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Generalized parabolic sheaves on an integral projective curve

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Abstract. We extend the notion of a parabolic vector bundle on a smooth curve to define generalized parabolic sheaves (GPS) on any integral projective curve X. We construct the moduli spaces $M(X)$ of GPS of certain type on X. If X is obtained by blowing up finitely many nodes in Y then we show that there is a surjective birational morphism from $M(X)$ to $M(Y)$. In particular, we get partial desingularisations of the moduli of torsion-free sheaves on a nodal curve Y.

Keywords. Generalized parabolic sheaf; projective curve.

1. Introduction

In [1] we defined and studied GPBs (generalized parabolic bundles) on an irreducible nonsingular projective curve. The notion easily generalizes to a GPS (= generalized parabolic sheaf) on an integral projective curve X. A GPS is a torsion-free sheaf E together with an additional structure called parabolic structure over disjoint effective Cartier divisors $\{D_j\}_{j\in J}$, J a finite set (see Definitions 1.3, 1.4). In [1] we constructed moduli spaces for GPBs with parabolic structure of certain type over a single divisor (i.e. J = singleton). Here we consider many divisors. Moreover, X being singular, the method used in [1] fails. Therefore we generalize the method of Simpson [4] for the construction of moduli spaces.

Theorem 1. *There exists a (coarse) moduli space $M_{X,J}(k,d)$ of semistable GPS F of rank k, degree d with parabolic structure over D_j given by a flag $\mathscr{F}^i : H^\circ(F\otimes\mathcal{O}_{Dj})\supset F_1^j(F)\supset 0, \forall_j\in J$ and weights $(0,\alpha)$, where $a_j = \dim F_1^j(F)$ and rational number α are fixed with $0<\alpha<1$. $M_{X,J} = M_{X,J}(k,d)$ is a projective variety of dimension $k^2(g-1)+1+\Sigma_j a_j(k \text{ degree } D_j - a_j)$, g = arithmetic genus of X.*

If X is nonsingular, then $M(k,d)$ is normal. If further $(k,d)=1$, a_j = multiple of k and α is close to 1 then $M(k,d)$ is nonsingular and is a fine moduli space.

Theorem 2. *Let X be the curve (proper transform) obtained by blowing up nodes $\{y_j\}_{j\in J}$ of an integral projective curve $Y, \pi_{XY}: X\to Y$ surjection. For $j\in J$, let $D_j = \pi_{XY}^{-1}(y_j)$, $a_j = k$. Then there exists a surjective birational morphism $f_{XY}: M_{X,J\cup J'}\to M_{Y,J'}$. In particular, if $J'=\phi, X$ = desingularization of Y, $(k,d)=1, \alpha$ close to 1, then $M_{X,J}$ is the desingularization of the moduli space $M_{Y,\phi}$ of semistable torsion-free sheaves on Y. Further, if X' is a partial desingularization of Y, obtained by blowing up $y_j, j\in J', \pi_{X,X'}$:*

$X \to X', \pi_{X'Y}: X' \to Y$, then (with suitable D_j and parabolic structure as above) $f_{XY} = f_{X'Y} \circ f_{XX'}$. Thus $M_{X'J'}$ is a partial desingularization of $M_{Y,\phi}$.

There is a close relationship between torsion-free sheaves on a singular curve Y and GPS on its desingularization. An analogue of Theorem 2 holds if $\{y_j\}$ are ordinary cusps, and hopefully also in case each y_i is an ordinary n-tuple point with linearly independent tangents.

1. Preliminaries

Let X be an integral projective curve defined over an algebraically closed field k. Let ω_X denote the dualising sheaf on X, it is a torsion-free sheaf. For a torsion-free sheaf E on X we denote by $r(E)$ and $d(E)$ respectively the rank and degree of E. Let $\{D_j\} j \in J$ be finitely many effective divisors on X such that supports of D_j are mutually disjoint.

DEFINITION 1.1

A quasi-parabolic structure on E over D_j is a flag $\mathscr{F}^j(E)$ of vector subspaces of $H^0(E \otimes \mathcal{O}_{D_j})$ viz.

$$\mathscr{F}^j(E): F_o^j(E) \equiv H^0(E \otimes \mathcal{O}_{D_j}) \supset F_1^j(E) \supset \cdots \supset F_r^j(E) = 0.$$

DEFINITION 1.2

Let $\mathscr{F}(E) = \{\mathscr{F}^j(E)\}_{j \in J}$. A QPS is a pair $(E, \mathscr{F}(E))$ where E is a torsion-free sheaf and $\mathscr{F}(E)$ is a quasiparabolic structure on $\{D_j\} j \in J$ as above.

