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Moduli for principal bundles over algebraic curves: I

A RAMANATHAN

Last Address: School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India

Abstract. We classify principal bundles on a compact Riemann surface. A moduli space for semistable principal bundles with a reductive structure group is constructed using Mumford's geometric invariant theory.

Keywords. Principal bundles; compact Riemann surface; geometric invariant theory; reductive algebraic groups.

1. Introduction

Let X be a projective nonsingular irreducible curve over $\mathbb{C}$ (or equivalently a connected compact Rieman surface) of genus $\geqslant 2$ and G a connected reductive algebraic group over $\mathbb{C}$. Our problem is to classify algebraic principal G-bundles on X. When $G = GL(r, \mathbb{C})$, i.e. for vector bundles, this has been done by Mumford, Narasimhan and Seshadri ([13], [15], [19]).

In [14] we have defined the notion of stable and semistable G-bundles on X (Definition 2.12) and have proved that a G-bundle is stable if and only if it is associated to certain representations of $\pi_1(X - x_0)$ (cf. Definition 3.14), and have constructed by local analytic methods, a moduli space, which is a normal complex space, for stable G-bundles on X ([14], Theorems 7.7 and 4.3). In this thesis we use global algebraic methods depending on Mumford's theory of stable and semistable points for actions of reductive groups on algebraic schemes to construct a moduli space, which is a normal projective algebraic variety, for semistable principal G-bundles under a suitable equivalence; see Definitions 3.1, 3.9 and Theorem 5.9 (in part II: Editor).

In § 2 we explain some notations and recall some preliminary results. In § 3 we prove a kind of Jordan–Hölder theorem for semistable G-bundles.

Professor Annamalai Ramanathan, who was a Fellow of the Academy and a co-editor of the Proceedings, passed away on 12 March 1993, at the young age of 46. For some reason his doctoral thesis (written in 1976) was never published. A manuscript had in fact been prepared for publication, but apparently he wanted to revise it, which unfortunately was not to be.

The results in the thesis have been found very useful by researchers in the area, especially more recently in view of the remarkable connection between Conformal Field Theory and moduli spaces of principal bundles on curves. It was suggested by his teachers Professors M S Narasimhan and S Ramanan that the thesis be published, in the available form, and the idea also found enthusiastic support among other mathematicians, as this would provide a much needed reference article for the material.

Professor Ramanathan, who was on the Faculty of the Tata Institute of Fundamental Research, Bombay, was an accomplished mathematician, a recipient of the Shanti Swarup Bhatnagar prize, and a fine person very helpful to students and colleagues. As a tribute to his memory we are publishing the thesis in the Proceedings, convinced that he would have appreciated it if he were to be with us.

For convenience the publication will be in two parts, the second part being scheduled for the next issue.

– Editor

Since the unipotent radical U of a parabolic subgroup $P = M \cdot U$, M a maximal reductive subgroup of P, can be shrunk to identity (Lemma 3.5.12), it follows that given a P-bundle we can make it 'jump' to the M-bundle obtained from it by the extension of structure group $P \to M$ (Propositions 3.5 (i) and 3.24 (ii)). For constructing moduli space, this then makes it necessary to identify a G-bundle E and a G-bundle E' obtained from E through a reduction of structure group to a parabolic subgroup $P = M \cdot U$ and followed by the extensions of structure group $P \to M$ and $M \subsetneq G$. If E is semistable and the reduction to P is admissible (Definition 3.3), we prove that the G-bundle E' obtained by this process is again semistable (Lemma 3.5.11) and define E and E' to be equivalent (Definition 3.6).

We then prove that given a semistable G-bundle E, there is an admissible reduction of structure group to a suitable parabolic subgroup $P = M \cdot U$ such that the M-bundle obtained by the extension of structure group $P \to M$ is a stable M-bundle (which is a Jordan–Holder series for E). Further the G-bundle obtained from this stable M-bundle by the extension of structure group $M \subsetneq G$ depends only on E and is denoted by gr$E$ (Proposition 3.12). Also E_1 is equivalent to E_2 if and only if $\mathrm{gr}E_1 \approx \mathrm{gr}E_2$ (Proposition 3.12 (iii)).

Next we prove a result (Proposition 3.24) which shows that the equivalence on semistable G-bundles introduced above is the right one in the sense that a semistable G-bundle can only tend to an equivalent bundle in the limit. Lemmas 3.21 and 3.23 provide the essential tools for proving this proposition.

The method of proof generally is to reduce the problem to a proper reductive subgroup M of G of maximal rank (a Levi component of a parabolic subgroup P) and to use induction on the semisimple rank of the structure group. Lemma 2.11 which says that given two reductions of structure group σ_1, σ_2 of a semistable G-bundle to the parabolic subgroups P_1, P_2 respectively (with P_1, P_2 in general position (cf. Remark 3.5.6)), we can get a common reduction $\sigma_1 \cap \sigma_2$ to the subgroup $P_1 \cap P_2$ which gives both σ_1 and σ_2 under the extension $P_1 \cap P_2 \subsetneq P_i$, $i = 1, 2$, is quite useful in this context.

In §4 we first show that there is a family of G-bundles $\xi \to T \times X$ with a group H acting on T and on ξ as a group of G-bundle isomorphisms compatible with its action on T with the properties that any other family of G-bundles is locally induced from ξ and for any two morphisms $t_1, t_2 : S \to T$ from any scheme S, the induced G-bundle $(t_i \times id_X)^* \xi$ on $S \times X$ are isomorphic if and only if there is a morphism $h: S \to H$ such that $t_1 = h[t_2]$ (where $h[t_2]$ is the composite $S \xrightarrow{h \times t_2} H \times T \xrightarrow{\alpha} T$, α being the action of H on T). Hence a good quotient of T modulo H (Definition 4.1) would give a moduli scheme for G-bundles. (Actually it is sufficient if such a G-bundle ξ existed locally with respect to the faithfully flat topology on T; see Definition 4.4 and Proposition 4.5.) Let us call the family of G-bundles $\xi \to T \times X$ with the above properties, a universal family of G-bundles (cf. Definition 4.6). Seshadri has constructed a universal family of semistable vector bundles (say, of rank r) $\mathscr{V} \to R \times X$, R being a subscheme of the Quot scheme so that we have a surjective homomorphism $I_n \to \mathscr{V} \to 0$, where I_n is the trivial vector bundle of rank n, with the group $GL(n, \mathbb{C})$ acting naturally on $\mathscr{V}$ and R ([19], §6; [20]; Prop. 4.11.5). To construct a universal family for G-bundles we take an embedding of G in a $GL(k, \mathbb{C})$ and look upon a G-bundle as a vector bundle (of rank k) with a G-structure, i.e. a $GL(k, \mathbb{C})$-bundle $E \to X$ with a reduction of structure group to G given by a section $X \to E/G$ of $E/G \to X$ so that the space S of sections of $\mathscr{V}_r \to r \times X = X$, as r varies through R, gives a family of

G-bundles $\xi \to S \times X$ (§4·8; Lemma 4.8.1). The group $GL(n, \mathbb{C})$ acts naturally on ξ and S and it can be shown that ξ is a universal family (Lemma 4.10). That the space S can be constructed as a scheme with suitable universal properties follows from the existence theorems on Hilbert schemes (Lemma 4.8.1). Then we have to prove the existence of a good quotient of S modulo $GL(n, \mathbb{C})$. For this it is convenient to take the adjoint representation $\mathrm{Ad}: G \to GL(\mathscr{G})$, $\mathscr{G}$ the Lie algebra of G. However, Ad is not injective in general and hence we construct first, as outlined above, a universal family $\xi' \to R' \times X$ for Ad G-bundles and then from ξ' a universal family for G-bundles. (To be more precise the construction is in three steps: from vector bundles to Aut G-bundles to Ad G-bundles to G-bundles.) To get from Ad G-bundles to G-bundles, the idea is to look upon a G-bundle E as an Ad G-bundle E' together with certain line bundles of suitable types on the associated bundle $E'/\bar{B}(= E/B)$ where B is a Borel subgroup of G and $\bar{B}$ its image in Ad G. This is analogous to the fact that a vector bundle V is determined by the projective bundle $\mathbb{P}(V)$ and the tautological line bundle on $\mathbb{P}(V)$ corresponding to V (Lemma 4.15.1). This involves the existence of the space S of line bundles on the fibres of the composite $\xi'/\bar{B} \to R' \times X \to R'$, and for this we make use of the existence theorems on Picard schemes ([TDTE, V]). The Picard functors in general are representable only after 'sheafification' with respect to the faithfully flat topology ([TDTE, V], §1; [1]) and hence a universal family of line bundles will exist only locally in the faithfully flat topology. This means that we will be able to construct universal families for G-bundles only locally in the faithfully flat topology by this method. However, this is sufficient for our purposes (cf. Definition 4.4 and Proposition 4.5).

In §5 we complete the proof of the existence of a coarse moduli scheme for semistable G-bundles by showing that a good quotient of S modulo $GL(n, \mathbb{C})$ exists. It follows from Lemma 5.1, that it is enough to show the existence of a good quotient of R' modulo $GL(n, \mathbb{C})$. For this we adopt the method of Mumford and Seshadri in the case of vector bundles where they reduce it to a problem on a product of Grassmannians as follows ([15], §5; [19], §§6, 7). The surjection $I_n \to \mathscr{V} \to 0$ makes the fibres of $\mathscr{V}$ points of the Grassmannian $G_{n,r} = Z$ of r-dimensional quotients of I_n and by evaluating at points $x_1, \ldots, x_N \in X$ we get a morphism $R \to Z^N$. Seshadri has proved that for a suitable choice of $x_1, \ldots, x_N$, $N \gg 0$, R maps into the set Z^N_{ss} of semistable points of Z^N for the natural action of $SL(n, \mathbb{C})$ on Z^N, and that the morphism $R \to Z^N_{ss}$ is a proper injection ([20], §3, Lemma 2). Since Z^N_{ss} has a good quotient modulo $GL(n, \mathbb{C})$ ([10], Theorem 1.10, p. 38; [22], Theorem 1.1(B)), it follows that R has a good quotient modulo $GL(n, \mathbb{C})$. In the case of G-bundles for any point $(r', x) \in R' \times X$ we not only have a fibre of $\mathscr{V}$ over (r, x), where $r \in R$ is the image of $r' \in R'$, which gives a point of $G_{n,r}$, but also a 'G-structure' on this quotient. Since we have taken the adjoint representation, this 'G-structure' is actually a Lie algebra structure. Let $Y = GL(\mathscr{G})/\mathrm{Ad}\, G$ and $Q \to G_{n,r}$ be the universal quotient bundle on the Grassmannian $G_{n,r}$. We then have, for any point $x \in X$, a morphism $R' \to Q(Y)$ where $Q(Y)$ is the associated bundle of Q, which is considered as a $GL(\mathscr{G})$-bundle, with fibre Y. We prove that for suitable choice of $x_1, \ldots, x_N \in X$ under the natural morphism $R' \to Q(Y)^N$, the image of R' is contained in $Q(Y)^N_{ss}$ (Lemma 5.5.3), the set of semistable points of $Q(Y)^N$ for the natural action of $SL(n, \mathbb{C})$ on $Q(Y)^N$ (and suitable polarization). This is a consequence of the fact that the tensor in $\mathscr{G}^* \otimes \mathscr{G}^* \otimes \mathscr{G} = \mathrm{Hom}(\mathscr{G} \otimes \mathscr{G}, \mathscr{G})$ corresponding to the Lie algebra structure is semistable for the action of $SL(\mathscr{G})$ (Lemma 5.5.1). We also prove $R' \to Q(Y)^N_{ss}$ is proper (Lemma 5.6). (By Lemma 5.1 it follows that R' has a good quotient modulo $GL(n, \mathbb{C})$ as required.) For this, the crucial fact needed is that if a sequence of isomorphic semisimple Lie algebra structures on a vector space V (considered as elements of $\mathrm{Hom}(V\otimes V, V)$) tends to a semisimple Lie algebra structure on V, then the limit also gives the same Lie algebra structure on V. In other words, if Y is the $GL(\mathscr{G})$ orbit of $x\in\mathrm{Hom}(V\otimes V, V)$ such that $x\colon V\otimes V\to V$ makes V into a semisimple Lie algebra and if $x_0\in\bar{Y}$, the closure of Y in $\mathrm{Hom}(V\otimes V, V)$, is such that $x_0\colon V\otimes V\to V$ also makes V into a semisimple Lie algebra, then $x_0\in Y$. (This rigidity result is a consequence of the vanishing theorem, $H^2(\mathscr{G},\mathscr{G})=0$ (cf. [16], §§3, 4).)

We make essential use (cf. Lemmas 5.5.2 and 5.5.3) of the fact that if we take a representation $G\to GL(n,\mathbb{C})$ such that center of G maps into scalars then the associated vector bundle of a semistable G-bundle is a semistable vector bundle (Proposition 3.17). We deduce this as a consequence of the result that stable G-bundles are unitary bundles. This is one of the reasons why we restrict to working over $\mathbb{C}$, and not over fields of arbitrary characteristic.

2. Notation and preliminaries

2.1. By a *scheme* we mean a separated scheme of finite type over the complex numbers $\mathbb{C}$ and by a point of a scheme we mean a closed (or $\mathbb{C}$-valued) point of the scheme. Terms such as open, closed, dense, etc. are used with reference to the Zariski topology.

By an algebraic group we mean an *affine* algebraic group.

2.2. Let A be an algebraic group and Y a scheme. A principal bundle over Y with structure group A (or an A-bundle over Y, for short) is a scheme E on which A operates (from the right) and an A-invariant morphism $\pi\colon E\to Y$ such that for any point $y\in Y$ there is a neighbourhood U and a faithfully flat morphism $f\colon U'\to U$, and an A-equivariant isomorphism $f^*(E)\xrightarrow{\approx} U'\times A$, over U', where A operates on $U'\times A$ by right translations on the second factor ([SGA, I], expose XI, Definition 4.1). Since A is affine and hence a linear group, it follows ([TDTE, I], Proposition 7.1 and the paragraph following it) that under these conditions $\pi\colon E\to Y$ is locally isotrivial i.e. we can take $f\colon U'\to U$ above to be an étale covering.

2.3. Let $\pi\colon E\to Y$ be an A-bundle. Let F be a quasi-projective scheme on which A operates from the left. Let the group A act on the product $E\times F$ by $a(e,f)=(e\cdot a, a^{-1}f)$, $a\in A$, $e\in E$, $f\in F$. From local isotriviality and the fact that any finite set of points of F is contained in an affine subset it follows by descent that there exists an unique scheme $E(F)$ and a morphism $E\times F\to E(F)$ which makes $E\times F$ an A-bundle over $E(F)$ for this action of A on $E\times F$ (cf. [17], § 3.2, Proposition 4, pp. 15–16; [SGA, I], exposé V, § 1). There is a natural morphism $E(F)\to Y$ and we call $E(F)$ the fiber bundle associated to E for the action of A on F.

If B is a subgroup of A we denote by E/B the fiber bundle associated to E for the action of A on A/B by left translations.

Let $\rho\colon A\to A'$ be a homomorphism of algebraic groups and let A act on A' by $a\cdot a'=\rho(a)a'$, $a\in A$, $a'\in A'$. The group A' acts on $E\times A'$ by right translation on the second factor and this action goes down to $E(A')$. This makes $E(A')$ into an A'-bundle. We sometimes denote $E(A')$ by $\rho_*(E)$.

