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Generalized parabolic bundles and applications—II

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Abstract. We prove the existence of the moduli space $M(n,d)$ of semistable generalised parabolic bundles (GPBs) of rank n, degree d of certain general type on a smooth curve. We study interesting cases of the moduli spaces $M(n,d)$ and find explicit geometric descriptions for them in low ranks and genera. We define tensor products, symmetric powers etc. and the determinant of a GPB. We also define fixed determinant subvarieties $M_L(n,d)$, L being a GPB of rank 1. We apply these results to study of moduli spaces of torsionfree sheaves on a reduced irreducible curve Y with nodes and ordinary cusps as singularities. We also study relations among these moduli spaces (rank 2) as polarization varies over $[0,1]$.

Keywords. Generalized parabolic bundles; nodal curves; torsionfree sheaves.

1. Introduction

This work is a generalization and continuation of our work in [B3] where (and in [B5]) we introduced the notion of generalized parabolic bundles (GPBs). They are vector bundles with flags (of vector spaces) over effective divisors. In [B3] we studied the special case of flags of length 2 and we consider here flags of sufficiently general type. In § 2 we study the generalities on semistable and stable GPBs and their properties. We prove the existence of the moduli space $M(n,d)$ of semistable GPBs of rank n, degree d of certain general type. In § 3, we study interesting cases of the moduli spaces $M(n,d)$ with flags of length 2. We define 1-stability and 1-semistability, compare these notions with the stability and semistability of GPBs and use them to prove the existence of fine moduli spaces for GPBs. We consider the question of defining tensor products, symmetric powers etc. and the determinant of a GPB. We also define fixed determinant subvarieties $M_L(n,d)$, L being a GPB of rank 1. We describe $M_L(2,d)$ explicitly when X is an elliptic curve. There is interesting geometry associated to this (Remark 4.8). Section 4 is the application of these results to the study of moduli spaces of torsion-free sheaves on a reduced irreducible curve Y with nodes and ordinary cusps as singularities. The case of a single node was considered in [B3]. Let X be the desingularization of Y, $p:X \to Y$ being the natural map. We give a correspondence between GPBs on X and torsionfree sheaves on Y. Unlike the correspondence given by Seshadri (Theorem 17, p. 178 [S]), this correspondence is not bijective, but it preserves rank and degree. Also it maps 1-stable (1-semistable) QPBs to stable (semistable) torsionfree sheaves and vice versa. We also consider relations among various moduli spaces (rank 2) as the weight α varies over $[0,1]$ (see 4.9). These relations are similar to those obtained for stable pairs by Bradlow, Thaddeus, Garcia-Prada and others.

Finally we introduce orthogonal GPBs and study their relation to orthogonal sheaves on Y (5·10, 5·11). We postpone the general case (principal G-bundles) to a future paper. On the one hand GPBs are generalizations of parabolic bundles ([SM], [B2]). On the other hand they generalize the presentation functor giving normalizations of compactified Jacobians studied in detail by Seshadri, Oda, Kleiman, Altman and others. GPBs are associated to representations of the group $\pi_1(X)*\mathscr{Z}*\cdots*\mathscr{Z}$, where $\pi_1(X)$ is the fundamental group of X and $*$ denotes the free product of groups (Theorems 1, 2 [B4]). GPBs have many applications. They have been used crucially for proving factorization rules of generalized theta functions [RN] and for proving Frobenius splitting for moduli varieties of vector bundles on ordinary curves [MR].

2. Generalized parabolic bundles

2A *Generalities on GPBs*

Let X be an irreducible nonsingular algebraic curve defined over an algebraically closed base field k. Let D be an effective divisor on X. Let E be a vector bundle of rank n and degree d on X.

DEFINITION 2.1

A quasi parabolic structure on E over the divisor D is a flag $\mathscr{F}$ of vector subspaces of $H^0(E\otimes\mathcal{O}_D)$ given by $\mathscr{F}:F_0(E)=H^0(E\otimes\mathcal{O}_D)\supset F_1(E)\supset\cdots\supset F_r(E)=0$.

DEFINITION 2.2

A quasiparabolic bundle (QPB in short) is a vector bundle E together with quasi-parabolic structures $\mathscr{F}^j$ over finitely many disjoint divisors D_j, $j=1,\ldots,J$. Let $\underline{\mathscr{F}}=(\mathscr{F}^1,\ldots,\mathscr{F}^J)$. Then a QPB is a pair $(E,\underline{\mathscr{F}})$.