DEFINITION 1.3

A parabolic structure on E over D_j is a quasiparabolic structure $\mathscr{F}^j(E)$ (See. 1.1) together with an r_j-tuple of real numbers $\alpha^j = (\alpha_1^j(E), \cdots, \alpha_{r_j}^j(E)), 0 \leqslant \alpha_1^j(E) < \cdots < \alpha_{r_j}^j \cdot (E) < 1$, called weights associated to $\mathscr{F}^j(E)$.

Let $m_i^j = \dim F_{i-1}^j(E) - \dim F_i^j(E), i = 1, \cdots, r_j$. Define $wt_j(E) = \Sigma_{i=1}^{r_j} m_i^j \alpha_i^j(E)$, $wt E = \Sigma_j wt_j(E)$. Let par $d(E) = d(E) + wt(E)$, par $\mu(E) = \text{par}\, d(E)/r(E)$.

DEFINITION 1.4

A GPS (generalized parabolic sheaf) is a triple $(E, \mathscr{F}(E), \alpha)$ with $\mathscr{F}, \alpha$ as in 1.1 and 1.3.

1.5

Let K be a subsheaf of E such that the quotient E/K is torsion-free in a neighbourhood of D. Let $h: K \to E$ be the inclusion map. Since D is a divisor and E/K is torsion-free, one has $\mathrm{Tor}_1^{\mathcal{O}}(E/K, \mathcal{O}_D) = 0$ and therefore $h|_D: K|_D \to E|_D$ is an injection. Hence $H^0(K \otimes \mathcal{O}_D)$ can be identified with a subspace $F_o^j(K)$ of $F_o^j(E)$. Define $F_i^j(K) = F_o^j(K) \cap F_i^j(E)$. This gives (after omitting repetitions) a flag $\mathscr{F}^j(K), j \in J$. The set $\{\alpha_i^j(K)\}$ of weights for K is a subset of $\{\alpha_i^j(K)\}$ defined as follows. One has $F_i^j(K) =$

$F_i^j(E) \cap F_o^j(K)$ for some i, let i_o be largest such i. Then $\alpha_i^j(K) := \alpha_{i_o}^j(E)$. Thus a subsheaf of a GPS with torsion-free quotient gets a natural structure of a GPS.

DEFINITION 1.6

A GPS$(E, \mathscr{F}(E), \boldsymbol{\alpha})$ is semistable (respectively stable) if for every (resp. proper) subsheaf K of E with torsion-free quotient, one has par $\mu(K) \leqslant$ (resp. $<$) par $\mu(E)$.

Remarks 1.7. (1) If E/K is not torsion-free, then we may still define $F_o^i(K) =$ image of $H^0(K \otimes \mathcal{O}_D)$ under $H^0(h|_D)$ and define $\mathscr{F}^j(K)$ by intersecting $F_o^j(K)$ with the flag $\mathscr{F}^j(E)$. Thus we can talk of wtK. If M is the largest subsheaf of E containing K, with E/M torsion-free and $r(K) = r(M)$ then par $\mu(K) \leqslant$ par $\mu(M)$. Then the condition of 1.6 is satisfied for every subsheaf K of E if $(E, \mathscr{F}(E), \boldsymbol{\alpha})$ is a semistable (resp. stable) GPS. (2) There exists a natural parabolic structure on a quotient sheaf also. Semistability and stability can also be defined equivalently using quotients instead of subsheaves. (See 3.4, 3.5 [1]).

Assumptions 1.8. In this paper we want to study moduli spaces of GPS$(F, \mathscr{F}(F), \boldsymbol{\alpha})$ of the form $\mathscr{F}^j(F): F_o^j(F) \supset F_1^j(F) \supset 0, \alpha^j = (0, \alpha), 0 < \alpha < 1$. We also assume that for all j support of D_j is contained in the set of nonsingular points of X. Henceforth we restrict ourselves to bundles of the above type. We also assume that the base field is that of complex numbers.

DEFINITION 1.9

A morphism of GPS is a morphism of torsion-free sheaves $f: F \to F'$ such that $(f|D_j)(F_1^j(F)) \subseteq F_1^j(F')$ for all j.

Lemma 1.10. *Let $(F, \mathscr{F}(F), \boldsymbol{\alpha})$ be a semistable GPS. Then there exists an integer n_1 dependent only on g(= arithmetic genus of X) and degree $D_j, j \in J$ such that if $\chi(F) = n > n_1$, then*

(1) $H^1(F) = 0, \mathbf{C}^n \approx H^0(F)$,
(2) *F is generated by global sections,*
(3) *$H^0(F) \to H^0(F \otimes \mathcal{O}_{D_j})$ is onto.*

Proof. This follows from $H^1(F') \approx H^0(X, \mathrm{Hom}(F', \omega_X))^*$ and the latter is zero if $p\mu(F')$ is sufficiently large (depending on $\chi(F'), g$). For (3) we need to take $F' = F(-D_j)$. (For details, see Lemma 3.7 [1]).