2.4. Suppose A operates on the quasi projective schemes F_1 and F_2 and $j\colon F_1\to F_2$ is an A-equivariant morphism. Then $id_E\times j\colon E\times F_1\to E\times F_2$ induces a morphism

$E(j): E(F_1) \to E(F_2)$. Using local isotriviality it follows easily that if j is an open (respectively closed, locally closed) immersion then so is $E(j)$ ([EGA, IV], Proposition 2.7.1 (x), (xii)).

2.5. Let B be an algebraic subgroup of A. A pair (E', φ) where E' is a B-bundle and φ is an isomorphism $i_* E' \to E$ of A-bundles is said to give a reduction of structure group of E to B. Reductions of structure group of E to B and the sections of the fiber bundle $E/B \to Y$ are in natural one-to-one correspondence ([17], Proposition 9; cf. also §4.8 and Lemma 4.8.1 below). $E \to E/B$ is a B-bundle over E/B, and to a section $\sigma: Y \to E/B$ we associate the B-bundle $\sigma^*(E)$ in this correspondence. We sometimes denote the B-bundle $\sigma^*(E)$ by $E[B]$. Moreover, if $\rho: B \to B'$ is a homomorphism we denote $\rho_* \sigma^*(E)$ by $E[B, B']$ also.

2.6. If $E \to Y$ is a $GL(n, \mathbb{C})$-bundle we often denote by the same letter E the associated vector bundle $E(\mathbb{C}^n)$ and if $V \to Y$ is a vector bundle again we denote by the same letter V the corresponding $GL(n, \mathbb{C})$-bundle (which can be constructed back from V) (cf. [TDTE, I], p. 28).

For a vector bundle $V \to Y$ we denote by $\mathbb{P}(V)$ the associated projective bundle of 1-dimensional sub-spaces of V.

If X is a projective non-singular irreducible algebraic curve and $V \to X$ is a vector bundle of rank n, we mean by the degree of V the degree of the line bundle $\overset{n}{\wedge} V$. We denote by $\mu(V)$ the number $\deg V/\mathrm{rk}\, V$.

2.7. We let X always stand for a projective non-singular irreducible curve of genus $g \geq 2$ (over $\mathbb{C}$) and G a connected reductive algebraic group (over $\mathbb{C}$). We denote by $\mathscr{G}$ the Lie algebra of G and by $\mathscr{G}'$ the commutator subalgebra $[\mathscr{G}, \mathscr{G}]$ of $\mathscr{G}$. The center of $\mathscr{G}$ is denoted by $\mathscr{z}$ so that $\mathscr{G} = \mathscr{z} \oplus \mathscr{G}'$.

For any group M, we let $Z[M]$ stand for its center and $Z_0[M]$ the connected component of identity of $Z[M]$. Let $Z = Z[G]$ and $Z_0 = Z_0[G]$.

2.8. A subgroup P of G is a parabolic subgroup of G if G/P is complete. It is convenient for us to consider G also as a parabolic subgroup. However, by a maximal parabolic subgroup of G, we mean a parabolic subgroup of G which is maximal among proper parabolic subgroups. (For comparison note that in [14] we had reserved the term parabolic to proper parabolic subgroups.)

2.9. For a parabolic subgroup P we generally use the notation $M \cdot U$ for a Levi decomposition of P with U the unipotent radical of P and M a maximal reductive subgroup of P. We call a maximal reductive subgroup of P a Levi component of P.

If P_1 is a parabolic subgroup of M, then $P_1 \cdot U$ is a parabolic subgroup of G. For any parabolic subgroup P' of G contained in P, $P' \cap M$ is a parabolic subgroup of M. This gives a bijective correspondence between the set of parabolic subgroups of M and those of G contained in P ([3], Proposition 4.6, p. 86).

2.10. Since G is a linear algebraic group ([17], §6.3) any analytic G-bundle on the compact Riemann surface X has a unique algebraic G-bundle structure and any analytic morphism between G-bundles is an algebraic morphism. The equivalence of the categories of algebraic and analytic G-bundles on X justifies our use of results from [14] though we work in the algebraic category now.

2.11. Let $k(X)$ be the function field of X. Then it follows from a result of Springer that any connected linear algebraic group defined over $k(X)$ has a Borel subgroup defined over $k(X)$ (cf. [23], Theorem 1.9, p. 57 and the remarks following it.) From this it can be deduced that any G-bundle over the curve X is locally trivial in the Zariski topology.

2.12. We denote by (Sch) the category of algebraic schemes (over $\mathbb{C}$). We use the faithfully flat topology on (Sch) and for any functor $\mathbf{F}$: (Sch)$\to$(Sets) we mean by the sheaf $\tilde{\mathbf{F}}$ associated to $\mathbf{F}$ the 'sheafification' of the presheaf $\mathbf{F}$ with respect to the faithfully flat topology ([1], Theorem 1.1, Chapter II, p. 24; [TDTE, V], p. 3).

We now recall a few things from [14].

2.13. DEFINITION

A G-bundle $E\to X$ is called *stable* (resp. *semistable*) if for any reduction of structure group $\sigma: X\to E/P$ to any maximal parabolic subgroup P of G we have $\deg\sigma^*(T_{E/P})>0$ (resp. $\geqslant 0$), where $T_{E/P}$ is the tangent bundle along fibers of $E/P\to X$ (cf. [14], Definition 1.1).

2.14. DEFINITION

Let P be a proper parabolic subgroup of G. A character $\chi: P\to\mathbb{C}^*$ is called *dominant* if it is given by a positive linear combination of fundamental weights for some choice of a Cartan subalgebra and positive system of roots (cf. [14], p. 131).

A dominant character is trivial on Z_0.

2.15. *Lemma. The G-bundle $E\to X$ is stable* (*resp. semistable*) *if and only if for any reduction $\sigma: X\to E/P$ to any proper parabolic subgroup, not necessarily maximal, we have* $\deg(\chi_*\sigma^*E)<0$ (*resp.* $\leqslant 0$) *for any nontrivial dominant character χ of P.*

For proof see ([14], Lemma 2.1, pp. 131–132).

3. Equivalence on semistable bundles

We will be interested in studying the following functor.

3.1. DEFINITION

Let $\mathbf{F}_{ss}$:(Sch)$\to$(Sets) be the functor which associates to a scheme S the set of isomorphism classes of G-bundles $\xi\to S\times X$ such that for every point $s\in S$ the restriction $\xi_s\to s\times X=X$ of ξ to $s\times X$ is a semistable G-bundle. For a morphism $f: S'\to S$, $\mathbf{F}_{ss}(f)(\xi)$ is defined to be the pull back $(f\times id_X)^*(\xi)$.

Given a topological G-bundle τ on X we denote by $\mathbf{F}^{\tau}_{ss}$ the sub-functor of $\mathbf{F}_{ss}$ defined by

$$\mathbf{F}^{\tau}_{ss}(S)=\{\xi\in\mathbf{F}_{ss}(S)\,|\,\xi_s\to X \text{ is topologically isomorphic to } \tau\,\forall\, s\in S\}.$$

We refer to a $\xi\in F_{ss}(S)$ as a family of semistable G-bundles parameterized by S.

3.1.1. *Remark.* We will prove later that for an arbitrary G-bundle $\xi\to S\times X$ the set $\{s\in S\,|\,\xi_s \text{ is semistable}\}$ is an open subset of S (Proposition 5.8, cf. also [14]; Proposition 4.1, p. 138).

We recall the definition of a coarse moduli scheme (cf. [10], Definition 5.6, p. 99).

3.2. DEFINITION

Let $\mathbf{F}$:(Sch)$\rightarrow$(Sets) be a functor. A scheme M and a morphism of functors φ from F to h_M, the functor represented by M (i.e. $h_M(S) = \mathrm{Hom}(S, M)$), is called a *coarse moduli scheme* for $\mathbf{F}$ if (i) the map $\varphi_{\mathbb{C}}:\mathbf{F}(\mathrm{Spec}\,\mathbb{C}) \rightarrow h_M(\mathrm{Spec}\,\mathbb{C})$ is a bijection; (ii) given any scheme N and any morphism $\psi:\mathbf{F} \rightarrow h_N$ there is an unique morphism $\chi:h_M \rightarrow h_N$ such that $\psi = \chi o \varphi$.

Proposition 3.5 below (see also Proposition 3.24) shows that $\mathbf{F}_{ss}$ cannot have a separated coarse moduli scheme and suggests an equivalence relation between semistable G-bundles to obtain a coarse moduli scheme.

3.3. DEFINITION

Let $\xi \rightarrow S \times X$ be a G-bundle. A reduction σ of structure group of ξ to a parabolic subgroup P is called *admissible* if for any character χ on P which is trivial on Z_0 and any point $s \in S$ the line bundle $\chi_* \sigma_s^*(\xi_s)$, given by the reduction σ_s of structure group of ξ_s induced by σ and the character χ, has degree zero.

3.4. *Remark.* For a $GL(n, \mathbb{C})$-bundle $E \rightarrow X$ a reduction σ of structure group to the parabolic subgroup P defined by a flag $0 = V_0 \subset \ldots \subset V_r = \mathbb{C}^n$ is equivalent to giving the sub-bundles $(\sigma^* E)(V_i)$, $i = 0, \ldots, r$. The reduction σ is admissible if and only if $\mu(\sigma^* E(V_i/V_{i-1})) = \mu(E(\mathbb{C}^n))$, $i = 1, \ldots, r$.

3.5. PROPOSITION

Let $\xi \rightarrow S \times X = Y$ be a G-bundle. Let σ be a reduction of structure group to a parabolic subgroup $P = M \cdot U$. Then there is a G-bundle $\xi' \rightarrow \mathbb{C} \times Y$ such that

i) $\xi'|_{\mathbb{C}^* \times Y} \approx (\pi_Y^* \xi)|_{\mathbb{C}^* \times Y}$, *where $\pi_Y:\mathbb{C} \times Y \rightarrow Y$ is the projection and $\xi'_0 \rightarrow Y$, the restriction of ξ' to $0 \times Y = Y$ is isomorphic to $j_* p_* \sigma^*(\xi)$ where $p:P \rightarrow M$ is the projection and $j:M \subsetneq G$ is the inclusion* ($\mathbb{C}^*$ *denotes* $\mathbb{C} - (0)$).

ii) *if $\xi \rightarrow S \times X$ is a family of semistable G-bundles and σ is an admissible reduction then $\xi'_0 \rightarrow S \times X$ is also a family of semistable G-bundles.*

Before giving a proof of this proposition we note down several remarks and lemmas.

3.5.1. DEFINITION

Let $\rho: G \rightarrow GL(V)$ be a representation. Let W be a subspace of V such that the stabilizer $\{g \in G \mid \rho(g)\, W = W\}$ of W in G is a parabolic subgroup. We call any subspace of V of the form $\rho(g)\, W$, for some $g \in G$, a *subspace of type W* (cf. [14], Definition 3.1, p. 135).

Note that the subspaces of type W form the orbit $\mathrm{Gr}(W)$ of W under the natural action of G on the Grassmannian Gr of all subspaces of V of rank = rank W. Since the stabilizer of W is a parabolic subgroup, $\mathrm{Gr}(W)$ is closed in Gr. Suppose $E \rightarrow X$ is a G-bundle. Taking $\mathrm{Gr}(W)$ with the canonical reduced subscheme structure we have a closed immersion $E(\mathrm{Gr}(W)) \subsetneq E(\mathrm{Gr})$ (§ 2.4). A section σ of $E(\mathrm{Gr}) \rightarrow X$ gives a sub-bundle of $E(V)$ in a natural way.

3.5.2. DEFINITION

If $\sigma: X \rightarrow E(\mathrm{Gr})$ factors through $E(\mathrm{Gr}(W)) \subsetneq E(\mathrm{Gr})$ we call the sub-bundle given by σ a *sub-bundle of type W.*

3.5.3. *Remark.* If P is the stabilizer of W in G then $\mathrm{Gr}(W) = G/P$. Therefore, such a section σ of $E(\mathrm{Gr}(W)) = E/P$ gives a reduction of structure group to P. The sub-bundle corresponding to σ is then $(\sigma E)(W)$. If P is a proper parabolic subgroup and E is stable (resp. semistable) then it follows, as in the proof of Lemma 3.3 of [14], that $\mu((\sigma^* E)(W)) <$ (resp. $\leqslant$) $\mu(E(V))$.

3.5.4. *Remark.* If a sub-bundle of $E(V)$ is generically of type W (i.e. for a nonempty open subset U, $\sigma(U) \subset E(\mathrm{Gr}(W))$) it is actually of type W everywhere (i.e. $\sigma(X) \subset E(\mathrm{Gr}(W))$) since $E(\mathrm{Gr}(W))$ is a closed subvariety of $E(\mathrm{Gr})$.

3.5.5. *Remark.* Let $V \to X$ be a vector bundle. Let W, W' be sub-bundles of V. We often identify a vector bundle with the sheaf of its sections, which is a locally free sheaf. That W is a sub-bundle of V is equivalent to saying that the sheaf W is a subsheaf of V such that the quotient sheaf V/W is torsion free (or equivalently locally free, since X is a curve). We denote by $W \cap W'$ the subsheaf of V which is the kernel of the natural homomorphism $W \to V/W'$ (or $W' \to V/W$). We denote by $\overline{W \cap W'}$ the inverse image of the torsion subsheaf of $V/W \cap W'$ under the projection $V \to V/W \cap W'$. Then $\overline{W \cap W'}$ is a sub-bundle of V. We call it the sub-bundle generated by $W \cap W'$. For $x \in X$, let V_x denote the fiber of V at x. Then $W_x \cap W'_x \subset (\overline{W \cap W'})_x$. There is a non-empty open subset $U \subset X$ such that for $x \in U$, $W_x \cap W'_x = (\overline{W \cap W'})_x$. Moreover $W \cap W' = \overline{W \cap W'}$ if and only if $\dim(W_x \cap W'_x)$ is constant as x varies over X.

3.5.6. *Remark.* Let σ_1, σ_2 be two reductions of structure group of $E \to X$ to the parabolic subgroups P_1, P_2 respectively. Let U be an open subset of X on which E is trivial (§ 1.11). Identifying $p^{-1}(U)$ with $U \times G$ the reductions σ_i give rise to morphisms $\sigma_i : U \to G/P_i$. Let $\varphi : U \to G/P_1 \times G/P_2$ be defined by $\varphi(x) = (\sigma_1(x), \sigma_2(x))$, $x \in U$. Since P_i is its own normalizer G/P_i can be identified with the set of conjugates of P_i in G by associating to the coset gP_i the conjugate $gP_i g^{-1}$. The group G acts diagonally on $G/P_1 \times G/P_2$ and an orbit of G on $G/P_1 \times G/P_2$ can be thought of as giving a relative position of conjugates of P_1 and P_2. Therefore, as can be seen by using Bruhat's lemma and the configuration of standard parabolic subgroups ([3], § 4) the number of orbits for this action is finite. Let $O_1, \ldots, O_r$ be the orbits of G in $G/P_1 \times G/P_2$. Then O_i are locally closed ([2], p. 98) and hence $\varphi^{-1}(O_i)$ are locally closed in U_i. Since U is 1-dimensional, the locally closed subsets are either open sets or finite set of points. Therefore, there is a unique orbit O_i such that $\varphi^{-1}(O_i)$ is a nonempty open subset of U. If $(P_1, P_2) \in O_i$, then P_1 and P_2 are in the relative position corresponding to the generic relative position determined by σ_1 and σ_2 and we then say that (P_1, P_2) is *compatible* with (σ_1, σ_2). It is easy to see that this notion does not depend on U and the trivialization of E over U.

