(E) + r(G) \leq k$, $h^0(G) \geq n - h, q_j(G) \leq q_j - q_j(E)$, $\dim P_j(H) \leq q_j(E)$. Substituting these in the expression for $\tau_G$ one gets

$$\left( h - nq_j \right) (kh - nr(E)) + \alpha \Sigma j (q_jh - nq_j(E)) > 0.$$ 

Since $H$ and hence $E$ runs over a bounded family we can find $M_3(n) \geq M_2(n)$ such that for $m \geq M_3(n)$, the term $kh - nr(E)$ can be replaced by $(hP(m) - n(\chi(X(m))/m = (hP(m) - n \dim P(H \otimes W))/m$. Thus we get $\sigma_H > 0$ contradicting the semistability of the image point in $Z$.

Lemma 2.7. There exists $n_2 \geq n_1$ such that for all semistable $GPS F$ with Euler characteristic $n \geq n_2$ the following holds
(1) If \( E \subset F \) then \( \tau_E \leq 0 \) where
\[
\tau_E = \left( \frac{\left( n - \alpha \sum q_j \right)}{k} \right) (kh^0(E) - nr(E)) + \alpha \sum q_j h^0(E) - nl_j(E).
\]
(2) If \( \tau_E = 0 \) for some \( E \subset F \), then \( \chi_E = 0 \).
(3) If \( \tau_E < 0 \), then \( \sigma_E(m) < 0 \) for \( m \geq M_A(n) \). If \( \tau_E = 0 \), then \( \sigma_E(m) = 0 \) for \( m \geq M_A(n) \).

Proof. (1) Let \( 0 = E_0 \subset E_1 \subset \cdots \subset E_n = E \) be the Harder-Narasimhan filtration of \( E \) considered as a torsion-free sheaf only, ignoring the parabolic structure. Let \( \Omega = E_i/E_{i-1} \), \( i = 1, \ldots, r \), \( \mu_i = \text{degree} \, \Omega_i \), \( Q_i, v = \inf \mu_i \). One has \( \mu_i > \mu_{i+1} \) \( \forall i < r \) (by definition), \( h^0(E) \leq \sum h^0(Q_i) \) (by induction). Using Corollary 2.5 [4], this implies \( h^0(E) \leq \sum r(Q_i) (\mu_i + B_1), B_1 \) constant. Since \( \Omega_1 \) is a subsheaf of a semistable GPS \( F \) we have \( \mu_i \leq \mu(F) + wv_i, w = (wtF)/k \). Since \( r_\infty, \sum r(Q_i) = r(E), v = v_\infty \) we get \( h^0(E) \leq \sum (\mu(F) + v + B_1 + (r(Q_i) - 1) (\mu(F) + w + B_1) \leq v + r(E) - 1) \).

(2) \( \tau_E = 0 \), then by the above argument one must have \( v \geq n/k - B_2 \), \( \chi(E) = h^0(E) \) and \( \tau_E = \chi_E \). Thus \( \chi_E = \sigma_E(m) = 0 \).

(3) Note that \( \tau_E = \lim_{m \to \infty} \sigma_E(m) \). Hence given \( \varepsilon > 0 \), \( \exists M_A(n) \) such that for \( m \geq M_A(n) \), \( \sigma_E(m) < \tau_E + \varepsilon \). If \( \tau_E < 0 \) then choosing \( \varepsilon \) such that \( \tau_E + \varepsilon < 0 \), we get \( \sigma_E(m) < 0 \) for \( m \geq M_A(n) \).

Theorem 1. (1) Let \( X \) be an integral projective curve of arithmetic genus \( g \) over \( C \). Let \( \{D_j\}_{j \in J} \) be finitely many effective Cartier divisors in \( X \) such that the support of \( D_j \) does not intersect the set of singular points of \( X \).\( \forall j \), supports of \( D_j \) are mutually disjoint and degree \( D_j = d_j, j \in J \). Let \( S \) denote the set of equivalence classes of semistable GPS \( F \) of rank \( q \) degree \( d \) with parabolic structure over \( D_j \) given by \( F_j(F) = H^0(F \otimes O_{D_j}) \supset F_j(F) \supset \cdots \supset F_1(F) = 0 \), co-dimension of \( F_j(F) \) in \( F_j(F) \) equal to \( q_j \) (fixed) for \( j \in J \) and weights \( (0, \alpha, 0 < \alpha < 1) \). Then \( S \) has the structure of a projective variety \( M(k, d) \) of dimension \( k^2 (q - 1) + 1 + \sum q_j (kd_j - q_j) \).