Lemma 1.11. *A morphism f of semistable GPS of same par μ is of constant rank. If the GPS have the same rank and one of them is stable, than either $f = 0$ or f is an isomorphism.*

Proof. This can be proved similarly as in Lemma 3.8 [1].

COROLLARY 1.12.

A stable GPS is simple i.e. its only endomorphisms are homothesies.

PROPOSITION 1.13.

The category S of all semistable GPS on X (of type described in 1.18) with a fixed par $\mu = m$ is an abelian category. Its simple objects are the stable GPS.

Proof. This follows from 1.11 and 1.12.

DEFINITION 1.14

In view of the above proposition, a semistable GPS$(E, \mathscr{F}, \alpha)$ in S has a filtration with successive quotients stable GPS with par $\mu = m$. We denote by $\mathrm{gr}(E, \mathscr{F}, \alpha)$ the associated graded object for the filtration. Up to isomorphism this object is independent of the choice of stable filtration. Define an equivalence relation on S by $(E, \mathscr{F}, \alpha)$ is equivalent to $(E', \mathscr{F}', \alpha)$ iff $\mathrm{gr}(E, \mathscr{F}, \alpha) \approx \mathrm{gr}(E', \mathscr{F}', \alpha)$.

Remark 1.5. We may (for convenience) use the terminology 'a GPS E'' when there is no confusion about parabolic structure possible.

2. Construction of the moduli space

2.1 Consider semistable GPS F of type described in 1.8 with rank k, Euler characteristic $n > n_1$ fixed. Let $P(m)$ be the Hilbert polynomial of F. Let $Q = Q(\mathcal{O}^n, P(m))$ be the quot scheme of coherent sheaves over X which are quotients of $\mathcal{O}^n$ and have Hilbert polynomial equal to P. Let $\mathscr{F}$ denote the universal quotient sheaf on $Q \times X$. Let R be the open subscheme of Q consisting of points $q \in Q$ such that $\mathscr{F}_q = \mathscr{F} | q \times X$ is torsion-free and the map $H^0(\mathcal{O}^n) \to H^0(\mathscr{F}_q)$ is an isomorphism. It follows that $H^1(\mathscr{F}_q) = 0$ for $q \in R$. For every j, let $p_j : R \times D_j \to R$ be the canonical map and define $V_j = (p_{j*})(\mathscr{F}_{|D_j})$. Let $G(V_j)$ be the flag bundle over R of the type determined by the parabolic structure over D_j. It is a relative Grassmannian bundle of quotients of rank q_j. Let $\tilde{R}$ denote the fibre product of $\{G(V_j)\}_j$ over R. Let $\tilde{R}^s$ (resp. $\tilde{R}^{ss}$) denote the subset of $\tilde{R}$ corresponding to stable (resp. semistable) GPS. Similarly we can define $\tilde{Q}$, $\tilde{Q}^s$ and $\tilde{Q}^{ss}$.

The quot scheme Q has a natural embedding in a Grassmannian. For $m \geqslant M_1(n)$, the natural map $H^0(\mathcal{O}_X^n(m)) \to H^0(\mathscr{F}_q(m))$ is surjective for all $q \in Q$. Let $W = H^0(\mathcal{O}_X(m))$, then $H^0(\mathscr{F}_q(m))$ is a quotient of $\mathbf{C}^n \otimes W$ of dimension $P(m)$ for $m \geqslant M_1$. This gives a closed embedding $Q \to \mathrm{Grass}_{P(m)}(\mathbf{C}^n \otimes W)$. A point q of $\tilde{Q}$ gives for each j, a q_j-dimensional quotient of $\mathbf{C}^n$. Hence we get an embedding $\tilde{Q} \to \mathbf{Z} = \mathrm{Grass}_{P(m)} \cdot (\mathbf{C}^n \otimes W) \times (\times_j \mathrm{Grass}_{qj}(\mathbf{C}^n))$. This embedding is equivariant under the action of PGL(n). The PGL(n) action on $\tilde{Q}$ and $\mathbf{C}^n$ is the natural one, while on W it acts trivially. On $\mathbf{Z}$ we take the polarization

$$a(n - \alpha \sum_j q_j)/km \times a\alpha \times \cdots \times a\alpha,$$

where 1 denotes $\mathcal{O}(1)$ and a is a sufficiently big integer to make all the numbers above integers, $n > \Sigma_j q_j$.