3.5.7. *Remark.* For $g \in G$ let $R_g : E \to E$ be the action of g on $E : R_g(e) = e \cdot g$ for $e \in E$. Then R_g induces a morphism $E/P \to E/g^{-1}Pg$. Hence any reduction σ of structure group to the subgroup P gives rise, by composition with $E/P \to E/g^{-1}Pg$, to a reduction σ_g of structure group to $g^{-1}Pg$. The $g^{-1}Pg$-bundle $\sigma_g^* E$ is obtained from the P-bundle $\sigma^* E$ by the extension of structure group $P \to g^{-1}Pg$, $p \mapsto g^{-1}pg$, $p \in P$. If $\chi^g : g^{-1}Pg \to \mathbb{C}^*$ is the character defined by $\chi^g(g^{-1}pg) = \chi(p)$ then $\chi_* \sigma^* E \approx \chi^g_* \sigma_g^* E$. Therefore the stability (resp. semistability) condition for the reduction σ is satisfied if and only if it is satisfied for σ_g.

3.5.8. *Lemma. Let $E \to X$ be a G-bundle and σ an admissible reduction of structure group to the proper parabolic subgroup P. Let $\rho: G \to GL(V)$ be a representation such that Z_0 acts by scalars. Let $W \subset V$ be a nonzero subspace left invariant by P. Then $\mu(\sigma^* E(W)) = \mu(E(V))$.*

Proof. Let $\det_V: GL(V) \to \mathbb{C}^*$ and $\det_W: GL(W) \to \mathbb{C}^*$ be the determinant characters. The representation $\rho: G \to GL(V)$ induces $\rho: P \to GL(W)$. Define $\chi_1: G \to \mathbb{C}^*$ to be $\det_V \circ \rho$ and $\chi_2: P \to \mathbb{C}^*$ to be $\det_W \circ \rho$. We then have $\deg(E(V)) = \deg(\chi_{1*}E)$ and $\deg(\sigma^* E(W)) = \deg(\chi_{2*}\sigma^* E)$. Also $\chi_{1*}E \approx \chi_{2*}\sigma^* E$. Let $r_1 = \text{rank } V$ and $r_2 = \text{rank } W$. Since Z_0 acts by scalars the character $\chi = \chi_1^{r_2} \cdot \chi_2^{-r_1}$ of P is trivial on Z_0. Therefore, σ being admissible, $\deg(\chi_* \sigma^* E) = 0$, i.e. $r_2 \deg(\chi_{1*}\sigma^* E) - r_1 \deg(\chi_{2*}\sigma^* E) = 0$, i.e. $\mu(\sigma^* E(W)) = \mu(E(V))$.

3.5.9. *Lemma. Let $P = M \cdot U$ be a parabolic subgroup of G. Let P_1 be a proper parabolic subgroup of M.*

i) *Let χ_1 be a dominant character of P_1. Extend the character χ_1 to a character χ_1' on $P_1 \cdot U$ by defining it to be trivial on U. Then there exists an integer $n > 0$ such that on $P_1 \cdot U$ we have $\chi_1'^n = \chi' \cdot \chi^{-1}$ where χ' is a dominant character of $P_1 \cdot U$ and χ is a dominant character of P.*
ii) *Let μ_1 be a character on $P_1 \cdot U$, trivial on Z_0. Then there exists an integer $n > 0$ such that $\mu_1^n = \mu' \cdot \mu$ where μ' is a character on $P_1 \cdot U$ which is trivial on $Z_0[M]$ and μ is a character on P which is trivial on Z_0.*

Proof. Let $\mathscr{h}$ be a Cartan subalgebra of $\mathscr{G}'$ and $\mathscr{G} = \mathscr{z} \oplus \mathscr{h} \oplus \Sigma_{\alpha \in \Delta} \mathscr{G}^\alpha$ be a root space decomposition. Let $\alpha_1, \ldots, \alpha_l$ be a system of simple roots and $\lambda_1, \ldots, \lambda_l$ the corresponding fundamental weights. We can assume, by conjugating if necessary that the Lie algebras of P, $P_1 \cdot U$ and M are $\mathscr{z} \oplus \mathscr{h} \oplus \Sigma_{\alpha \in D} \mathscr{G}^\alpha$, $\mathscr{z} \oplus \mathscr{h} \oplus \Sigma_{\alpha \in D_1} \mathscr{G}^\alpha$ and $\mathscr{z} \oplus \mathscr{h} \oplus \Sigma_{\alpha \in D^s} \mathscr{G}^\alpha$ respectively where

$$D = \{\alpha \in \Delta \mid \alpha = \Sigma m_i \alpha_i \quad \text{with } m_i \geqslant 0 \quad \text{for } i = 1, \ldots, r\}$$

$$D_1 = \{\alpha \in \Delta \mid \alpha = \Sigma m_i \alpha_i \quad \text{with } m_i \geqslant 0 \quad \text{for } i = 1, \ldots, s\}$$

and

$$D^s = \{\alpha \in \Delta \mid \text{both } \alpha \text{ and } -\alpha \text{ are in } D\}.$$

Since $P_1 U \subset P$ we have $D_1 \subset D$. The roots $\alpha_{r+1}, \ldots, \alpha_l$ constitute a simple system of roots for $\mathscr{M}$, the Lie algebra of M. Since the vector space spanned by $\lambda_1, \ldots, \lambda_r$ is the orthogonal complement of the space spanned by $\alpha_{r+1}, \ldots, \alpha_l$, we can determine constants a_{ij} such that $\lambda_i' = \lambda_i + \Sigma_{j=1}^r a_{ij} \lambda_j$, $i = r+1, \ldots, l$, belong to the space spanned by $\alpha_{r+1}, \ldots, \alpha_l$. Clearly then λ_i', $i = r+1, \ldots, l$ are the fundamental weights of $\mathscr{M}$ corresponding to the simple system of roots $\alpha_{r+1}, \ldots, \alpha_l$. Therefore, $\tilde{\chi}_1$, being a dominant form of P_1 in M, is of the form $\tilde{\chi}_1 = \Sigma_{k=r+1}^l b_k \lambda_k'$, $b_k \geqslant 0$. Clearly the form $\tilde{\chi}_1'$ corresponding to the character χ_1' on $P_1 \cdot U$ is also $\Sigma_{k=r+1}^l b_k \lambda_k' = \tilde{\chi}_1$. Substituting for λ_k' in terms of λ_i we have

$$\tilde{\chi}_1' = \sum_{k=r+1}^{s} b_k \lambda_k + \sum_{j=1}^{r} a_j \lambda_j.$$

Let

$$\tilde{\chi}' = \sum_{k=r+1}^{s} b_k \lambda_k + \sum_{a_j \geqslant 0} a_j \lambda_j$$

and

$$-\tilde{\chi} = \sum_{a_j<0} a_j\lambda_j$$

so that

$$\tilde{\chi}_1' = \tilde{\chi}' - \tilde{\chi}.$$

Note that since $\tilde{\chi}_1'$ corresponds to a character on $P_1 \cdot U, b_k, a_j \in \mathbb{Z}$. The linear forms $\tilde{\chi}'$ and $\tilde{\chi}$ may not go down to give characters on the corresponding groups since G may not be simply connected. However, we can find an integer $n > 0$ such that $n\tilde{\chi}'$ and $n\tilde{\chi}$ give rise to characters χ' and χ on $P_1 U$ and P respectively. Then $\chi_1'^n = \chi' \cdot \chi^{-1}$ gives the required decomposition.

To prove (ii) note that if the finite group $Z_0[M] \cap [M, M]$ is of order, say n then the character μ_1^n restricted to $Z_0[M]$ is trivial on $Z_0[M] \cap [M, M]$ and hence can be extended to a character μ of P by defining it to be trivial on $[M, M] \cdot U$. Then the character $\mu' = \mu_1^n \cdot \mu^{-1}$ of $P_1 \cdot U$ is trivial on $Z_0[M]$ and gives the required decomposition $\mu_1^n = \mu' \cdot \mu$.

3.5.10. *Lemma. Let $P = M \cdot U$ be a parabolic subgroup of G and F a P-bundle on X. Let $F' = p_*(F)$ where $p: P \to M$ is the projection. Let P_1 be a parabolic subgroup of M. Then*

i) *if σ_1 is a reduction of structure group of F' to the subgroup P_1 of M then there exists a reduction σ of structure group of F to $P_1 \cdot U$ such that $p_{1*}\sigma^* E \simeq \sigma_1^* F'$ where $p_1: P_1 \cdot U \to P_1$ is the projection.*
ii) *if τ is a reduction of structure group of F to $P_1 \cdot U$ then there exists a reduction of structure group τ_1 of F' to P_1 such that $\tau_1^* F' \approx p_{1*}\tau^* F$.*

Suppose E is a G-bundle and $F = \sigma'^* E$ for an admissible reduction σ' of structure group of E to P. Let $q: (\sigma'^* E)/P_1 \cdot U = \sigma'^*(E/P_1 \cdot U) \to E/P_1 \cdot U$ be the projection. Then

i)′ *in the notation of* (i) *if σ_1 is admissible then $q \circ \sigma$ is an admissible reduction of structure group of E to $P_1 \cdot U$.*
ii)′ *in the notation of* (ii) *if $q \circ \tau$ is admissible then τ_1 is also admissible.*

Proof. The natural morphism $\psi: P/P_1 U \to M/P_1$ has a section $\varphi: M/P_1 \to P/P_1 \cdot U$ induced by the inclusion $M \subsetneq P$. The morphism ψ induces $\tilde{\psi}: F/P_1 \cdot U \to F'/P_1$ and φ induces a section $\tilde{\varphi}: F'/P_1 \to F/P_1 \cdot U$ for $\tilde{\psi}$ since $F/P_1 \cdot U$ and F'/P_1 are the associated fiber bundles of F with fibers $P/P_1 \cdot U$ and M/P_1 respectively.

For (i) take $\sigma = \tilde{\varphi} \circ \sigma_1$ and for (ii) take $\tau_1 = \tilde{\psi} \circ \sigma$.

(i)′ Let χ be a character of $P_1 \cdot U$. By 3.5.9(ii) for some $n > 0$ we can write $\chi^n = \chi' \cdot \chi$ with χ' a character of P_1, trivial on $Z_0[M]$ and χ a character of P, trivial on Z_0. We have therefore,

$$\chi_*^n (q \circ \sigma)^* E \approx (\chi' \cdot \chi)_* (q \circ \sigma)^* E \approx (\chi'_* \sigma^* F) \otimes (\chi_* F)$$
$$\approx (\chi'_* \sigma_1^* F') \otimes (\chi_* F).$$

Since σ_1 is admissible, $\deg(\chi'_* \sigma_1^* F') = 0$. Since σ' is admissible, $\deg(\chi_* F) = 0$. Therefore $\deg(\chi_* (q \circ \sigma)^* E) = 0$ which proves that $q \circ \sigma$ is admissible.
(ii)′ Let χ_1 be a character of P_1, trivial on $Z_0[M]$. Let χ_1' be the character on $P_1 \cdot U$ got by extending χ_1 to $P_1 \cdot U$ by setting it to be trivial on U. We then have

$\chi_{1*}\tau_1^* F' \approx \chi_{1*} p_1^* \tau^* F$ by (ii). Also $\chi_{1*} p_1^* \tau^* F \approx \chi'_{1*} \tau^* F \approx \chi'_1 (q \circ \tau)^* E$. Since $q \circ \tau$ is admissible $\deg(\chi'_{1*}(q \circ \tau)^* E) = 0$, which proves τ_1 is admissible.

3.5.11. *Lemma. Let $E \to X$ be a G-bundle and σ an admissible reduction of structure group of E to a proper parabolic subgroup $P = M \cdot U$. Let $p: P \to M$ be the projection, and $j: M \subsetneq G$ the inclusion. Then*

i) *E is a semistable G-bundle if and only if $p_* \sigma^* E$ is a semistable M-bundle.*
ii) *if E is semistable then $j_* p_* \sigma^* E$ is semistable.*

Proof. Assume $p_* \sigma^* E$ is a semistable M-bundle. Let σ' be a reduction of structure group of E to a proper parabolic subgroup P' of G. We have to show that σ' satisfies the condition for semistability. We can assume by conjugating that (P, P') is compatible with (σ, σ') (see 3.5.6 and 3.5.7).

Let $\mathscr{P}, \mathscr{P}'$ be the subalgebras of $\mathscr{G}$ corresponding to the subgroups P, P' respectively. By ([14], Remark 2.2, p. 132) semistability condition is equivalent to $\deg((\sigma'^* E)(\mathscr{P}')) \leqslant 0$.

Let $0 = V_0 \subset V_1 \subset \cdots \subset V_r = \mathscr{G}$ be a flag in $\mathscr{G}$ such that V_j is invariant under P and U acts trivially on V_j / V_{j-1} for $j = 1, \ldots, r$. The stabilizer of $\operatorname{Im}(\mathscr{P}' \cap V_j \to V_j / V_{j-1})$ in M is a parabolic subgroup of M since $P \cap P'$ leaves $\mathscr{P}' \cap V_j$ invariant and $(P \cap P') \cap M$ is a parabolic subgroup of M (§ 2.9). Since (P, P') is compatible with (σ, σ') the sub-bundle W_j generated by $\operatorname{Im}((\sigma'^* E(\mathscr{P}')) \cap (\sigma^* E(V_j)) \to \sigma^* E(V_j / V_{j-1}))$ is a sub-bundle (generically and hence everywhere cf. 3.5.4) of type $\operatorname{Im}(\mathscr{P}' \cap V_j \to V_j / V_{j-1})$. Note that since U acts trivially on V_j / V_{j-1}, $\sigma^* E(V_j / V_{j-1}) \approx p_* \sigma^* E(V_j / V_{j-1})$ where $p: P \to M$ is the projection. Since $p_* \sigma^* E$ is a semistable M-bundle

$$\mu(W_j) \leqslant \mu(p_* \sigma^* E(V_j / V_{j-1})). \tag{1}$$

(cf. Remark 3.5.3; if the stabilizer of $\operatorname{Im}(\mathscr{P}' \cap V_j \to V_j / V_{j-1})$ is M itself, use the fact that σ is admissible). Since σ is admissible, by Lemma 3.5.8

$$\mu(p_* \sigma^* E(V_j / V_{j-1})) = \mu(E(\mathscr{G})) = 0.$$

Therefore,

$$\mu(W_j) \leqslant 0 \quad \text{for all } j. \tag{2}$$

Denote by $\underline{V}_j$ the sub-bundle $\sigma^* E(V_j)$ of $E(\mathscr{G})$ and by $\underline{\mathscr{P}'}$ the sub-bundle $\sigma'^* E(\mathscr{P})$ of $E(\mathscr{G})$.

We shall show that for $1 \leqslant j \leqslant r - 1$

$$\deg(\overline{\underline{\mathscr{P}'} \cap \underline{V}_j}) \leqslant 0 \Rightarrow \deg(\overline{\underline{\mathscr{P}'} \cap \underline{V}_{j+1}}) \leqslant 0 \tag{*}$$

(see Remark 3.5.5 for notation). Since $\overline{\underline{\mathscr{P}'} \cap \underline{V}_1} = W_1$ and $\deg W_1 \leqslant 0$ by (2) and $\underline{\mathscr{P}'} \cap \underline{V}_r = \underline{\mathscr{P}'}$ it will then follow by induction that $\deg(\underline{\mathscr{P}'}) \leqslant 0$ as required to be shown.