(2) If \( X \) is nonsingular, then \( M(k, d) \) is normal. If \( \text{further} (k, d) = 1, q_j \) is a multiple of \( k \) and \( \alpha \) is sufficiently near \( 1 \) then \( M(k, d) \) is nonsingular and it is a fine moduli space.

Proof. Let \( w_j \) denote the dualising sheaf of \( X_j \), it is a torsion-free sheaf. Fix \( n > \max (n_2, kh^0(w) + \alpha \sum q_j) \) and \( m \geq M_A(n) \). We keep the notations of 2.1. We shall show that a geometric invariant theoretic quotient of \( \tilde{R} \) modulo \( PGL(n) \) exists. Our required moduli space \( M(k, d) \) will be this quotient. \( \tilde{R} \) is an open subset of \( \tilde{Q} \), \( \tilde{Q} \) is embedded in \( Z \) (with \( m, n \) as above) by a \( PGL(n) \) equivariant embedding. We first claim that if \( (p, p_j) \in \mathbb{R}^2 \) (resp. \( \mathbb{R}^3 \)) then its image belongs to \( Z^n \) (resp. \( Z^2 \)). This follows immediately from Lemma 2.7 and Lemma 2.5. Let \( F \) correspond to a point in \( \tilde{R}^n - \tilde{R}^2 \). Then \( F \) has a subsheaf \( E \) which is a torsion-free stable GPS with \( \text{par} \mu(E) = \text{par} \mu(F) \) i.e. \( \chi_E = 0 \). For such an \( E \), \( \sigma_{H^0(E)} = \chi_E \) (Lemma 1.10), hence the image in \( Z \) belongs to \( Z^n - Z^2 \).
Conversely we shall now check that if a point in \( \mathcal{Q} \) is such that its image belongs to \( Z^a \), then the point is in \( \mathcal{Q}^a \) i.e. if \( F \) is the corresponding quotient, then \( F \) is torsion-free, the map \( C^a \to H^0(F) \) is an isomorphism and \( F \) is a semistable GPS. Lemma 2.6 implies that \( C^a \to H^0(F) \) is injective and for every rank 1 torsion-free quotient \( G \) of \( F \), \( n \leq kh^0(G) + \alpha \Sigma_j q_j \) (as \( \tau_0 \leq 0 \)). We claim that \( H^1(F) = 0 \). Otherwise there exists a nontrivial homomorphism \( F \to w_X \). If \( G \) is the sheaf image of this morphism, \( h^0(w_X) \geq h^0(G) \) and hence \( n \leq kh^0(w_X) + \alpha \Sigma_j q_j \), contradicting the assumptions on \( n \). Thus \( h^0(F) = n \) and \( C^a \to H^0(F) \) is an isomorphism. Let \( \tau \) be the torsion subsheaf of \( F \), \( \tau = \tau_\alpha + \Sigma_j \tau_j \), support \( \tau_j \subseteq \text{supp} \ D_j \), (supp \( \tau_\alpha \) \( \cap \ (\cup \text{supp} \ D_j) = \emptyset \). Taking \( H = H^0(\tau_\alpha), H^0(\tau_j), \sigma_H \leq 0 \) gives \( H^0(\tau_\alpha) = 0, H^0(\tau_j) = 0 \). Here \( \alpha < 1 \) is crucial since \( \sigma_H = n(h - \alpha \dim P_j(H)) \). Thus \( H^0(\tau) = H^0(\tau_\alpha) + \Sigma_j H^0(\tau_j) = 0 \) i.e. \( \tau = 0 \).