We denote a point of Z by $(P, (P_j)_j)$ where $P : \mathbf{C}^n \otimes W \to U, P_j : C^n \to U_j$ are surjective maps, $\dim U = P(m), \dim U_j = q_j$ for all j. Similarly a point of $\tilde{Q}$ is denoted by $(p, (p_j)_j)$

where $p:\mathcal{O}^n \to F, p_j: H^0(F|D_j) \to Q_j^F$ are surjections, $\dim Q_j^F = q_j \forall j$. For a subsheaf E of F, we define $Q_j^E = p_j(H^0(E|D_j))$. For a quotient $q:F \to G$, define $Q_j^G = H^0(G|D_j)/q(\mathrm{Ker}\, p_j)$. For simplicity of notation, we denote $\dim(Q_j^E)$ ($\dim(Q_j^G)$) by $q_j(E)$ (by $q_j(G)$). In particular, $q_j = q_j(F)$.

PROPOSITION 2.2

For a nontrivial proper subspace $H \subset \mathbf{C}^n$ of dimension h define σ_H by

$$\sigma_H = \left(\left(n - \alpha\sum_j q_j\right)\Big/ km\right)(hP(m) - n \dim P(H \otimes W))$$

$$+ \alpha \sum_j (q_j h - n \dim P_j(H)).$$

Then a point $(P,(P_j))$ of Z is semistable (resp. stable) for $PGL(n)-$action (with the above polarization) if and only if $\sigma_H \leqslant 0$ (respectively < 0).

Proof. See [3, Proposition 5.1.1] and [2, Proposition 4.3].

DEFINITION 2.3

Let F be a torsion-free sheaf of rank k on X. For every subsheaf E of F and $m \geqslant 0$ integer define

$$\chi_E(m) = \left(\left(n - \alpha\sum_j q_j\right)\Big/ km\right)(\chi(E)P(m) - n\chi(E(m))$$

$$+ \alpha \sum_j (q_j \chi(E) - n q_j(E)),$$

$$\sigma_E(m) = \left(\left(n - \alpha\sum_j q_j\right)\Big/ km\right)(h^0(E)P(m) - n\chi(E(m))$$

$$+ \alpha \sum_j (q_j h^0(E) - n q_j(E)).$$

Lemma 2.4. Let F be a torsion-free sheaf corresponding to a point $(p,(p_j)_j)$ of $\tilde{Q}$. Then F is semistable (respectively stable) if and only if for every subsheaf E of F we have $\chi_E = \chi_E(m) \leqslant$ (resp. < 0) for any integer m.

Proof. Let E be a subsheaf of F with F/E torsion-free. Substituting $P(m) = km + n$, $\chi(E(m)) = \chi(E) + mr(E)(r(E) = \text{rank of } E)$ in the expression for χ_E and simplifying one gets

$$\chi_E(m) = nr(E)\left[\chi(E)/r(E) - n/k + \alpha\sum_j q_j/k - \alpha\sum_j q_j(E)/r(E)\right].$$

By definition F is semistable (respectively stable) if and only if the expression in the square bracket is $\leqslant 0$ (resp. < 0).

Suppose now that F/E is not torsion-free. Then there exists $\tilde{E}, E \subset \tilde{E}$ such that $F/\tilde{E}$ is torsion-free, rank $E =$ rank $\tilde{E}$. Let $\tilde{E}/E = \tau = \tilde{\tau} + \Sigma_j \tau_j$, where $\tau_{|D_j} = \tau_{j|D_j}$. By the above argument, $\chi_{\tilde{E}}(m) \leqslant 0$. We claim that $\chi_E(m) < \chi_{\tilde{E}}(m)$. Using $\chi(\tilde{E}(m)) - \chi(E(m)) = h^0(\tau)$ for $m \geqslant 0$, $q_j(\tilde{E}) - q_j(E) \leqslant h^0(\tau_j)$ we get $\chi_E(m) - \chi_{\tilde{E}}(m) = n(\alpha \Sigma_j h^0(\tau_j) - h^0(\tau)) < 0$ since $\alpha < 1$.