To prove (*) factorize the natural homomorphism $\overline{\underline{\mathscr{P}'} \cap \underline{V}_{j+1}} \to \underline{V}_{j+1} / \underline{V}_j$ (cf. [13], § 4, p. 547)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{\underline{\mathscr{P}'} \cap \underline{V}_j} & \longrightarrow & \overline{\underline{\mathscr{P}'} \cap \underline{V}_{j+1}} & \longrightarrow & Q & \to & 0 \\
 & & & & & & \downarrow & & \\
0 & \longleftarrow & Q' & \longleftarrow & \underline{V}_{j+1} / \underline{V}_j & \longleftarrow & W_{j+1} & \longleftarrow & 0.
\end{array}$$

Since $Q \to W_{j+1}$ is a generic isomorphism $\deg Q \leqslant \deg W_{j+1}$. Using (2) $\deg Q \leqslant 0$. By the hypothesis of (*) $\deg(\overline{\underline{\mathscr{P}'} \cap \underline{V}_j}) \leqslant 0$. Therefore $\deg(\overline{\underline{\mathscr{P}'} \cap \underline{V}_{j+1}}) \leqslant 0$, as was to be shown.

Now suppose, conversely, E is a semistable G-bundle. Let σ_1 be a reduction of structure group of $p_*\sigma^*E = E'$ to a parabolic subgroup P_1 of M. Let χ_1 be a dominant character on P_1. By Lemma 3.5.10 there is a reduction of structure group σ' of σ^*E to the parabolic subgroup $P_1 \cdot U$ such that $\sigma_1^*E' \approx p_{1_*}\sigma'^*\sigma^*E$ where $p_1: P_1 \cdot U \to P_1$ is the projection. By Lemma 3.5.9

(i) for the character χ_1 extended to $P_1 \cdot U$ we have $\chi_1^n = \chi' \cdot \chi^{-1}$ for some $n > 0$ with χ' a dominant character on $P_1 \cdot U$ and χ a dominant character on P. We have $\chi_*^n \sigma_1^* E' \approx (\chi' \cdot \chi^{-1})_*(q \circ \sigma')^*E$ (where $q: \sigma^*(E/P_1 \cdot U) \to E/P_1 \cdot U$ is the projection). Also $\chi_*(q \circ \sigma')^*E \approx \chi_*\sigma^*E$. Since σ is admissible $\deg(\chi_*\sigma^*E) = 0$ and since E is semistable $\deg(\chi'_*(q \circ \sigma')^*E) \leqslant 0$. Therefore $\deg(\chi_*\sigma_1^*E') = (1/n)\deg(\chi_*^n\sigma_1^*E') = (1/n)\ \{\deg\ (\chi'_*(q \circ \sigma')^*E) - \deg(\chi_*\sigma^*E)\} \leqslant 0$. This shows $p_*\sigma^*E$ is a semistable M-bundle.
(ii) By (i) the M-bundle $p_*\sigma^*E$ is semistable. But $p_*\sigma^*E$ gives a reduction of structure group of $j_*p_*\sigma^*E$ to the subgroup M. Again by (i) it follows $j_*p_*\sigma^*E$ is a semistable G-bundle.

3.5.12. *Lemma. Let $P = M \cdot U$ be a proper parabolic subgroup of G. Then there is a 1-parameter subgroup $\lambda: \mathbb{C}^* \to Z_0[M]$ such that the morphism $\mathbb{C}^* \times P \to P$ defined by $(z, p) \mapsto \lambda(z)p\lambda(z)^{-1}$, $z \in \mathbb{C}^*$, $p \in P$, extends to a morphism $\varphi: \mathbb{C} \times P \to P$ such that $\varphi(0, p) = m$ where $p = m \cdot u$, $m \in M$, $u \in U$.*

Proof. We use the notation of [3], § 4. Let $P = P_\theta$ where θ is a subset of a system of simple roots of G with respect to a maximal torus T. Then $M = Z_\theta$, $U = V_\theta$ and $p: \Pi_{b \in \alpha_\theta} U_b \to V_\theta$ given by group multiplication is an isomorphism of algebraic varieties ([2], p. 327, § 14.4) where U_b is the radical group corresponding to the root b, i.e. there is an isomorphism of algebraic groups $\theta_b: \mathbb{C} \to U_b$ such that

$$t\theta_b(x)t^{-1} = \theta_b(t^b x)(*)$$

([3], § 2.3, p. 64).

It follows from ([3], Proposition 3.6, p. 75) that we can find a 1-parameter subgroup $\lambda: \mathbb{C}^* \to Z_0(M)$ such that $\langle \lambda, b \rangle > 0$ for every $b \in \alpha_\theta$, where $\langle \lambda, b \rangle$ is the integer such that the composite $\mathbb{C}^* \to T \xrightarrow{b} \mathbb{C}$ is given by $z \mapsto z^{\langle \lambda, b \rangle}$.

Let $\varphi_b: \mathbb{C}^* \times U_b \to U_b$ be given by $\varphi_b(z, u) = \lambda(z)u\lambda(z)^{-1}$, $z \in \mathbb{C}^*$, $u \in U_b$. Define $\varphi'_b: \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$ such that the diagram

$$\begin{array}{ccc} \mathbb{C}^* \times U_b & \xrightarrow{\varphi_b} & U_b \\ {\scriptstyle id \times \theta_b}\uparrow & & \uparrow{\scriptstyle \theta_b} \\ \mathbb{C}^* \times \mathbb{C} & \xrightarrow[\varphi'_b]{} & \mathbb{C} \end{array}$$

commutes.

Using $(*)$ we get $\varphi'_b(z, \zeta) = z^{\langle \lambda, b \rangle} \cdot \zeta$, $z \in \mathbb{C}^*$, $\zeta \in \mathbb{C}$. Since $\langle \lambda, b \rangle > 0$, $\mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$ can be extended as a morphism $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that $(0, \zeta) \mapsto 0$. Therefore, φ_b extends to a morphism $\mathbb{C} \times U_b \to U_b$, again denoted by φ_b, such that $\varphi_b(0, u) = 1$, $\forall u \in U_b$. Using the isomorphism $p: \Pi_{b \in \alpha_\theta} U_b \to V_\theta$ we can write $P = M \cdot \Pi_{b \in \alpha_\theta} U_b$. Then the morphism

$\varphi:\mathbb{C}\times(M\cdot\Pi_{b\in\alpha_0}U_b)\to M\cdot\Pi_{b\in\alpha_0}U_b$ defined by $\varphi(z,m\cdot\Pi_{b\in\alpha_0}u_b)=m\cdot\Pi_{b\in\alpha_0}\varphi_b(z,u_b)$ $z\in\mathbb{C}$, $m\in M$, $u_b\in U_b$ satisfies the requirements of the lemma.

We can now give the proof of Proposition 3.5.

Proof of Proposition 3.5. Consider $\mathbb{C}\times Y\times P$ as the trivial group scheme over $\mathbb{C}\times Y$ determined by P. Then $\mathbb{C}\times(\sigma^*E)$ is a principal homogeneous space over $\mathbb{C}\times Y$ under the group scheme $\mathbb{C}\times Y\times P$, in the obvious way (in the sense of [SGA, I], expose XI, Definition 4.1). Let $\tilde{\varphi}:\mathbb{C}\times Y\times P\to\mathbb{C}\times Y\times P$ be the homomorphism of group schemes defined by $\varphi(z,y,p)=(z,y,\varphi(z,p))$ where $\varphi:\mathbb{C}\times P\to P$ is the morphism given by Lemma 3.5.12. Then taking the associated principal homogeneous space of $\mathbb{C}\times\sigma^*E\to\mathbb{C}\times Y$ under the extension $\tilde{\varphi}$ (which exists, since $\mathbb{C}\times Y\times P$ is an affine algebraic group scheme over $\mathbb{C}\times Y$, see [SGA I], expose XI, § 4, p. 11) we get a principal homogeneous space ξ'' over $\mathbb{C}\times Y$ under the trivial group scheme $\mathbb{C}\times Y\times P$. We can consider ξ'' as a P-bundle over $\mathbb{C}\times Y$ in the obvious way. Let $i:P\subsetneq G$ be the inclusion. Then $\xi'=i_*\xi''$ gives a family of G-bundles for which the assertion (i) of Proposition 3.5 is easily seen to hold. Then (ii) follows from 3.5.11 (ii).

3.6. DEFINITION

Two families of semistable G-bundles $\xi\to S\times X$ and $\xi'\to S\times X$ parametrized by the scheme S are said to be *related* if there is an admissible reduction of structure group σ (resp. σ') of ξ (resp. ξ') to a parabolic subgroup $P=M\cdot U$ (resp. $P'=M'\cdot U'$) such that the G-bundles $j_*p_*\sigma^*\xi$ and $j'_*p'_*\sigma'^*\xi'$ are isomorphic, where $p:P\to M$, $p':P'\to M'$ are the projections and $j:M\subsetneq G, j':M'\subsetneq G$ are the inclusions. We say ξ is *equivalent* to ξ' if there exist families of semistable G-bundles $\xi_i\to S\times X$, $i=1,\ldots,r$ such that ξ is related to ξ_1, ξ_r is related to ξ' and ξ_i is related to ξ_{i+1} for $i=1,\ldots,r-1$.

3.7. *Lemma. If the semistable G-bundles $E\to X$ and $E'\to X$ are equivalent then they are topologically isomorphic.*

Proof. It is enough to prove that if σ is a (admissible) reduction of structure group of E to the parabolic subgroup $P=M\cdot U$ then E and $j_*p_*\sigma^*E$, where $p:P\to M$ is the projection and $j:M\subsetneq G$ is the inclusion, are topologically isomorphic.

By Proposition 3.5 (i) there is a family of G-bundles $\xi\to\mathbb{C}\times X$ such that $\xi_z\approx E$ for $z\neq 0$ and $\xi_0\approx j_*p_*\sigma^*E$. Since $\mathbb{C}$ is connected, E and $j_*p_*\sigma^*E$ are topologically isomorphic.

3.8. *Remark.* We can also prove the lemma directly without using Proposition 3.5 by using the topological classification of bundles on X ([14], § 5, pp. 142–143).

3.9. DEFINITION

Let $\tilde{\mathbf{F}}_{ss}$:(Sch)$\to$(Sets) be the functor which associates to $S\in$(Sch) the set of equivalence classes of families of semistable G-bundles parametrized by S. On morphisms $\tilde{\mathbf{F}}_{ss}$ is defined in the obvious way. Given a topological G-bundle τ on X let $\tilde{\mathbf{F}}^{\tau}_{ss}$ be the sub-functor of $\tilde{\mathbf{F}}_{ss}$ which associates to S the set of equivalence classes of families of semistable G-bundles of the topological type τ parametrized by S.

We prove in this thesis that the functors $\tilde{\mathbf{F}}^{\tau}_{ss}$ have coarse moduli schemes which are projective (see Theorem 5.9).

We shall now study the equivalence of semistable G-bundles on X and pick out representatives for the equivalence classes. We shall need some more lemmas.

3.10. *Lemma. Let $P_i = M_i \cdot U_i$ be proper parabolic subgroups of G, $i=1,2$. Given a character χ on the parabolic subgroup $P_1 \cap P_2 \cdot U_1$* (§1.9; [3], Proposition 4.4, p. 86) *which is trivial on Z_0, there is an integer $n>0$ such that $\chi^n = \chi_1 \cdot \chi_2$ with χ_i a character on P_i, trivial on Z_0, $i=1,2$.*

Proof. By Bruhat's lemma there is a maximal torus $T \subset P_1 \cap P_2$. Let $\mathcal{T}$ be the Lie algebra of T and $\mathcal{G} = z \oplus h \oplus \Sigma_{\alpha\in\Delta} \mathcal{G}^\alpha$ be a root space decomposition with $z + h = \mathcal{T}$.

Let L be the lattice $\ker(\exp: h \to T)$ and L^* the lattice of linear forms on h which take integral values on L. Then the characters of a parabolic subgroup P containing T correspond to the linear forms in L^* which are orthogonal (with respect to a Weyl group invariant form which can be taken to be Q-valued on L^*) to all the roots α such that both $\mathcal{G}^\alpha$ and $\mathcal{G}^{-\alpha}$ are contained in the Lie algebra of P. Let V_i (resp. V) be the Q-vector space spanned by the elements of L^* which are orthogonal to those roots α for which both $\mathcal{G}^\alpha$ and $\mathcal{G}^{-\alpha}$ are contained in the Lie algebra of P_i (resp. $P_1 \cap P_2 \cdot U_1$), $i=1,2$. Then clearly $V = V_1 + V_2$. Let $\tilde{\chi}$ be the linear form corresponding to the character χ. Then $\tilde{\chi} \in V$. Write $\tilde{\chi} = \tilde{\chi}_1 + \tilde{\chi}_2$ with $\tilde{\chi}_i \in V_i$. We can find an integer $n>0$ such that $n\tilde{\chi}_i \in L^*$, $i=1,2$. Then $n\tilde{\chi}_i$ give characters χ_i on P_i and we have $\chi^n = \chi_1 \cdot \chi_2$.

3.11. *Lemma. Let E be a semistable G-bundle and σ_i an admissible reduction of structure group of E to the parabolic subgroup $P_i = M_i \cdot U_i$, $i=1,2$. Let $p_i: P_i \to M_i$ be the projections. Suppose (P_1, P_2) is compatible with (σ_1, σ_2). Then*

i) *there is a reduction of structure group $\sigma = \sigma_1 \cap \sigma_2$ of E to the subgroup $P_1 \cap P_2$ such that $\sigma_i = \pi_i \circ \sigma$* where *$\pi_i: E/P_1 \cap P_2 \to E/P_i$, $i=1,2$ is the natural morphism.*
ii) *letting $q_i: E/P_1 \cap P_2 \to E/P_1 \cap P_2 \cdot U_i$ be the natural morphism, the reductions $q_i \circ \sigma$* ($i=1,2$) *are admissible.*
iii) *if $p_{1*}\sigma_1^* E$ is a stable M_1-bundle then $P_1 \cap P_2$ contains a Levi component of P_1.*
iv) *let $P_1 \cap P_2 = M_3 \cdot U_3$ be a Levi decomposition of $P_1 \cap P_2$ such that $M_3 \subset M_1$. Note that M_3 is a Levi component for both $P_1 \cap P_2 \cdot U_1$ and $P_2 \cap M_1$. By* 3.5.10 (ii) *and* (ii)′ *the admissible reduction of structure group $q_1 \circ \sigma$ of E induces an admissible reduction of structure group σ' of the M_1-bundle $p_{1*}\sigma^* E = F'$ to the parabolic sub-group $P_2 \cap M_1$. We then have $p_{3*}\sigma^* E \approx p'_{3*}\sigma'^* F'$ where $p_3: P_1 \cap P_2 \to M_3$ and $p'_3: P_2 \cap M_1 \to M_3$ are the projections.*

Proof. Let $\mathcal{P}_i$ be the subalgebra of $\mathcal{G}$ corresponding to the sub-group P_i, $i=1,2$. Let $0 = V_0 \subset V_1 \subset \cdots \subset V_p = \mathcal{P}_1 \ldots V_r = \mathcal{G}$ be a flag in $\mathcal{G}$ such that each V_j is invariant under P_1 and U_1 acts trivially on V_j/V_{j-1}, $j=1,\ldots,r$. Let $0 = W_0 \subset W_1 \subset \cdots \subset W_q = \mathcal{P}_2 \subset \cdots \subset W_s = \mathcal{G}$ be a flag with the same properties with respect to P_2. We denote by $\underline{V}_j$ (resp. $\underline{W}_j$) the sub-bundle $\sigma_1^* E(V_j)$ (resp. $\sigma_2^* E(W_j)$) of $E(\mathcal{G})$. We shall show that $\underline{\mathcal{P}_1 \cap \mathcal{P}_2}$ is a sub-bundle of $E(\mathcal{G})$, i.e. $\underline{\mathcal{P}_1 \cap \mathcal{P}_2} = \underline{\mathcal{P}}_1 \cap \underline{\mathcal{P}}_2$ (cf. Remark 3.5.5). We note that since σ_1 and σ_2 are admissible by Lemma 3.5.8. $\mu(\underline{V}_j/\underline{V}_{j-1}) = \mu(E(\mathcal{G}))$ and $\mu(\underline{W}_j/\underline{W}_{j-1}) = \mu(E(\mathcal{G}))$. Moreover $\mu(E(\mathcal{G})) = 0$ (cf. [14], Remark 2.2).