DEFINITION 2.3

An isomorphism of QPBs $(E,\underline{\mathscr{F}})$ and $(E',\underline{\mathscr{F}}')$ is an isomorphism $f:E\to E'$ which maps the flag $\mathscr{F}_j$ to the flag $\mathscr{F}'_j$ for all j.

DEFINITION 2.4

A (generalized) parabolic structure on a vector bundle E over an effective divisor D consists of

(1) a quasiparabolic structure on E over D
(2) real numbers $\alpha_1,\ldots,\alpha_r$ with $0\leqslant\alpha_1<\cdots<\alpha_r<1$ called weights associated to the flag.

DEFINITION 2.5

Let $\underline{\alpha}=(\alpha_1,\ldots,\alpha_r)$, $m_i=\dim F_{i-1}(E)/F_i(E)$, $i=1,\ldots,r$. Define $wt_D E=\sum_{i=1}^r m_i\alpha_i$. If we consider parabolic structures over divisors $D_1,\ldots,D_J$, we define $wtE=\sum_J wt_{D_j}E$. Define pardeg $E=$ degree $E+wtE$ and par $\mu(E)=$ pardeg E/rank E. These are called respectively the parabolic degree and the parabolic slope of E.

DEFINITION 2.6

A generalized parabolic bundle (abbreviated as GPB) is a vector bundle E together with parabolic structures over finitely many disjoint divisors. We denote it by a triple $(E,\underline{\mathscr{F}},\underline{\alpha})$. Here $\underline{\alpha}=(\underline{\alpha}^1,\ldots,\underline{\alpha}^j)$, $\underline{\alpha}^j=(\alpha_1^j,\ldots,\alpha_{rj}^j)$, $j=1,\ldots,J$.

DEFINITION 2.7

Every subbundle K of E gets a natural structure of a GPB (see 3.2 [B3]). By a subbundle of a GPB, we will mean a subbundle with this induced parabolic structure. A GPB $(E,\mathscr{F},\underline{\alpha})$ is semistable (respectively stable) if for every (respectively proper) subbundle K of E, one has par $\mu(K)\leqslant$ (resp. $<$) par $\mu(E)$.

DEFINITION 2.8

Let $(E,\mathscr{F},\underline{\alpha})$ be a GPB. Let $\alpha^{r_j+1}=1$, $\alpha_0^j=\alpha_{r_j}^j-1$. For real number α with $0\leqslant\alpha<1$, define $E_\alpha^j=F_{i-1}^j(E)$ if $\alpha_{i-1}^j<\alpha\leqslant\alpha_i^j$, $i=1,\dots,r_j+1$.

A morphism of GPBs is a homomorphism $f:E\to M$ of underlying vector bundles such that $f(E_\alpha^j)\subseteq M_\alpha^j$ for $0\leqslant\alpha<1$ and $j=1,\dots,J$.

PROPOSITION 2.9

(1) *If $h:I\to K$ is a generic isomorphism of GPBs which is not an isomorphism, then* par $\mu(I)<$ par $\mu(K)$.
(2) *If $f:E_1\to E_2$ is a morphism of semistable GPBs of the same rank and same parabolic degree (the divisors D_j being fixed for all j), then f is of constant rank.*
(3) *If, in addition, one of E_1,E_2 is a stable GPB then f is either zero or an isomorphism.*

Proof. Since (2) and (3) follow from (1) by standard arguments, we only prove (1). For $1\leqslant j\leqslant J$, let N_j denote the kernel of $h|D_j:I|D_j\to K|D_j$. One has $\deg K\geqslant\deg I+\sum_j h^0(N_j)$, because h is a generic isomorphism, with equality if and only if h is an isomorphism away from the union of D_j. Define

$$wtN_j=\sum_{i=0}^{r_j-1}\alpha_i^j(\dim(H^0(N_j)\cap F_i^j(I)/H^0(N_j)\cap F_{i+1}^j(I)).$$

Since $\alpha_i^j<1$ for all i,j one has

$$wtN_j\leqslant\sum_i(\dim H^0(N_j)\cap F_i^j(I)-\dim H^0(N_j)\cap F_{i+1}^j I))=h^0(N_j).$$

Thus $wtN\equiv\sum_j wtN_j\leqslant\sum_j h^0(N_j)$ with equality if and only if all $N_j=0$. Since h is a morphism of GPBs and $h|D_j$ induces an injection $(I|D_j)/N_j\to K|D_j$ it follows that $wtI-wtN\leqslant wtK$. Thus, par $\deg I=\deg I+wtI\leqslant\deg\ I+wtN+wtK\leqslant\deg\ I+\sum_j h^0(N_j)+wtK\leqslant$ par $\deg K$. The last two inequalities cannot be equalities unless h is an isomorphism. Since I and K have the same rank, (1) follows.