Lemma 2.5. *There exists an integer $M_2(n) \geqslant M_1(n)$ such that if $(p,(p_j)) \in \tilde{Q}$ is a point satisfying the following two conditions, then the image of the point in Z is semistable (resp. stable).*

(1) *The canonical map $\mathbf{C}^n \to H^0(F)$ is an isomorphism.*

(2) *For every subsheaf E of F generated by global sections, $\sigma_E(m) \leqslant 0$ (resp. < 0) for $m \geqslant M_2(n)$.*

Proof. Let $H \subset \mathbf{C}^n$ be a subspace. Let E be the subsheaf of F generated by H and let K be the kernel of the surjection $H \otimes \mathcal{O}_X \to E$. As H varies over subspaces of $\mathbf{C}^n$ and F varies over Q, the sheaves E and hence K form a bounded family. Hence there exists $M_2(n)$ such that for $m \geqslant M_2(n), h^1(E(-D_j)(m)) = 0$, $h^1(E(m)) = 0$ and $h^1(K(m)) = 0$ for all such E and K. It follows that dim $P(H \otimes W) = \chi(E(m))$. Clearly dim $H \leqslant h^0(E)$, $\dim P_j(H) = q_j(E)$. Therefore $\sigma_H \leqslant \sigma_E(m) \leqslant 0$ (resp. < 0). Thus the image $(P,(P_j))$ of $(p,(p_j))$ is semistable (resp. stable).

Lemma 2.6. *One can find $M_3(n) \geqslant M_2(n)$ such that for $m \geqslant M_3(n)$ the following holds. If $(p, p_j)) \in \tilde{Q}$ is a point whose image in Z is semistable then* (i) *$\mathbf{C}^n \to H^0(F)$ is injective and* (ii) *for all torsion-free quotients $F \to G \to 0$, one has $\tau_G \leqslant 0$. Here τ_G is defined by*

$$\tau_G = \left(\frac{n - \alpha \Sigma_j q_j}{k}\right)\left(- kh^0(G) + nr(G) \right) + \alpha \sum_j (nq_j(G) - q_j h^0(G)).$$

Proof. Note that if H_0 is the kernel of the map $\mathbf{C}^n \to H^0(F)$, then $\sigma_{H_0} > 0$ contradicting the semistability of the image point in Z (Proposition 2.2). Hence (i) follows. For (ii), suppose that there exists a torsion-free quotient G with $\tau_G > 0$. Then $h^0(G) < n$, for $h^0(G) \geqslant n$ implies $\tau_G \leqslant 0$. Let H be the kernel of the composite $\mathbf{C}^n \to H^0(F) \to H^0(G)$. Let E denote the subsheaf of F generated by H. Clearly we have $r(E) + r(G) \leqslant k$, $h^0(G) \geqslant n - h, q_j(G) \leqslant q_j - q_j(E)$, dim $P_j(H) \leqslant q_j(E)$. Substituting these in the expression for τ_G one gets

$$\left(\left(n - \alpha \sum_j q_j\right) \Big/ k\right)(kh - nr(E)) + \alpha \sum_j (q_j h - nq_j(E)) > 0.$$

Since H and hence E runs over a bounded family we can find $M_3(n) \geqslant M_2(n)$ such that for $m \geqslant M_3(n)$, the term $kh - nr(E)$ can be replaced by $(hP(m) - n\chi(E(m))/m = (hP(m) - n \dim P(H \otimes W))/m$. Thus we get $\sigma_H > 0$ contradicting the semistability of the image point in Z.

Lemma 2.7. *There exists $n_2 \geqslant n_1$ such that for all semistable GPS F with Euler characteristic $n \geqslant n_2$ the following holds*

(1) *If* $E \subset F$ *then* $\tau_E \leqslant 0$ *where*

$$\tau_E = \left(\left(n - \alpha\sum_j q_j\right)\Big/ k\right)(kh^0(E) - nr(E)) + \alpha\sum_j (q_j h^0(E) - nq_j(E)).$$

(2) *If* $\tau_E = 0$ *for some* $E \subset F$, *then* $\chi_E = 0$.

(3) *If* $\tau_E < 0$, *then* $\sigma_E(m) < 0$ *for* $m \geqslant M_4(n)$. *If* $\tau_E = 0$, *then* $\sigma_E(m) = 0$ *for* $m \geqslant M_4(n)$.