We prove that each $\underline{V}_i \cap \underline{W}_j$ is a sub-bundle by induction. Assume that $\underline{V}_i \cap \underline{W}_j$ is a sub-bundle with $\deg(\underline{V}_i \cap \underline{W}_j) = 0$ for (i,j) such that either $i \leqslant m-1$ and j arbitrary or $i=m$ and $j \geqslant n+1$. Under this assumption we shall prove that $\underline{V}_m \cap \underline{W}_n$ is a sub-bundle of degree zero. It will then follow by induction that $\underline{V}_i \cap \underline{W}_j$ is a sub-bundle of degree zero for all i,j.

If $\underline{V}_m \cap \underline{W}_{n+1} = 0$ there is nothing to prove since $\underline{V}_m \cap \underline{W}_n = 0$ in that case. Suppose $\underline{V}_m \cap \underline{W}_{n+1} \neq 0$. Factorize the natural homomorphism $\underline{V}_m \cap \underline{W}_{n+1} \to \underline{W}_{n+1}/\underline{W}_n$.

$$\begin{array}{c} 0 \to A_1 \to \underline{V}_m \cap \underline{W}_{n+1} \to A_2 \to 0 \\ \downarrow \\ 0 \leftarrow A_4 \leftarrow \underline{W}_{n+1}/\underline{W}_n \leftarrow A_3 \leftarrow 0. \end{array}$$

Since $\underline{W}_{n+1}/\underline{W}_n = p_{2*}\sigma_2^* E(W_{n+1}/W_n)$ and $p_{2*}\sigma_2^* E$ is a semistable M_2-bundle (Lemma 3.5.11(i)) and A_3 is a sub-bundle of type $\mathrm{Im}(V_m \cap W_{n+1} \to W_{n+1}/W_n)$ which has as stabilizer a parabolic subgroup in M_2, we have $\deg A_3 \leqslant 0$ (cf. proof of Lemma 3.5.11). Therefore $\deg A_2 \leqslant 0$ and hence

$$\deg A_1 \geqslant 0, \tag{1}$$

noting that $\deg(\underline{V}_m \cap \underline{W}_{n+1}) = 0$ by the induction hypothesis. Now consider the sub-bundle A_1 of $E(\mathscr{G})$. If $A_1 = 0$ it is easy to see that $V_m \cap W_n = A_1 = 0$ and we are through. If $A_1 \neq 0$ we can find a $t \leqslant m$ such that $A_1 \subset \underline{V}_t$ and $A_1 \not\subset \underline{V}_{t-1}$. Then factorize the non-zero homomorphism $A_1 \to \underline{V}_t/\underline{V}_{t-1}$

$$\begin{array}{c} 0 \to B_1 \to A_1 \to B_2 \to 0 \\ \downarrow \\ 0 \leftarrow B_4 \leftarrow \underline{V}_t/\underline{V}_{t-1} \leftarrow B_3 \leftarrow 0. \end{array}$$

Since by the induction hypothesis $\underline{V}_{t-1} \cap \underline{W}_n$ is a sub-bundle, we see that $B_2 \approx B_3$ and $B_1 = \underline{V}_{t-1} \cap \underline{W}_n$. Since σ_1 is admissible as before we have $\deg B_3 \leqslant 0$. Therefore $\deg B_2 \leqslant 0$. Also $\deg B_1 = 0$. Therefore

$$\deg A_1 \leqslant 0. \tag{2}$$

By (1) and (2) $\deg A_1 = 0$. This implies $\deg A_2 = \deg A_3 = 0$. Then $A_2 \to A_3$ being a generic isomorphism of vector bundles of the same degree becomes an isomorphism. Therefore $A_1 = \underline{V}_m \cap \underline{W}_n$ which shows that $\underline{V}_m \cap \underline{W}_n$ is a sub-bundle.

In particular we have proved that $\underline{\mathscr{P}_1 \cap \mathscr{P}_2}$ is a sub-bundle of $E(\mathscr{G})$.

(i) Since (P_1, P_2) is compatible with (σ_1, σ_2) there is a nonempty open subset U of X over which E is trivial such that, choosing a trivialization, the morphism $\sigma_1 \times \sigma_2 : U \to G/P_1 \times G/P_2$ given by the sections σ_1 and σ_2 has its image in the G orbit O of $(P_1, P_2) \in G/P_1 \times G/P_2$ (Remark 3.5.6). Since the stabilizer in G of $(P_1, P_2) \in G/P_1 \times G/P_2$ is $P_1 \cap P_2$ we have that O is naturally isomorphic to $G/P_1 \cap P_2$. It then follows easily that there is a section σ of $E/P_1 \cap P_2 \to X$ over U such that $\sigma_i | U = \pi_i \circ \sigma$, $i = 1,2$. The complement $X - U$ of U in X is a set of a finite number of points. Let $x \in X - U$. Let U' be a neighbourhood of x in X, over which E is trivial, and choose a trivialization. Then σ_1, σ_2 and σ give morphisms $\sigma'_1 \times \sigma'_2 : U' \to G/P_1 \times G/P_2$ and $\sigma' : U \cap U' \to G/P_1 \cap P_2$. By our choice of U, $(\sigma'_1 \times \sigma'_2)(U \cap U') \subset O$. Therefore $(\sigma'_1 \times \sigma'_2)(x)$ is in the closure $\bar{O}$ of O in $G/P_1 \times G/P_2$. Suppose $(\sigma'_1 \times \sigma'_2)(x) \notin O$. Since $\bar{O} - O$ is a union of orbits of dimension strictly less than that of O ([2], §(1.8), p. 98), the stabilizer of $(\sigma'_1 \times \sigma'_2)(x)$ in G will have dimension strictly greater than $P_1 \cap P_2$. However, letting $\underline{\mathscr{P}_{i_x}}$ be the fiber over x of the vector bundle $\mathscr{P}_i$ it is easy to see that the Lie algebra of this stabilizer corresponds to $\underline{\mathscr{P}_{1_x} \cap \mathscr{P}_{2_x}}$, which has the same dimension as $P_1 \cap P_2$ since $\underline{\mathscr{P}_1 \cap \mathscr{P}_2}$ is a sub-bundle (Remark 3.5.6). This contradiction proves that $(\sigma'_1 \times \sigma'_2)(x) \in \bar{O}$. This implies that we can extend the section σ over U to x and hence, by

the same argument at other points of $X - U$, to the whole of X. Then $\sigma_i = \pi_i \circ \sigma$ on the whole of X since both the sides agree on the dense open subset U.
(ii) Let χ be a character on $P_1 \cap P_2 \cdot U_1$ which is trivial on Z_0. By Lemma 3.10 for some integer $n > 0$, we can write $\chi^n = \chi_1 \cdot \chi_2$ where χ_i is a character on P_i trivial on Z_0. Clearly $\chi^n_*(q_1 \circ \sigma)^* E \approx (\chi_{1*}\sigma_1^* E) \otimes (\chi_{2*}\sigma_2^* E)$. Since both σ_1 and σ_2 are admissible $\deg(\chi_{i*}\sigma_i^* E) = 0$, $i = 1, 2$. Therefore $\deg(\chi_*(q_1 \circ \sigma)^* E) = 0$.
(iii) The reduction σ gives rise to a reduction of structure group of the P_1-bundle $\sigma_1^* E$ to the subgroup $P_1 \cap P_2$ and hence to $P_1 \cap P_2 \cdot U_1$. By Lemma 3.5.10 (ii) and (ii)′ we have a reduction of structure group of $p_{1*}\sigma_1^* E$ to $(P_1 \cap P_2 \cdot U_1) \cap M_1$ which is admissible. If $p_{1*}\sigma_1^* E$ is a stable M_1-bundle then we should have $(P_1 \cap P_2 \cdot U_1) \cap M_1 = M_1$. This implies that $P_1 \cap P_2$ contains a Levi component of P_1 ([3], Proposition 4.4, p. 86).
(iv) Let $F = \sigma_1^* E$. By Lemma 3.5.10 (ii) $\sigma'^* F' \approx p_*(q_1 \circ \sigma)^* F$ where $p:(P_1 \cap P_2) \cdot U_1 = (P_2 \cap M_1) \cdot U_1 \to P_2 \cap M_1$ is the projection. Therefore applying p'_{3*} to both sides $p'_{3*}\sigma'^* F' \approx p'_{3*} p_*(q_1 \circ \sigma)^* F$. But the latter is isomorphic to $p_{3*}\sigma^* E$.

3.12. PROPOSITION

Let E be a semistable G-bundle on X. Then there exists a semistable G-bundle grade E, denoted by gr E, *uniquely determined up to isomorphism by the condition that there exists an admissible reduction of structure group σ of E to a parabolic subgroup $P = M \cdot U$ such that $p_* \sigma^* E$ (where $p: P \to M$ is the projection) is a stable M-bundle and* $\mathrm{gr}\, E \approx j_* p_* \sigma^* E$ *(where j: $M \subsetneq G$ is the inclusion). We then have* i) *E and* gr E *are equivalent,* ii) $\mathrm{gr}(\mathrm{gr}\, E) \approx \mathrm{gr}\, E$ *and* iii) *two semistable G-bundles E_1 and E_2 are equivalent if and only if* $\mathrm{gr}\, E_1 \approx \mathrm{gr}\, E_2$.

3.12.1. COROLLARY

The set of isomorphism classes of semistable G-bundles E such that $E \approx$ gr E *forms a set of representatives for the equivalence classes of semistable G-bundles.*

Proof. The corollary follows immediately from the proposition.

We shall first prove that given E there exists a semistable G-bundle gr E satisfying the above condition. If E is stable we have only to take $\mathrm{gr}\, E = E$. If E is not stable then there exists an admissible reduction of structure group of E to a proper parabolic subgroup $P = M \cdot U$. By Lemma 3.5.11(i) $E' = E[P, M]$ (cf. §2.5 for notation) is a semistable M-bundle. To prove the existence of gr E we now use induction on the semi-simple rank (i.e. the rank of the commutator subgroup) of the structure group. If the semisimple rank is zero then the group is a torus group and any bundle with a torus group as structure group is stable and hence we can start the induction. Since the semisimple rank of M is strictly less than that of G by the induction hypothesis there exists an admissible reduction of structure group of E' to a parabolic subgroup $P_1 = M_1 \cdot U_1$ of M such that $E'[P_1, M_1]$ is a stable M_1-bundle. Then by Lemma 3.5.10 (ii) and (ii)′ there is an admissible reduction of structure group of E to the parabolic subgroup $P_1 \cdot U$ of G such that $E[P_1 U, P_1] \approx E'[P_1]$. Therefore $E[P_1 U, P_1](M_1) \approx E'[P_1, M_1]$. Note that M_1 is a Levi component of $P_1 \cdot U$ also and $E[P_1 U, P_1](M_1) \approx E[P_1 U, M_1]$. Therefore we can set $\mathrm{gr}\, E = E[P_1 \cdot U, M_1](G)$.

We now proceed to show the uniqueness of a bundle satisfying the condition of the proposition.

First note that for any reduction of structure group of E to $P = M \cdot U$ the isomorphism class of $E[P, M](P)$ and hence of $E[P, M](G)$, does not depend on the choice of the Levi component of P since any two Levi components are conjugate by an element of U ([3], § 0.8, p. 59).

If E is stable, uniqueness is obvious. On the other hand suppose σ_1 and σ_2 are two admissible reductions of structure group of E to the proper parabolic subgroups $P_1 = M_1 \cdot U_1$ and $P_2 = M_2 \cdot U_2$ respectively such that $E[P_i, M_i]$ is a stable M_i-bundle, $i = 1, 2$. Then we have to show that the G-bundles $E_i = E[P_i, M_i](G)$ are isomorphic.

We can assume, by conjugating if necessary, that (P_1, P_2) is compatible with (σ_1, σ_2) (cf. Remark 3.5.7). Then by Lemma 3.11, it follows that there is a reduction of structure group σ of E to the subgroup $P_1 \cap P_2$ such that $\sigma_i = p_i \circ \sigma$ where $p_i: E/P_1 \cap P_2 \to E/P_i$, $i = 1, 2$. By Lemma 3.11 (iii) $P_1 \cap P_2$ contains a Levi component of both P_1 and P_2. Let $P_1 \cap P_2 = M \cdot U$ be a Levi decomposition for $P_1 \cap P_2$. Then M is a Levi component of both P_1 and P_2. Also $U \subset U_i$, $i = 1, 2$. Therefore both E_1 and E_2 are isomorphic to $E[P_1 \cap P_2, M](G)$ proving the uniqueness.

The fact that if E_1 and E_2 are equivalent then $\mathrm{gr}\, E_1 \approx \mathrm{gr}\, E_2$ follows from Lemma 3.13 (i) below. The other assertions of the proposition are clear.

3.13. *Lemma. Let σ (resp. σ') be an admissible reduction of structure group of the semistable G-bundle E (resp. E') to the parabolic subgroup $P = M \cdot U$. Then we have the following.*

i) $\mathrm{gr}\, E \approx (\mathrm{gr}(E[P, M]))(G)$
ii) $\mathrm{gr}\, E \approx \mathrm{gr}(E[P, M](G))$
iii) *If* $\mathrm{gr}(E[P, M]) \approx \mathrm{gr}(E'[P, M])$ *then* $\mathrm{gr}\, E \approx \mathrm{gr}\, E'$.
iv) *If the M-bundles $E[P, M]$ and $E'[P, M]$ are equivalent then the G-bundles E and E' are equivalent.*

Proof. Let σ_1 be an admissible reduction of structure group of E to a parabolic subgroup $P_1 = M_1 \cdot U_1$ such that the M_1-bundle $E[P_1, M_1]$ is stable. Then $\mathrm{gr}\, E \approx E[P_1, M_1](G)$. We can assume that (P_1, P) is compatible with (σ_1, σ). Then by Lemma 3.5.10 (i) there is a reduction of structure group s of E to $P_1 \cap P_2$ such that $\sigma_1 = p_1 \circ s$ and $\sigma = p \circ s$ where $p_1: E/P_1 \cap P \to E/P_1$ and $p: E/P_1 \cap P \to E/P$ are the natural morphisms. Also by Lemma 3.5.10(iii) $P_1 \cap P$ contains a Levi component of P_1. Therefore, conjugating if necessary, we can assume $M_1 \subset M$. We have a Levi decomposition $P_1 \cap P = M_1 \cdot U'$ for $P_1 \cap P$ where U' is the unipotent radical of $P_1 \cap P$. Also $U' \subset U$ and $(P_1 \cap P) \cdot U = M_1 \cdot U$ is a Levi decomposition for $(P_1 \cap P) \cdot U$. Therefore,

$$\mathrm{gr}(E[P_1, M_1]) \approx E[P_1 \cap P, M_1] \tag{1}$$

and M_1 is a Levi component of $P_1 \cap M$. It follows from Lemma 3.5.10 (iv) that

$$F'[P_1 \cap M, M_1] \approx E[P_1 \cap P, M_1]. \tag{2}$$

Using (1), $F'[P_1 \cap M, M_1]$ is a stable M_1-bundle. Therefore

$$\mathrm{gr}\, F' \approx F'[P_1 \cap M, M_1](M) \approx E[P_1 \cap P, M_1](M). \tag{3}$$

From (1) and (3) $(\mathrm{gr}\, F')(G) \approx \mathrm{gr}\, E$ which proves (i). The reduction of structure group of F' to $P_1 \cap M$ gives in the obvious way a reduction of structure group of $F'(G)$ to $P_1 \cap M$ such that

$$F'[P_1 \cap M] \approx (F'(G))[P_1 \cap M]. \tag{4}$$

Since M_1 is a Levi component of $(P_1 \cap M) \cdot U$, considering the reduction of structure group of $F'(G)$ to $P_1 \cap M$ as a reduction to the subgroup $(P_1 \cap M)\cdot U$ we have from (4),

$$(F'(G))[P_1 \cap M)\cdot U, M_1] \approx F'[P_1 \cap M, M_1].$$

Since $F'[P_1 \cap M, M_1]$ is a stable M_1-bundle, $\mathrm{gr}(F'(G)) \approx F'[P_1 \cap M, M_1](G) \approx (\mathrm{gr}\, F')(G)$ and the last bundle is gr E by (i). This proves (ii), and (iii) follows immediately from (i). Proposition 3.12 (iii) and part (iii) above imply (iv).