Suppose that \( F \) is not semistable. Then there exists a subsheaf \( E \) of \( F \) such that \( E \) is a semistable GPS with \( \text{par} \mu(E) > \text{par} \mu(F) \) i.e. \( \chi_E > 0 \). By Lemma 1.10, \( \sigma_{h^{\text{G}(E)}} = \chi_E > 0 \) contradicting the semistability of the image point in \( Z \).

It follows that the (geometric invariant theoretic) quotient \( M(k,d) \) of \( \mathcal{R} \) mod \( \text{PGL}(n) \) is the same as that of \( \mathcal{Q} \) and it exists if and only if the quotient of image of \( \mathcal{Q} \) in \( Z \) exists. It is well known that the latter exists. The quotient \( M(k,d) \) is a projective variety as \( \mathcal{Q} \) is so. It is easy to check that the points of \( M(k,d) \) correspond to equivalence classes of semistable GPS(3.15, [1]; [4]).

(2) If \( X \) is nonsingular \( \mathcal{R} \) is known to be nonsingular and hence \( M(k,d) \) is normal. If \( (k,d) = 1, \alpha \) is sufficiently near to \( 1 \) and \( q_j \) is an integral multiple of \( k \), then GPS is semistable if and only if it is stable by Lemma 3.3 (or Lemma 3.17, [1]). The nonsingularity of \( \mathcal{R} \) together with corollary 1.12 then imply that \( M(k,d) \) is nonsingular.

One can show that \( M(k,d) \) is then a fine moduli space, by proving the universal bundle on \( \mathcal{R} \) descends to a universal bundle on \( M(k,d) \) after twisting by a line bundle (see [1], Proposition 3.18).

3. Application

3.1. Let \( Y \) be an integral projective curve with only singularities ordinary double points \( \{y_j\}_{j \in J} \). Let \( J' \subseteq J \) be a subset. Let \( X \) be the curve (proper transform) obtained by blowing up \( \{y_j\}_{j \in J'} \). Let \( \pi_{xy}: X \to Y \) be the natural morphism. Let \( D_j \) denote the divisor \( \pi_{xy}^{-1}(y_j), j \in J' \). All the QPS and GPS that we consider are assumed to be of the type described in 1.8. We also assume that \( \dim F^1_j(F) = r(F) \) for \( j \in J' \).

**DEFINITION 3.2**

Let \( \alpha \) be a real number in \( [0,1] \). A QPS \( (F, \mathcal{F}) \) on \( X \) is \( \alpha \)-stable (resp. \( \alpha \)-semistable) if for any proper subsheaf \( K \) of \( F \) with torsion-free quotient, one has

\[
(d(K) + \alpha \Sigma_{j \in J'} \dim F^1_j(K))/r(K) < (\leq)(d(F) + \alpha' r(F))/r(F).
\]

**Remark.** For \( 0 < \alpha < 1 \), the above condition is same as that for stability (resp. semistability) of the GPS\((F, \mathcal{F}, \alpha)\) with \( \alpha' = (0, \alpha), j \in J' \).
Lemma 3.3. (1) Suppose that $1 - 1/J’r(F)(r(F) - 1) < \alpha < 1$. Then $(F, \mathcal{F})$ is $\alpha$-semistable implies that it is 1-semistable. If the QPS is 1-stable then it is also $\alpha$-stable.

(2) Assume that $(r(F), d(F)) = 1$. Then the QPS is 1-stable if and only if it is 1-semistable. Thus under the assumptions of (1) and (2) 1-stability, $\alpha$-stability and $\alpha$-semistability are all equivalent.

Proof. This is a straightforward generalization of Lemma 3.17 [1].

PROPOSITION 3.4.