Proof. (1) Let $0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$ be the Harder-Narasimhan filtration of E considered as a torsion-free sheaf only, ignoring the parabolic structure. Let $Q_i \doteq E_i/E_{i-1}$, $i = 1, \cdots, r, \mu_i =$ degree $Q_i/$ rank $Q_i, v = \inf \mu_i$. One has $\mu_i > \mu_{i+1} \forall i < r$ (by definition), $h^0(E) \leqslant \Sigma_j h^0(Q_i)$ (by induction). Using Corollary 2.5 [4], this implies $h^0(E) \leqslant \Sigma^i r(Q_i)$ $(\mu_i + B_1), B_i$ constant. Since Q_1 is a subsheaf of a semistable GPS F we have $\mu_i \leqslant \mu_1 \leqslant \mu(F) + w\forall_i, w = (wtF)/k$. Since $n \geqslant n_1, \Sigma_i r(Q_i) = r(E), v = \mu_{i_o}$ we get $h^0(E) \leqslant \Sigma_{i \neq i_o} r(Q_i)(\mu(F) + w + B_1) + (v + B_1) + (r(Q_{i_o}) - 1)(\mu(F) + w + B_1) \leqslant v + (r(E) - 1)\cdot$ $n/k + B_2, B_2$ constant. Hence $\tau_E \leqslant n(v + B_2 - n/k)$. Therefore if $v \leqslant n/k - B_2$ (resp. $<$) then $\tau_E \leqslant 0$ (resp. <0). We can choose n_2 large enough so that for $n \geqslant n_2$, we have $h^1(Q(m)) = 0$ for all $m \geqslant 0$ and for all stable torsion-free sheaves Q of rank $\leqslant k$ and $\mu \geqslant n/k - B_2$. Hence if $v \geqslant n/k - B_2$ for E, then $h^1(Q_i(m) = 0 \forall i$, therefore $h^1(E(m)) = 0$ and $\chi(E(m)) = h^0(E(m))$ for $m \geqslant 0$. Then $\tau_E = \chi_E$ and $\chi_E \leqslant 0$ by Lemma 2.4. Thus $\tau_E \leqslant 0$ for all $E \subset F$.

(2) If $\tau_E = 0$, then by the above argument one must have $v \geqslant n/k - B_2, \chi(E) = h^0(E)$ and $\tau_E = \chi_E$. Thus $\chi_E = \sigma_E(m) = 0$.

(3) Note that $\tau_E = \lim_{m\to\infty} \sigma_E(m)$. Hence given $\varepsilon > 0, \exists M_4(n)$ such that for $m \geqslant M_4(n)$, $\sigma_E(m) < \tau_E + \varepsilon$. If $\tau_E < 0$ then choosing ε such that $\tau_E + \varepsilon < 0$, we get $\sigma_E(m) < 0$ for $m \geqslant M_4(n)$.

Theorem 1. (I) *Let X be an integral projective curve of arithmetic genus g over C. Let $\{D_j\}_{j\in J}$ be finitely many effective Cartier divisors in X such that the support of D_j does not intersect the set of singular points of X for all j, supports of D_j are mutually disjoint and degree $D_j = d_j, j \in J$. Let S denote the set of equivalence classes of semistable GPS F of rank k degree d with parabolic structure over D_j given by $F_o^j(F) = H^0(F \otimes \mathcal{O}_{Dj}) \supset F_1^j(F) \supset 0$, co-dimension of $F_1^j(F)$ in $F_o^j(F)$ equal to q_j (fixed) for $j \in J$ and weights $(0, \alpha), 0 < \alpha < 1$. Then S has the structure of a projective variety $M(k, d)$ of dimension $k^2(g-1) + 1 + \Sigma_j q_j(kd_j - q_j)$.*

(II) *If X is nonsingular, then $M(k, d)$ is normal. If further $(k, d) = 1, q_j$ is a multiple of k and α is sufficiently near 1 then $M(k, d)$ is nonsingular and it is a fine moduli space.*

Proof. Let w_X denote the dualising sheaf of X, it is a torsion-free sheaf. Fix $n >$ max $(n_2, kh^0(w_X) + \alpha\Sigma_j q_j)$ and $m \geqslant M_4(n)$. We keep the notations of 2.1. We shall show that a geometric invariant theoretic quotient of $\tilde{R}$ modulo PGL(n) exists. Our required moduli space $M(k, d)$ will be this quotient. $\tilde{R}$ is an open subset of $\tilde{Q}, \tilde{Q}$ is embedded in Z (with m, n as above) by a PGL(n) equivariant embedding. We first claim that if $(p, p_j)) \in R^{ss}$ (resp. $\mathbf{R}^s$) then its image belongs to Z^{ss} (resp. Z^s). This follows immediately from Lemma 2.7 and Lemma 2.5. Let F correspond to a point in $\tilde{R}^{ss} - \tilde{R}^s$. Then F has a subsheaf E which is a torsion-free stable GPS with par $\mu(E) =$ par $\mu(F)$ i.e. $\chi_E = 0$. For such an $E, \sigma_{H^0(E)} = \chi_E$ (Lemma 1.10), hence the image in Z belongs to $Z^{ss} - Z^s$.