We will now relate the algebraic notion of equivalence classes of semistable bundles to the transcendental notion of unitary bundles. We recall the definition of unitary bundles on the compact Riemann surface X (cf. [14], § 1, § 6). Let $x_0 \in X$ be a fixed point. Let $\Gamma = \pi_1(X - x_0)$ be the fundamental group of $X - x_0$ (with respect to some base point, the base point does not count since we will be concerned only with the isomorphism classes in the constructions below). Let $\gamma \in \Gamma$ be the element corresponding to the 'loop around x_0' ([14], p. 144). Let $\rho:\Gamma \to G$ be a homomorphism such that $\rho(\gamma) = C \in Z_0$. Let $\tilde{Z}_0$ be the Lie algebra of Z_0 and $\tilde{C} \in \tilde{Z}_0$ such that $\exp(\tilde{C}) = C$. Let E_ρ be the G-bundle on $X - x_0$ associated to the universal covering $\widetilde{X - x_0} \to X - x_0$, which is a Γ-bundle, by the homomorphism ρ. Let D be a neighbourhood of x_0 in X isomorphic to the unit disc. Let H be the upper half plane. Then $H \to D - x_0$, $z \mapsto \exp 2\pi i z$, is the universal covering and $\psi: H \to Z_0 \subset G$ defined by $\psi(z) = \exp(-z\tilde{C})$ gives a section of E_ρ over $D - x_0$. We define $E(\rho, \tilde{C})$ to be the G-bundle obtained by patching up the trivial G-bundle on D with E_ρ on $X - x_0$ with the help of the trivialization of E_ρ on $D - x_0$ given by ψ (cf. [14], § 6 for more details).

3.14. DEFINITION

We call a G-bundle a *unitary G-bundle* if it is isomorphic to a G-bundle $E(\rho, \tilde{C})$ constructed as above for some $\rho:\Gamma \to G$ such that $\rho(\Gamma) \subset K$, a maximal compact subgroup of G.

3.15. PROPOSITION

A semistable G-bundle E is unitary if and only if $E \approx \mathrm{gr}\, E$.

Proof. Let $E = E(\rho, \tilde{C})$ be an unitary bundle. Suppose ρ is irreducible, i.e. the only elements of $\mathscr{G}$ fixed by ad $\rho(h)$ for every $h \in \Gamma$ are those of the center of $\mathscr{G}$ ([14, § 1). Then by ([14], Theorem 7.1) E is stable. Therefore in this case $E \approx \mathrm{gr}\, E$. Suppose ρ is not irreducible. Then by ([14], Proposition 2.1) $\rho(\Gamma)$ leaves a proper parabolic subalgebra $\mathscr{P}$ of $\mathscr{G}$ invariant and hence $\rho(\Gamma) \subset P$, the subgroup corresponding to $\mathscr{P}$. Since ρ is unitary $\rho(\Gamma) \subset K \cap P \subset M$ where M is a Levi component of P. This implies that $E(\rho, \tilde{C})$ has a reduction of structure group to M. This reduction considered as a reduction to P is admissible. For let $\chi: P \to \mathbb{C}^*$ be a character on P which is trivial on Z_0. Then by ([14], Remark 6.1, p. 145), $\chi_* E(\rho, \tilde{C}) \approx E(\chi \circ \rho, \tilde{\chi}(\tilde{C}))$ where $\tilde{\chi}$ is the morphism induced by χ on the universal coverings. Since $\tilde{C} \in \tilde{Z}_0$ and χ is trivial on Z_0, $\tilde{\chi}(\tilde{C}) = 0$. Therefore $\deg(\chi_*(E(\rho, \tilde{C})) = 0$. We have for this reduction $E[P, M](G) \approx E$. If $E[P, M]$ is a stable M-bundle we are through. If not we use induction on the semisimple rank of the structure group to conclude that $\mathrm{gr}\, E(\rho, \tilde{C}) \approx E(\rho, \tilde{C})$.

Conversely suppose $E \approx \mathrm{gr}\, E$. If E is stable by ([14], Theorem 7.1) E is an unitary G-bundle. If E is not stable, let σ be an admissible reduction of structure group of E to the proper parabolic subgroup $P = M \cdot U$, such that $E' = E[P, M]$ is a stable M-bundle.

Then $\operatorname{gr} E \cong E'(G)$ (Proposition 3.12). By ([14], Theorem 7.1) $E' \cong E'(\rho', \tilde{C}')$ for some $\rho': \Gamma \to K' \subset M$, where K' is a maximal compact subgroup of M, $\rho(\gamma) = C' \in Z_0[M]$ and $\tilde{C}' \in \tilde{Z}_0[M]$ such that $\exp \tilde{C}' = C'$. We then make the following

Claim. $\tilde{C}' \in \tilde{Z}_0 \subset \tilde{Z}_0[M]$.

Since the reduction σ is admissible $\deg(\chi_* E[P, M]) = 0$ for any character χ on P which is trivial on Z_0. But $\chi_*(E[P, M]) \approx E'(\chi \circ \rho', \tilde{\chi}(\tilde{C}'))$ by ([14], Remark 6.1). Therefore $\tilde{\chi}(\tilde{C}') = 0$. Since any character of $Z_0[M]$ which is trivial on the finite group $Z_0[M] \cap [M, M]$ extends to a character of M and hence of P we see that the group of characters of P (resp. the group of characters of P which are trivial on Z_0) is a subgroup of finite index in the group of characters of $Z_0[M]$ (resp. in the group of characters of $Z_0[M]$ which are trivial on Z_0). This shows that if $\tilde{C}' \notin \tilde{Z}_0$ there will be a character χ on P, trivial on Z_0, such that $\tilde{\chi}(\tilde{C}') \neq 0$. This contradiction proves the claim.

Let $\rho = i \cdot \rho'$ where $i: M \subsetneq G$ is the inclusion. Then $E \approx \operatorname{gr} E \approx E'(G) \approx E(\rho, \tilde{C})$ where $\tilde{C} = \tilde{\imath}(\tilde{C}')$.

3.15.1. COROLLARY

For any semistable G-bundle E the G-bundle gr E is unitary. Associating E to gr *E gives a bijection between the set of equivalence classes of semistable G-bundles and the set of isomorphism classes of unitary G-bundles.*

Proof. This follows immediately from Proposition 3.12 (ii) and the preceding proposition.

3.16. *Lemma. Let P be a maximal parabolic subgroup of G and χ the dominant character of P which generates the group of characters of P/Z_0. Then there is an irreducible representation $\rho: G \to SL(V)$ such that $\rho(Z_0) = 1$. There is a line $\{v\} \subset V$ whose stabilizer is precisely P and P acts by the character χ on the line $\{v\}$.*

Proof. Let $G' = G/Z_0$ and P' = image of P in G'. Let $\pi: \tilde{G}' \to G'$ be the universal covering group of G'. Then $\tilde{P}' = \pi^{-1}(P')$ is a maximal parabolic subgroup of $\tilde{G}'$. Let $\rho': \tilde{G}' \to SL(W)$ be the fundamental representation corresponding to $\tilde{P}'$. Then there is a line $\{w\} \subset W$ whose stabilizer is $\tilde{P}'$ and $\tilde{P}'$ acts on $\{w\}$ by the dominant character $\tilde{\chi}$ which generates the character group of $\tilde{P}'$. Since χ is dominant $\chi \circ \pi = \tilde{\chi}^n$, for an integer $n > 0$. Then the irreducible G-subspace V generated by $v = w \otimes w \otimes \cdots \otimes w$ in $W \otimes W \otimes \cdots \otimes W$ (n factors) is actually a representation space for G' (since the highest weight $\tilde{\chi}^n$ goes down to G') and hence for G. Thus v and V satisfy the conditions of the lemma.

3.17. PROPOSITION

Let $\rho: G_1 \to G_2$ be a homomorphism between the reductive, connected algebraic groups G_1 and G_2 such that $\rho(Z_0[G_1]) \subset Z_0(G_2)$. Then

(i) *if E is a semistable G_1-bundle then the associated G_2-bundle $\rho_* E$ is semistable.*
(ii) *if the kernel of the homomorphism $G_1/Z_0[G_1] \to G_2/Z_0[G_2]$ induced by ρ is finite, then E is semistable if and only if $\rho_* E$ is semistable.*

Proof. (i) Let E be a semistable G_1-bundle. We first prove the case when $G_2 = GL(V)$. So let $\rho: G_1 \to GL(V)$ be a representation such that $Z_0[G_1]$ acts on V through the character $\chi: Z_0[G_1] \to \mathbb{C}^*$. Since E is semistable there is an admissible reduction of structure group of E to a parabolic subgroup $P = M \cdot U$ such that $E[P, M]$ is a stable M-bundle. Then by ([14], Theorem 7.1, p. 146) $E[P, M] \approx E'(h, \tilde{C})$ for an unitary representation $h: \Gamma \to K', K'$ a maximal compact subgroup of M and $\tilde{C} \in \tilde{Z}_0[M]$. Since σ is admissible $\tilde{C} \in \tilde{Z}_0[G_1]$ (as in the claim in the proof of Proposition 3.15).

Let $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ be the flag in V defined by the condition that V_i/V_{i-1} is the largest subspace of $V/V_{i-1}, i = 1, \ldots, n$ on which U acts trivially. This flag is invariant under the action of P. Since σ is admissible by Lemma 3.5.8

$$\mu(E[P](V_i/V_{i-1})) = \mu(E(V)). \tag{1}$$

Let $P' = M' \cdot U'$ be the parabolic subgroup of $GL(V)$ which is the stabilizer of the flag $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$. We can assume (by taking a conjugate of M' if necessary) that $\rho(M) \subset M'$. The flag of sub-bundles $E[P](V_0) \subset E[P](V_1) \subset \ldots \subset E(V)$ gives a reduction of structure group of the $GL(V)$-bundle $E_2 = \rho_* E$ to the subgroup P' which is admissible because of (1) (Remark 3.4). Now $E_2[P', M']$ is isomorphic to $E[P, M](M')$, the M'-bundle got from $E[P, M]$ by the extension of structure group $\rho: M \to M'$. By ([14], Remark 6.1, p. 145) $E_2[P', M'] \approx E(\rho \circ h, \tilde{\rho}(\tilde{C}))$. (Note that since $\tilde{C} \in \tilde{Z}_0[G_1]$ and $Z_0[G_1]$ acts by scalars $\tilde{\rho}(\tilde{C}) \in \tilde{Z}_0[GL(V))]$). Therefore $E_2[P', M']$ is semistable by ([14], Proposition 2.2). Therefore by Lemma 3.5.11, E_2 is semistable.

Now let G_2 be arbitrary, and E a semistable G_1-bundle. We shall show $E_2 = \rho_* E$ is semistable. Let σ be a reduction of structure group of E_2 to a maximal parabolic subgroup P of G_2. Let λ be the dominant character of P which generates the character group of $P/Z_0[G_1]$. By Lemma 3.16, there is a representation $h: G_2 \to SL(V)$ of G_2 on a vector space V such that $Z_0[G_2]$ acts trivially on V, there is a line $\{v\} \subset V$ whose stabilizer is precisely P, and P acts on the line $\{v\}$ by the character λ. Then as proved above $h_* E_2 = h_* \rho_* E$ is a semistable vector bundle (of degree 0 since $h(G_2) \subset SL(V)$). Further $\lambda_* E_2[P]$ is a sub-bundle of $h_* E_2$ (corresponding to $\{v\} \subset V$). Therefore $\deg(\lambda_* E_2[P]) \leqslant 0$, which shows that σ satisfies the semistability condition.

(ii) Now suppose $G_1/Z_0[G_1] \to G_2/Z_0[G_2]$ has finite kernel and $E_2 = \rho_* E$ is semistable. Let $h: G_2 \to SL(V)$ be a representation such that $\ker h = Z_0[G_2]$. Let $P = M \cdot U$ be a maximal parabolic subgroup of G_1 and σ a reduction of structure group of E to P. Clearly there is a G_2-invariant subspace V_1 of V such that V_1 is neither 0 nor V and U acts nontrivially on V_1. Let V_2 be the largest subspace of V_1 on which U acts trivially. Then $V_2 \neq 0$ by Lie–Kolchin theorem ([2], (10.5), p. 243) and $V_2 \neq V_1$. Also V_2 is stable under P. Let $r = \text{rank of } V_2$. Let $W = \overset{r}{\wedge} V_1$ and $\{w\} \subset W$, the line $\overset{r}{\wedge} V_2$ corresponding to $V_2 \subset V_1$. Then P acts on the line $\{w\}$ by a character λ^n for some $n > 0$, where λ is the dominant character of P which generates the group of characters of $P/Z_0[G_1]$. Now since E_2 is semistable by part (i) $E_2(W)$ is a semistable vector bundle (of degree 0).

Also $E_2(W) \approx E[P](W)$ and $\lambda^n_* E[P] \approx E[P](\{w\}) \subset E[P](W)$. Since $E[P](W)$ is semistable we have $\deg \lambda^n_* E[P] \leqslant 0$ which shows that σ satisfies the semistability condition.

3.18. COROLLARY

Let E be a G-bundle and Ad E *be its adjoint bundle. Then E is semistable if and only if* Ad E *is semistable.*

Proof. This follows immediately from the preceding proposition.

3.19. *Lemma. Let A and B be $\mathbb{C}$-algebras of finite type. Let J be an ideal of $A \otimes_{\mathbb{C}} B$. Then there is an unique ideal J_1 of A such that for any $\mathbb{C}$-algebra C and any homomorphism $f: A \to C$, $\ker f$ contains J_1 if and only if $\ker(f \otimes id_B)$ contains J.*

Moreover, if A is an $\mathbb{N}^r$-graded algebra ($\mathbb{N}$ = set of natural numbers) and J is a homogeneous ideal of $A \otimes_{\mathbb{C}} B$ where $A \otimes_{\mathbb{C}} B$ is given the gradation taking B as an algebra with trivial gradation, then J_1 is a homogeneous ideal of A.