Let $Q$ denote the set of isomorphism classes of QPS $(F, \mathcal{F})$ on $X$ of given type (3.1). Let $r(F) = k, d(F) = d$ be fixed. Let $S$ be the set of isomorphism classes of torsion-free sheaves of rank $k$ and degree $d$ on $Y$. Let $S_k$ denote the subset of $S$ corresponding to sheaves which are locally free at $y_j$ for $j \in J'$. Then (a) there is a surjective map $f_{XY}: Q \rightarrow S$ such that its restriction to $f_{XY}^{-1}(S_k)$ is a bijection onto $S_k$. (b) $(F, \mathcal{F})$ is 1-stable (1-semistable) iff its image under $f_{XY}$ is stable (semistable).

Proof. Let $D_j = x_j + z_j$. Then $((\pi_{XY}), F) \otimes k(y_j) = (k(x_j) \otimes k(z_j))^{(F)} = H^0(F \otimes O_{D_j}) = F^0_j(F)$. Thus we have a surjective $O_Y$-linear map $(\pi_{XY}), F \rightarrow F^0_j(F)$. Let $F^* = \ker$ the inverse of this map with the surjection $F^0_j(F) \rightarrow F^0_j(F)/F^1_j(F)$. Since $d(F) = 0(F) - r(F)\chi(O_Y), d(F^*) = 0(F^*) - r(F)\chi(O_Y), \chi(F) = \chi((\pi_{XY}), F), \chi(O_Y) = \chi(O_X) - J'$, it follows that if $(F, \mathcal{F}) \in Q$ then $F \in S$. We define $f_{XY}(F, \mathcal{F}) = F^*$. If $F \in S_k$ then $F = \pi_{XY}^* F'$ and $F^1_j(F) = F^0 \otimes k(y_j) \subset F^0_j(F)$ gives the bijection. Surjectivity of $f$ can be proved as in 4.5 [1] while the last assertion follows exactly as in 4.2 [1].

Theorem 2. (I) Let $M_{X,J}$ be the moduli space of semistable GPS on $X$ of type described in 3.1. Assume that $\alpha$ satisfies the conditions of Lemma 3.3(1). Then there is a surjective birational morphism $f_{XY}: M_{X,J} \rightarrow M_{Y,\phi} (= \text{moduli space of torsion-free sheaves on } Y)$.

(II) Let $Z$ be the desingularization of $Y$. Then the morphism $f_{XY}^*: M_{Z,J} \rightarrow M_{Y,\phi}$ factors as $f_{XY} \circ f_{XX}$. If the conditions of Lemma 3.3 are satisfied then $M_{X,J}$ is a desingularization of $M_{X,\phi}$ and $M_{X,J}(J' \subset J)$ are partial desingularizations.

Proof. (I) This follows easily from Lemma 3.3 and Proposition 3.4 since it is easy to globalise the construction (of $f_{XY}$) to families of GPS. (See Theorem 2 [1] for details).

(II) Let $f_{XX}(F, \mathcal{F}) = F^*$. Notice that $\pi_{XX}$ is an isomorphism outside $J - J'$. Hence $F^*$ has a parabolic structure $\mathcal{F}'$ over $D_j$ for $j \in J$ viz. $F^1(F^*) \approx F^1_j(F^*)$ for $i = 0, 1, j \in J'$. Thus $f_{XX}(F, \mathcal{F}) = (F^*, \mathcal{F}') \in M_{X,J}$. Let $f_{XY}(F^*, \mathcal{F}') = F^*$. Then we have the exact sequences (defining $F^*, \mathcal{F}'$)

$$0 \rightarrow F^* \rightarrow (\pi_{XY}), F \rightarrow \bigoplus_{j \in J'} F^0_jj(F)/F^1_j(F) \rightarrow 0$$

$$0 \rightarrow F^* \rightarrow (\pi_{XY}), F^* \rightarrow \bigoplus_{j \in J'} F^0_jj(F^*)/F^1_j(F^*) \rightarrow 0.$$ 

Using these and $\pi_{XY} = \pi_{XY} \pi_{XX}$, one gets

$$0 \rightarrow F^* \rightarrow (\pi_{XY}), F \rightarrow \bigoplus_{j \in J'} F^0_jj(F)/F^1_j(F) \rightarrow 0$$

proving $f_{XY} = f_{XX} f_{XY}$. The last assertion follows from Theorem 1 (II).
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References