Conversely we shall now check that if a point in $\tilde{Q}$ is such that its image belongs to Z^{ss}, then the point is in $\tilde{R}^{ss}$ i.e. if F is the corresponding quotient, then F is torsion-free, the map $\mathbf{C}^n \to H^0(F)$ is an isomorphism and F is a semistable GPS. Lemma 2.6 implies that $\mathbf{C}^n \to H^0(F)$ is injective and for every rank 1 torsion-free quotient G of $F, n \leqslant kh^0(G) + \alpha \Sigma_j q_j$ (as $\tau_G \leqslant 0$). We claim that $H^1(F) = 0$. Otherwise there exists a nontrivial homomorphism $F \to w_X$. If G is the sheaf image of this morphism, $h^0(w_X) \geqslant h^0(G)$ and hence $n \leqslant kh^0(w_X) + \alpha \Sigma_j q_j$ contradicting the assumptions on n. Thus $h^0(F) = n$ and $\mathbf{C}^n \to H^0(F)$ is an isomorphism. Let τ be the torsion subsheaf of $F, \tau = \tau_o + \Sigma_j \tau_j$, support $\tau_j \subseteq \text{supp } D_j$, $(\text{supp } \tau_o) \cap (\cup \text{ supp } D_j) = \phi$. Taking $H = H^0(\tau_0), H^0(\tau_j), \sigma_H \leqslant 0$ gives $H^0(\tau_o) = 0, H^0(\tau_j) = 0$. Here $\alpha < 1$ is crucial since $\sigma_H = n(h - \alpha \dim P_j(H))$. Thus $H^0(\tau) = H^0(\tau_o) + \Sigma_j H^0(\tau_j) = 0$ i.e. $\tau = 0$.

Suppose that F is not semistable. Then there exists a subsheaf E of F such that E is a semistable GPS with par $\mu(E) >$ par $\mu(F)$ i.e. $\chi_E > 0$. By Lemma 1.10, $\sigma_{H^0(E)} = \chi_E > 0$ contradicting the semistability of the image point in Z.

It follows that the (geometric invariant theoretic) quotient $M(k,d)$ of $\tilde{R}$ mod $\mathrm{PGL}(n)$ is the same as that of $\tilde{Q}$ and it exists if and only if the quotient of image of $\tilde{Q}$ in Z exists. It is well known that the latter exists. The quotient $M(k,d)$ is a projective variety as $\tilde{Q}$ is so. It is easy to check that the points of $M(k,d)$ correspond to equivalence classes of semistable GPS(3.15, [1]; [4]).

(2) If X is nonsingular $\tilde{R}$ is known to be nonsingular and hence $M(k,d)$ is normal. If $(k,d) = 1, \alpha$ is sufficiently near to 1 and q_j is an integral multiple of k, then GPS is semistable if and only if it is stable by Lemma 3.3 (or Lemma 3.17, [1]). The nonsingularity of $\tilde{R}$ together with corollary 1.12 then imply that $M(k,d)$ is nonsingular. One can show that $M(k,d)$ is then a fine moduli space, by proving the universal bundle on R descends to a universal bundle on $M(k,d)$ after twisting by a line bundle (see [1], Proposition 3.18).

3. Application

3.1. Let Y be an integral projective curve with only singularities ordinary double points $\{y_j\} j \in J$. Let $J' \subseteq J$ be a subset. Let X be the curve (proper transform) obtained by blowing up $\{y_j\}_{j \in J'}$. Let $\pi_{XY}: X \to Y$ be the natural morphism. Let D_j denote the divisor $\pi_{XY}^{-1}(y_j), j \in J'$. All the QPS and GPS that we consider are assumed to be of the type described in 1.8. We also assume that $\dim F_1^j(F)) = r(F)$ for $j \in J'$.

DEFINITION 3.2

Let α be a real number in $[0,1]$. A QPS $(F, \mathscr{F})$ on X is α-stable (resp. α-semistable) if for any proper subsheaf K of F with torsion-free quotient, one has

$$(d(K) + \alpha \sum_{j \in J'} \dim F_1^j(K))/r(K) < (\leqslant)(d(F) + \alpha J' r(F))/r(F).$$

Remark. For $0 < \alpha < 1$, the above condition is same as that for stability (resp. semistability) of the $\mathrm{GPS}(F, \mathscr{F}, \boldsymbol{\alpha})$ with $\alpha^j = (0, \alpha), j \in J'$.