Proof. Let $\{I_i\}$ be the collection of all ideals of A with the property $I_i \otimes_{\mathbb{C}} B \supset J$. Let $J_1 = \bigcap_i I_i$. For any collection of subspaces $\{S_j\}$ of A, $\left(\bigcap_j S_j\right) \otimes_{\mathbb{C}} B = \bigcap_j \left(S_j \otimes_{\mathbb{C}} B\right)$ we have $J_1 \otimes_{\mathbb{C}} B \supset J$. It is easy to check that J_1 has the required properties.

If A is graded by the subspaces $\{A_n\}_{n \in \mathbb{N}^r}$ and J is homogeneous then $J = \bigoplus_n J \cap \left(A_n \otimes_{\mathbb{C}} B\right)$. Since $J \cap \left(A_n \otimes_{\mathbb{C}} B\right) \subset \left(J_1 \otimes_{\mathbb{C}} B\right) \cap \left(A_n \otimes_{\mathbb{C}} B\right) = (J_1 \cap A_n) \otimes_{\mathbb{C}} B$, for the homogeneous ideal $J'_1 = \bigoplus_n J_1 \cap A_n \subset J_1$ we have $J'_1 \otimes_{\mathbb{C}} B \supset J$. Therefore $J'_1 = J_1$, proving the homogeneity of J_1.

3.20. *Lemma. Let H and Y be schemes and D a closed subscheme of $H \times Y$. Then there exists a unique closed subscheme H_1 of H such that, for any morphism $f: S \to H$ from a scheme S, the morphism $f \times id_Y: S \times Y \to H \times Y$ factors through $D \hookrightarrow H \times Y$ if and only if f factors through $H_1 \hookrightarrow H$.*

Proof. We can assume H, Y to be affine and then apply the previous lemma.

We shall now prove a lemma which is essentially the same as ([14], Lemma 4.1). We need it in this form.

Let Y be a projective scheme and L a very ample line bundle on Y. Let T be a scheme and $V_i \to T \times X, i = 1, \ldots, r$, vector bundles on $T \times X$. Let $\underline{V}_i$ be the sheaf of sections of V_i. Then the scheme V_i is $\mathrm{Spec}(S(\underline{V}_i^*))$ where $\underline{V}_i^*$ is the dual sheaf of $\underline{V}_i$ and $S(\underline{V}_i^*)$ is the symmetric algebra of $\underline{V}_i^*$. Let $\underline{V} = \oplus_{i=1}^r \underline{V}_i$. Then $V = \mathrm{Spec}(S(\underline{V}_1^*) \otimes \cdots \otimes S(\underline{V}_r^*))$, since $S(\underline{V}^*) = S(\underline{V}_1^*) \otimes \cdots \otimes S(\underline{V}_r^*)$. The $S(\underline{V}_i^*)$ are $\mathbb{N}$-graded algebras and hence $S(\underline{V}^*)$ has a natural $\mathbb{N}^r$-gradation. Let $C \subset V$ be a closed subscheme of V and $\mathscr{I}$ the sheaf of ideals of C in V. We call C a *multicone* if $\mathscr{I}$ is homogeneous with respect to the $\mathbb{N}^r$-gradation of $S(\underline{V}^*)$.

Let (Sch/T) be the category of schemes over T. Let $\pi = \pi(C/T \times Y): (\mathrm{Sch}/T) \to (\mathrm{Sets})$ be the functor which associates to an $(S \to T) \in (\mathrm{Sch}/T)$ the set $\mathrm{Hom}_{T \times Y}(S \times Y, C)$ of $T \times Y$-morphisms of $S \times Y$ into C. For a T-morphism $f: S' \to S$, $\pi(f)$ is defined to be pulling back by f.

Let $p_T: T \times Y \to T$ and $p_Y: T \times Y \to Y$ be the projections. Fix an integer m sufficiently large such that $H^1(Y, V_{j,t}(m)) = 0$ for all $t \in T$, where $V_{j,t}(m)$ is the restriction of $V_j(m) = V_j \otimes p_Y^*(L^m)$ to $t \times Y = Y$([11], Lecture 7, p. 49).

Let $\mathscr{H}_j = p_{T*}(V_j(m))$ and $\mathscr{H} = p_{T*}(V(m)) = \oplus_{j=1}^r \mathscr{H}_j$. Then $\mathscr{H}_j$ and $\mathscr{H}$ are locally free ([11], Lecture 7, p. 51). Let $H_j = \mathrm{Spec}(S(\mathscr{H}_j^*))$ be the vector bundle on T corresponding to the locally free sheaf $\mathscr{H}_j$ and $H = \mathrm{Spec}(S(\mathscr{H}^*)) = \mathrm{Spec}(S(\mathscr{H}_1^*) \ldots S(\mathscr{H}_r^*))$.

3.21. *Lemma. The functor π defined above is representable by a closed subscheme N of the scheme H.*

If C is a multicone in $V = \oplus_{j=1}^{r} V_j$ then N is a multicone in $H = \oplus_{j=1}^{r} H_j$.

3.21.1. COROLLARY

If C is a multicone then the set

$$\left\{ t \in T \;\middle|\; \begin{array}{l} \exists(\sigma_1,\ldots,\sigma_r) \in \pi(t \hookrightarrow T) \subset H^0(Y, V_t) = \bigoplus_{j=1}^{r} H^0(Y, V_{j,t}) \text{ such} \\ \text{that } \sigma_j \neq 0 \text{ as a section of } V_{j,t} \forall j = 1,\ldots,r \end{array} \right\}$$

is a closed subset of T.

Proof. Let $s \in H^0(Y, L^m)$ be a non-zero section and D the divisor of zeros of s. We then have an exact sequence

$$0 \to O \to L^m \to L^m \otimes O_D \to 0, \tag{1}$$

where O (resp. O_D) is the structure sheaf of Y (resp. D) (see [11], p. 63). Pulling back (1) by p_Y and tensoring by V we get the following exact (p_Y is flat, V is locally free) sequence on $T \times Y$.

$$0 \to V \to V(m) \to \mathscr{T} \to 0, \tag{2}$$

where $\mathscr{T} = V(m) \otimes p_Y^* O_D$.

Let $\pi_m = \pi(V(m)/T \times Y)$ (resp. $\pi_0 = \pi(V/T \times Y)$) be the functor from (Sch/T) to (Sets) which associates to $(f: S \to T)$ the set $\mathrm{Hom}_{T \times Y}(S \times Y, V(m)) = H^0(S \times Y, (f \times id_Y)^*(V(m))$ (resp. $\mathrm{Hom}_{T \times Y}(S \times Y, V) = H^0(S \times Y, (f \times id_Y)^*(V))$).

Pulling back (2) by $f \times id_Y$ to $S \times Y$ (which is equivalent to pulling back (1) by the flat morphism $p_Y \circ (f \times id_Y)$ and tensoring by the locally free sheaf $(f \times id_Y)(V)$) we get the following exact sequence on $S \times Y$

$$0 \to (f \times id_Y)^*(V) \xrightarrow{\alpha} (f \times id_Y)^*(V(m)) \xrightarrow{\beta} (f \times id_Y)^* \mathscr{T} \to 0. \tag{3}$$

The cohomology exact sequence of (3) gives $\pi_0(S) \subset \pi_m(S)$ so that π_0 is a subfunctor of π_m. Since C is a subscheme of V clearly $\pi(S) \subset \pi_0(S)$. We shall prove that π_m is represented by the T-scheme $H = \mathrm{Spec}(S(\mathscr{H}^*)) \xrightarrow{p} T$ and π_0 and π are represented by closed subschemes of H.

Since p_T is a projective morphism and $V(m)$, being locally free, is flat over T and $H^1(Y, V_t(m)) = 0$ for every $t \in T$ we have ([11], Lecture 7, pp. 50–51) $\mathscr{H}$ to be locally free and for any morphism $f: S \to T$ the natural morphism

$$f^*(\mathscr{H}) \to p_{S*}((f \times id_Y)^*(V(m))) \tag{$*$}$$

is an isomorphism. We then have the following sequence of natural isomorphisms:

$$\begin{aligned} \pi_m(S) &\simeq H^0(S \times Y, (f \times id_Y)^*(V(m))) \\ &\simeq H^0(S, p_S((f \times id_Y)^*(V(m))) \\ &\simeq H^0(S, f^*(\mathscr{H})), \text{(by } (*)) \end{aligned}$$

$$\simeq \mathrm{Hom}_S(S, f^*(H))$$
$$\simeq \mathrm{Hom}_T(S, H).$$

It follows that π_m is represented by H.

Let σ be the universal section in $H^0(H \times Y, (p \times id_Y)^*(V(m)))$ (i.e. the element corresponding to identity in $\mathrm{Hom}_T(H, H)$ in the above identifications). Let σ_D be the restriction of σ to $H \times D$. On $H \times Y$ we have the exact sequence (corresponding to (3))

$$0 \to (p \times id_Y)^*(V) \xrightarrow{\alpha'} (p \times id_Y)^*(V(m)) \xrightarrow{\beta'} (p \times id_Y)^*(\mathscr{T}) \to 0. \qquad (3)'$$

The sheaf $(p \times id_Y)^*\mathscr{T}$ has support on $H \times D$ and its restriction to $H \times D$ is a vector bundle which we denote by W. For the vector bundle $W \to H \times D$, $\beta' \circ \sigma_D$ gives a section. Let M be the closed subscheme of $H \times D$ which is the pull back of the zero section of W by $\beta' \circ \sigma_D$. Apply Lemma 3.20 to get a closed subscheme N_0 of H such that any morphism $f: S \to H$ factors through N_0 if and only if $f \times id_D: S \times D \to H \times D$ factors through M. It is easy to see that the ideal sheaf of M in $H \times D$ is homogeneous with respect to the natural $\mathbb{N}^r$-gradation induced from $S(\mathscr{H}^*)$. Therefore by Lemma 3.19 the ideal sheaf of N_0 in H is also homogeneous.

We claim that N_0 represents the functor π_0. It is enough to show that the element $(f \times id_Y)^*(\sigma)$ in $\pi_m(S)$ corresponding to $f \in \mathrm{Hom}(S, H)$ lies in $\pi_0(S)$ if and only if f factors through N_0. Since the restriction of $\beta' \circ \sigma$ to $N_0 \times Y$ (i.e. pull back by $N_0 \times Y \subsetneq H \times Y$) is zero we have that if f factors through N_0 then $(f \times id_Y)^*(\sigma)$ is zero. This implies by (3), $(f \times id_Y)^*(\sigma) \in \pi_0(S)$. Conversely if $(f \times id_Y)^*(\sigma) \in \pi_0(S)$ using (3), $\beta \circ ((f \times id_Y)^*(\sigma)) = (f \times id_Y)^*(\beta' \circ \sigma)$ is zero. Hence $(f \times id_D)^*(\beta' \circ \sigma_D)$ is zero. But this implies that $f \times id_D$ factors through M and hence f factors through N_0.

Therefore $p_0: N_0 \to T$ represents π_0 where p_0 is the restriction of $p: H \to T$. The universal section in $H^0(N_0 \times Y, (p_0 \times id_Y)^*(V))$ is σ_0 given by the pull back of σ by $N_0 \times Y \subsetneq H \times Y$. Let M_1 be the subscheme of $N_0 \times Y$ which is the pull back of the subscheme $(p_0 \times id_Y)^*(C) \subset (p_0 \times id_Y)^*(V)$ by σ_0. If C is a multicone in V the sheaf of ideals of M_1 in $N_0 \times Y$ is homogeneous with respect to the natural $\mathbb{N}^r$-gradation of the structure sheaf of $N_0 \times Y$. (Since the ideal sheaf of N_0 in H is homogeneous the structure sheaf of N_0 has a natural $\mathbb{N}^r$-gradation which gives the $\mathbb{N}^r$-gradation on the structure sheaf of $N_0 \times Y$.)

Again applying Lemma 3.20 we get a closed subscheme $N \subset N_0$ such that any morphism $f: S \to N_0$ factors through N_1 if and only if $f \times id_Y: S \times Y \to N_0 \times Y$ factors through M_1. It is then easy to see that N represents the functor π. By Lemma 3.19 if C is a multicone the ideal sheaf of N in N_0 (and hence in H) is homogeneous. This completes the proof of the lemma.

To prove the corollary note that if C is a multi-cone in V we have proved that

$$N = \mathrm{Spec}(S(\mathscr{H}_1^*) \otimes \cdots \otimes S(\mathscr{H}_r^*)/J)$$

where J is an $\mathbb{N}^r$-graded ideal of $S(\mathscr{H}_1^*) \otimes \cdots \otimes S(\mathscr{H}_r^*)$. Such an ideal defines a closed subscheme Q of $\mathrm{Proj}(S(\mathscr{H}_1^*)) \times \cdots \times \mathrm{Proj}(S(\mathscr{H}_r^*))$ and clearly the set defined in the corollary is $\tau(Q)$ where $\tau: \mathrm{Proj}(S(\mathscr{H}_1^*)) \times \cdots \times \mathrm{Proj}(S(\mathscr{H}_r^*)) \to T$ is the projection. Since τ is a proper morphism $\tau(Q)$ is closed in T.

3.22. *Lemma. Let $\xi \to S \times X$ be a family of semistable G-bundles. Let P be a maximal parabolic subgroup of G. Let χ be the dominant character on P which generates the group* *of characters of P which are trivial on Z_0. Let L be a line bundle of degree zero on X. Then the set*

$$S_{P,L} = \left\{ s \in S \,\middle|\, \begin{array}{l} \xi_s \text{ has an admissible reduction } \sigma \text{ of structure} \\ \text{group to } P \text{ such that } \chi_* \sigma^* E \approx L \end{array} \right\}$$

is a closed subset of S.

Proof. Let $Y = S \times X$ and $\xi_1 = S \times L$. Then $\xi_1 \underset{Y}{\times} \xi$ is a $\mathbb{C}^* \times G$-bundle on Y. Let $\bar{P}$ be the subgroup $\{(\chi(p), p) | p \in P\}$ of $\mathbb{C}^* \times G$. It is a closed algebraic subgroup of $\mathbb{C}^* \times G$ since it is the image of the homomorphism $P \to \mathbb{C}^* \times G$, $p \mapsto (\chi(p), p)$ ([2], (1.4), p. 88). Under the projection $\mathbb{C}^* \times G \to G$, $\bar{P}$ maps onto P. Therefore, the projection $\xi_1 \underset{Y}{\times} \xi \to \xi$ induces $\xi_1 \underset{Y}{\times} \xi/\bar{P} \to \xi/P$. A reduction of structure group of $\xi_1 \underset{Y}{\times} \xi$ to $\bar{P}$ gives, by composing with $\xi_1 \underset{Y}{\times} \xi/\bar{P} \to \xi/P$, a reduction of structure group of ξ to P and it is easily checked that $\chi_*(\xi[P])$ is isomorphic to ξ_1. Conversely one can check that given a reduction of structure group σ of ξ to P such that $\chi_* \sigma^* \xi$ is isomorphic to ξ_1 then there is a reduction of structure group of $\xi_1 \underset{Y}{\times} \xi$ to $\bar{P}$ which when composed with $\xi_1 \underset{Y}{\times} \xi/\bar{P} \to \xi/P$ gives back σ. By Lemma 3.16 there is an irreducible representation $\rho: G \to SL(V)$ with a line $\{v\} \subset V$ such that $P = \{g \in G / \rho(g)\{v\} = \{v\}\}$ and $\rho(p)v = \chi(p)v$ for $p \in P$. Let $\mathbb{C}^* \times G$ act on $\mathrm{Hom}(\mathbb{C}, V)$ by $(z, g)f = \rho(g) \circ f \circ z^{-1}$ where $z \in \mathbb{C}^*$, $g \in G$, $f \in \mathrm{Hom}(\mathbb{C}, V)$ and we have denoted by z^{-1} the multiplication by the scalar $z^{-1} \in \mathbb{C}^*$. Let $f_0 \in \mathrm{Hom}(\mathbb{C}, V)$ be defined by $f_0(z) = z \cdot v$. Then the stabilizer of f_0 in $\mathbb{C}^* \times G$ is $\bar{P}$ and hence $\mathbb{C}^* \times G \to \mathrm{Hom}(\mathbb{C}, V)$ is given by $(z, g) \mapsto (z, g)\, f_0$ which induces an isomorphism of schemes of $\mathbb{C}^* \times G/\bar{P}$ with the orbit C of f_0 under $\mathbb{C}^* \times G$ (C is locally closed in $\mathrm{Hom}(\mathbb{C}, V)$ and endows it with the canonical reduced subscheme structure, (cf. [2], (1.8), p. 98)). Let $\bar{C}$ be the closure of C in $\mathrm{Hom}(\mathbb{C}, V)$. Since C is invariant under scalar multiplication in $\mathrm{Hom}(\mathbb{C}, V)$, $\bar{C}$ taken with the canonical reduced scheme structure is a cone in $\mathrm{Hom}(\mathbb{C}, V)$. We claim that $\bar{C} = C \cup \{0\}$. Clearly $0 \in \bar{C}$. Suppose $0 \neq f \in \bar{C}$. Since C is locally closed, $\bar{C}$ is also the closure of C with respect to the strong (Hausdorff) topology. Therefore, there is a sequence $\rho(g_n) \circ f_0 \circ z_n^{-1}$ tending to f, with $g_n \in G$, $z_n \in \mathbb{C}^*$. Let K be a maximal compact subgroup of G. Then $G = K \cdot P$ ([14], proof of Proposition 2.1, p. 130). Write $g_n = k_n p_n$ with $k_n \in K$, $p_n \in P$. Since K is compact we can assume that $\lim_{n \to \infty} k_n = k$. Then $\lim_{n \to \infty} \rho(p_n) \circ f_0 \circ z_n^{-1} = \rho(k)^{-1} \circ f$ so that $(\rho(p_n) \circ f_0 \circ z_n^{-1})(z) = (\chi(p_n) z_n^{-1} z)v$ tends to $\rho(k)^{-1} f(z)$. This implies $\lim_{n \to \infty} \chi(p_n) z^{-1} = z_0$. Therefore $\rho(k)^{-1} f(z) = z_0 z \cdot v = (f_0 \circ z_0)(z)$, since $f \neq 0$, $z_0 \neq 0$ and we have $f = (z_0^{-1}, k) f_0$. Therefore $f \in C$. Thus we have proved $\bar{C} = C \cup \{0\}$.

A reduction of structure group of $\xi_1 \underset{Y}{\times} \xi$ to $\bar{P}$ is given by a section of $\xi_1 \underset{Y}{\times} \xi/\bar{P} = (\xi_1 \underset{Y}{\times} \xi)(\mathbb{C}^* \times G/\bar{P}) = (\xi_1 \underset{Y}{\times} \xi)(C)$. The inclusions $C \subset \bar{C} \subset \mathrm{Hom}(\mathbb{C}, V)$ induce the inclusions $(\xi_1 \underset{Y}{\times} \xi)(C) \subset (\xi_1 \underset{Y}{\times} \xi)(\bar{C}) \subset (\xi_1 \underset{Y}{\times} \xi)(\mathrm{Hom}(\mathbb{C}, V))$ (See §2.4).

By Lemma 3.21, the set $S'_{P,L}$ of points $s \in S$ such that $(\xi_1 \underset{Y}{\times} \xi)_s(\bar{C}) = (L \times \xi_s)(\bar{C})$ has a non-zero section is a closed subset of S. We shall now show that $S'_{P,L} = S_{P,L}$. Clearly $S_{P,L} \subset S'_{P,L}$. Suppose $s \in S'_{P,L}$. Then we have a non-zero homomorphism $\sigma: L \to \xi_s(V)$. Since $\deg L = 0$ and $\xi_s(V)$ is a semistable vector bundle of degree zero (Proposition 3.17) it follows that σ is an injection ([19], Proposition 3.1, p. 306). This implies that σ factors through the open immersion $(L \times \xi_s)(C) \hookrightarrow (L \times \xi_s)(\bar{C})$. Therefore $L \times \xi_s$ has a reduction to $\bar{P}$ and hence $s \in S_{P,L}$.

3.23. *Lemma. Let $\xi \to S \times X$ be a family of semistable G-bundles. Suppose there exist*

(i) *a dense open subset T of S*
(ii) *a reductive subgroup M_0 of G of maximal rank and*
(iii) *a stable M_0-bundle E_0*

with the following property: For every $t \in T$ the G-bundle $\xi_t \to X$ has an admissible reduction of structure group σ to a parabolic subgroup Q having M_0 as a Levi component such that the M_0-bundles $\xi_t[Q, M_0]$ and E_0 are isomorphic. Then for all $s \in S$ $\mathrm{gr}\, \xi_s$ is isomorphic to $E_0(G)$ and hence the G-bundles ξ_s for any $s \in S$ are all equivalent.

Proof. We can assume without loss of generality that S is reduced, irreducible and affine. We shall prove the lemma by induction on the semisimple rank of the structure group.

Suppose $M_0 = G$. Let $S \times E_0 \to S \times X$ be the trivial family of G-bundles given by E_0. Let $\rho: G \hookrightarrow GL(V)$ be a faithful representation and $V = V_1 \oplus \cdots \oplus V_r$ be a decomposition into irreducible subspaces. Let $\rho_i: G \to GL(V_i)$ be the representation of G on V_i induced by ρ. Define

$$C = \{(\rho_1 \lambda_1(g), \ldots, \rho_r \lambda_r(g)) \in GL(V) \mid \lambda_i \in \mathbb{C}^*, g \in G\}.$$

Let $\bar{C}$ be the closure of C in $\oplus_{i=1}^r \operatorname{End} V_i$. We consider $\bar{C}$ as a closed subscheme of $\oplus \operatorname{End} V_i$ with the reduced structure. Note that C is an open subscheme in $\bar{C}$ and $\bar{C}$ is a multicone in $\oplus \operatorname{End} V_i$. By the hypothesis of the lemma and assumption $M_0 = G$, for $t \in T$ we have an isomorphism $\varphi_t: E_0 \to \xi_t$. We can interpret φ_t as a section of the fiber bundle $\left(E_0 \underset{X}{\times} \xi_t\right)(G)$ with fiber G associated to the $G \times G$ bundle $E_0 \underset{X}{\times} \xi_t$ for the action of $G \times G$ on G given by $(g_1, g_2)(h) = g_2 h g_1^{-1}$, $g_1, g_2, h \in G$ (cf. [17], § 3.5, Example, p. 1–19). Since ρ is faithful one can identify G with the locally closed subscheme (taken with reduced structure) $\rho(G)$ of $\bar{C} \subset \oplus \operatorname{End} V_i$. We extend the action of $G \times G$ to $\oplus \operatorname{End} V_i$ by setting $(g_1, g_2) f = \rho(g_2) \circ f \circ \rho(g_1)^{-1}$, $g_1, g_2 \in G$ and $f \in \oplus \operatorname{End} V_i$. Then φ_t gives a section of $(E_0 \times \xi_t)(\bar{C}) = \left((S \times E_0) \underset{S \times X}{\times} \xi\right)_t (\bar{C})$. By Corollary 3.21.1 it follows that $\left((S \times E_0) \underset{S \times X}{\times} \xi\right)_s (\bar{C})$ has a section for any $s \in S$ which induces a non-zero endomorphism of $E_0(V_i)$ for every i. Then by ([14], Proposition 3.1) such a section gives rise to a G-bundle isomorphism of E_0 with ξ_s. This proves the lemma when $M_0 = G$. In particular if G is of semisimple rank zero then $G \approx \mathbb{C}^{*n}$ and any reductive subgroup of maximal rank has to be $\mathbb{C}^{*n}$. Therefore we have proved the lemma in the case of semisimple rank zero. (When $G = \mathbb{C}^{*n}$, alternatively, the lemma follows by observing that the $\mathbb{C}^{*n}$-bundle ξ is merely an n-tuple of line bundles $\mathscr{L}_i \to S \times X$ and by the hypothesis the morphism from S into the Jacobian (of suitable degree) of X determined by $\mathscr{L}_i$ is constant on T and hence on S.)

Now assume M_0 is a proper subgroup of G. Let $P_1, \ldots, P_r$ be a set of representatives of conjugacy classes of parabolic subgroups of G containing M_0 as a Levi component. Let $P'_1, \ldots, P'_r$ be maximal parabolic subgroups such that $P_i \subset P'_i$, $i = 1, \ldots, r$. Let χ_i be the dominant character of P_i' and let $L_i = \chi_{i*}(E_0)$. Since a reduction of structure group to P_i can be considered in a natural way as a reduction of structure group to $P_i' \supset P_i$, the hypothesis of the lemma implies that $T \subset \cup_{i=1}^r S_{P_i', L_i}$ where S_{P_i', L_i} is the set of points $s \in S$ such that ξ_s has a reduction of structure group to P_i' such that $\chi_{i*}\sigma^*\xi_s \approx L_i$. Since T is dense in S and each S_{P_i', L_i} is closed (Lemma 3.22) we have $S \subset \cup_{i=1}^r S_{P_i', L_i}$. But we have assumed that S is irreducible. Therefore $S = S_{P,L}$ for $P = P_i'$, $L = L_i$ for some i. Let $P = M \cdot U$ be a Levi decomposition chosen such that $M_0 \subset M$.

For this parabolic subgroup P and its dominant character χ let the cone $\bar{C} \subset \mathrm{Hom}(\mathbb{C}, V)$ be defined as in the proof of Lemma 3.22. Then $\left(\xi_1 \underset{Y}{\times} \xi\right)(\bar{C})$ is a cone in the vector bundle $\left(\xi_1 \underset{Y}{\times} \xi\right)(\mathrm{Hom}(\mathbb{C}, V)) \to S \times X$, where $Y = S \times X$. By Lemma 3.21, we have an S-scheme $p: H \to S$ representing the functor π defined by $\pi(S' \to S) = \mathrm{Hom}_{S \times X}\left(S' \times X, \left(\xi_1 \underset{Y}{\times} \xi\right)(\bar{C})\right)$. Moreover H is of the form $\mathrm{Spec}(S(\mathscr{F})/J)$ for a locally free sheaf $\mathscr{F}$ on S and a homogeneous ideal J of $S(\mathscr{F})$. Let $\varepsilon: S \to H$ be the zero section ([EGA II], § 8·3) and $H' = H - \varepsilon(S)$. Let $p': H' \to S$ be the restriction of p to H'. Let σ be the universal element in $\pi(H)$. Let $\xi_1' = (p' \times id_X)^*(\xi_1)$ and $\xi' = (p' \times id_X)^*(\xi)$. Then σ gives a section of $\left(\xi_1' \underset{Y}{\times} \xi'\right)(\bar{C}) \to H' \times X$. As in the proof of Lemma 3.22, this section factors through $\left(\xi_1' \underset{Y}{\times} \xi'\right)(C) \subsetneq \left(\xi_1' \underset{Y}{\times} \xi\right)(\bar{C})$. It follows that the G-bundle $\xi' \to H' \times X$ has an admissible reduction of structure group $\mathscr{K}'$ to $P = M \cdot U$. Let $\xi'' = \xi'[P, M]$. Since $\xi_h' \approx \xi_{p(h)}$ for $h \in H$, the reduction of structure group $\mathscr{K}_h'$ of ξ_h' induced by $\mathscr{K}'$ gives canonically a reduction of structure group of $\xi_{p(h)}$ which we denote by $\mathscr{K}_h$.

Let $\pi': H' \to \mathrm{Proj}(S(\mathscr{F})/J)$ and $\mathscr{F}: \mathrm{Proj}(S(\mathscr{F})/J) \to S$ be the natural morphisms ([EGA, II], § 8.3). Since $S = S_{P,L}$, we have that p' is surjective. Since $p' = \tau \circ \pi'$, τ is surjective. Therefore, τ being proper and surjective and S being irreducible there is an irreducible component of $\mathrm{Proj}(S(\mathscr{F})/J)$ which maps onto S. It follows from ([EGA, II], Corollary 8.3.6, p. 165) that the irreducible components of $\mathrm{Proj}(S(\mathscr{F})/J)$ are the images under π' of the irreducible components of H'. Therefore, there is an irreducible component H'' of H' which maps onto S. Let p'' be the restriction of p to H''. Let us denote by the same letter ξ'' the restriction of the M-bundle $\xi'' = \xi'[P, M]$ to $H'' \times X$. The open subset $T'' = p''^{-1}(T)$ of H'' is dense since H'' is irreducible. Let $t'' = p''^{-1}(t)$ for $t \in T$. Then $\xi_{t''}'' \approx q_* \mathscr{K}_{t''}^* \xi_t$ where $q: P \to M$ is the projection. Since $t \in T$ there is a reduction of structure group w of ξ_t to a parabolic subgroup Q with M_0 as a Levi component such that $j_* q_{0*} w^* \xi_t \approx j_* E_0$ where $q_0: Q \to M_0$ is a projection and $j: M_0 \subsetneq G$ is the inclusion. We can assume that (P, Q) is compatible with $(\mathscr{K}_{t''}, w)$. Using Lemma 3.11 we see that we have reduction of structure group $\mathscr{K}_{t''} \cap w$ of the M-bundle $q_* \mathscr{K}_{t''}^* \xi_t \approx \xi_{t''}$ to the parabolic subgroup $M \cap Q$ of M which has M_0 as a Levi component. Also for this reduction $\xi_{t*}''[M \cap Q, M_0] \approx E_0$ by 3.11 (iv). This shows that for the M-bundle $\xi'' \to H'' \times X$ the conditions (i), (ii) and (iii) of the hypothesis of the lemma are satisfied (with 'T' $= T''$, 'M_0' $= M_0$ and 'E_0' $= E_0$). Since the semisimple rank of M is one less than that of G we can apply the induction hypothesis to conclude that $\mathrm{gr}\, \xi_h'' \approx E_0(M)$ for all $h \in H''$. But this implies that $\mathrm{gr}(q_* \mathscr{K}_h^* \xi_{p(h)}) \approx E_0(M) \approx \mathrm{gr}(E_0(M))$. Therefore by 3.13 (ii) $\mathrm{gr}\, \xi_s \approx E_0(G)$ for all $s \in S$.

3.24. PROPOSITION

Let E be a semistable G-bundle. Then

(i) *if in a family of semistable G-bundles* $\xi \to S \times X$ *we have* $\xi_t \approx E$ *for t varying in a dense open subset of S then* ξ_s *is equivalent to E for all* $s \in S$.
(ii) *there exists a family of semistable G-bundles* $\xi \to \mathbb{C} \times X$ *such that for* $0 \neq z \in \mathbb{C}$, $\xi_z \approx E$ *and* $\xi_0 \approx \mathrm{gr}\, E$.

Proof. (i) follows from the preceding lemma and (ii) follows from Propositions 3.5 and 3.12.

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