Lemma 3.3. (1) *Suppose that* $1 - 1/J'r(F)(r(F)-1) < \alpha < 1$. *Then* $(F,\mathscr{F})$ *is* α*-semistable implies that it is* 1*-semistable. If the QPS is* 1*-stable then it is also* α*-stable.*

(2) *Assume that* $(r(F), d(F)) = 1$. *Then the QPS is* 1*-stable if and only if it is* 1*-semistable. Thus under the assumptions of* (1) and (2) 1*-stability,* α*-stability and* α*-semistability are all equivalent.*

Proof. This is a straightforward generalization of Lemma 3.17 [1].

PROPOSITION 3.4.

Let Q denote the set of isomorphism classes of QPS $(F,\mathscr{F})$ *on X of given type* (3.1). *Let* $r(F) = k, d(F) = d$ *be fixed. Let S be the set of isomorphism classes of torsion-free sheaves of rank k and degree d on Y. Let* S_k *denote the subset of S corresponding to sheaves which are locally free at* y_j *for* $j \in J'$. *Then* (*a*) *there is a surjective map* $f_{XY}: Q \to S$ *such that its restriction to* $f_{XY}^{-1}(S_k)$ *is a bijection onto* S_k. (*b*) $(F,\mathscr{F})$ *is* 1*-stable* (1*-semistable*) *iff its image under* f_{XY} *is stable* (*semistable*).

Proof. Let $D_j = x_j + z_j$. Then $((\pi_{XY})_* F) \otimes k(y_j) = (k(x_j) \oplus k(z_j))^{r(F)} = H^0(F \otimes \mathcal{O}_{Dj}) \equiv F_o^j(F)$. Thus we have a surjective $\mathcal{O}_Y$-linear map $(\pi_{XY})_* F \to F_o^j(F)$. Let F' be the kernel of the composite of this map with the surjection $F_o^j(F) \to F_o^j(F)/F_1^j(F)$. Since $d(F) = \chi(F) - r(F)\chi(\mathcal{O}_X), d(F') = \chi(F') - r(F')\chi(\mathcal{O}_Y), \chi(F) = \chi((\pi_{XY})_* F), \chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) - J'$, it follows that if $(F,\mathscr{F}) \in Q$ then $F' \in S$. We define $f_{XY}(F,\mathscr{F})) = F'$. If $F' \in S_k$ then $F = \pi_{XY}^* F'$ and $F_1^j(F) = F' \otimes k(y_j) \subset F_o^j(F)$ gives the bijection. Surjectivity of f can be proved as in 4.5 [1] while the last assertion follows exactly as in 4.2 [1].

Theorem 2. (I) *Let* $M_{X,J'}$ *be the moduli space of semistable GPS on X of type described in* 3.1. *Assume that* α *satisfies the conditions of Lemma* 3.3(1). *Then there is a surjective birational morphism* $f_{XY}: M_{X,J'} \to M_{Y,\phi}$ (= *moduli space of torsion-free sheaves on Y*).

(II) *Let Z be the desingularization of Y. Then the morphism* $f_{ZY}: M_{Z,J} \to M_{Y,\phi}$ *factors as* $f_{XY} \circ f_{ZX}$. *If the conditions of Lemma 3.3 are satisfied then* $M_{Z,J}$ *is a desingularization of* $M_{Y,\phi}$ *and* $M_{X,J'}(J' \subset J)$ *are 'partial desingularizations.*

Proof. (I) This follows easily from Lemma 3.3 and Proposition 3.4 since it is easy to globalise the construction (of f_{XY}) to families of GPS. (See Theorem 2 [1] for details).

(II) Let $f_{ZX}(F,\mathscr{F}) = F'$. Notice that π_{ZX} is an isomorphism outside $J - J'$. Hence F' has a parabolic structure $\mathscr{F}'$ over $D_j, j \in J'$ viz. $F_i^j(F') \approx F_i^j(F)$ for $i = 0, 1, j \in J'$. Thus $f_{ZX}(F,\mathscr{F}) = (F',\mathscr{F}') \in M_{X,J'}$. Let $f_{XY}(F',\mathscr{F}') = F''$. Then we have the exact sequences (defining $F', \mathscr{F}''$)

$$0 \to F' \to (\pi_{ZX})_* F \to \oplus_{J-J'} F_o^j(F)/F_1^j(F) \to 0$$

$$0 \to F'' \to (\pi_{XY})_* F' \to \oplus_{J'} F_o^j(F')/F_1^j(F') \to 0.$$

Using these and $\pi_{ZY} = \pi_{XY}\pi_{ZX}$, one gets

$$0 \to F'' \to (\pi_{ZY})_* F \to \oplus_J F_o^j(F)/F_1^j(F) \to 0$$

proving $f_{ZY} = f_{XY_o} f_{ZX}$. The last assertion follows from Theorem 1 (II).

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