PROPOSITION 2.10

Let $\mathscr{C}$ denote the category of semistable GPBs $(E,\mathscr{F},\underline{\alpha})$ on an irreducible non-singular curve X with parabolic structures over fixed divisors $D_1,\dots,D_J$ on X and with fixed par $\mu=m$. *Then $\mathscr{C}$ is an abelian category whose simple objects are stable GPBs.*

By the Jordan–Holder theorem, for any $(E,\mathscr{F},\underline{\alpha})$ in $\mathscr{C}$, there exists a filtration of $(E,\mathscr{F},\underline{\alpha})$ in $\mathscr{C}$ with successive quotients stable GPBs with par $\mu=m$. The associated graded object for this filtration is unique up to isomorphism. Denote this object by $\mathrm{gr}(E,\mathscr{F},\underline{\alpha})$.

DEFINITION 2.11

We define an equivalence relation in $\mathscr{C}$ by $(E, \mathscr{F}, \underline{\alpha})$ and $(E', \mathscr{F}', \underline{\alpha}')$ are equivalent if $\mathrm{gr}(E, \mathscr{F}, \underline{\alpha})$ and $\mathrm{gr}(E', \mathscr{F}', \underline{\alpha}')$ are isomorphic in $\mathscr{C}$.

2B *Existence and properties of the moduli space*

Our aim is to construct a moduli space for equivalence classes of semistable GPBs of a 'fixed type'.

Theorem 1. *Let X be an irreducible nonsingular projective curve of genus $g (g \geqslant 0)$ defined over an algebraically closed field.*

Let $D_1, \ldots, D_J$ be finitely many (fixed) divisors on X such that their supports are mutually disjoint. Consider the set of semistable GPBs $(E, \mathscr{F}, \underline{\alpha})$ on X of fixed rank n and fixed degree with $\mathscr{F}^j : F_0^j(E) = H^0(E \otimes \mathscr{O}_{D_j}) \supset \cdots \supset F_r^j(E) = 0$ flags of length r (r independent of j) and weights $\underline{\alpha}^j = (\alpha_1, \ldots, \alpha_r)$ fixed independent of j. For each j, we assume the flag type of $\mathscr{F}^j$ fixed (varying with j). Denote this set modulo the equivalence relation (2.11) by M. Then M has the structure of a normal projective variety of dimension $n^2(g-1) + 1 + \sum_j \dim G_j$ where G_j is the flag variety of flags of type $\mathscr{F}^j$, $j = 1, \ldots, J$. The subset of M corresponding to stable GPBs is a nonsingular open subvariety.

Proof. The construction of the moduli space M is done using geometric invariant theory generalizing [B3]. We only sketch the proof as it is very similar to that in [B3]. We first construct a universal space $\tilde{R}$ for GPBs of the above type with an action of $PGL(N)$ on it. Then we show that there exists a good quotient M of $\tilde{R}$ by $PGL(N)$ in the sense of geometric invariant theory. We denote by S the set of all semistable GPBs $(E, \mathscr{F}, \underline{\alpha})$ of the above type. Without loss of generality, we may assume that $-n \leqslant \deg E < 0$. Let $C_j = \deg D_j, j = 1, \ldots, J$. For $(E, \mathscr{F}, \underline{\alpha}) \in S$, let $b = \mathrm{par}\deg(E)$. Since the GPB's $(E, \mathscr{F}, \underline{\alpha}) \in S$ are semistable, there exists m_0 such that for $m \geqslant m_0$, $h^1(E(m)) = 0$ and the canonical map $H^0(E(m)) \to \oplus_{j=1}^J H^0(E(m) \otimes \mathscr{O}_{D_j})$ is surjective. Given an integer b_0, one can choose $m \gg g$ such that for $F \in S$ (i.e. F such that $(F, \mathscr{F}, \underline{\alpha}) \in S$) or for $F \subset E, E \in S$ and $\mathrm{par}\deg F > b_0$ one has $H^1(F(m)) = 0$ and the canonical map $H^0(F(m)) \to \oplus_j H^0(F(m) \otimes \mathscr{O}_{D_j})$ is surjective. This can be done by arguments similar to those on p. 226 [SM]. We shall choose b_0 suitably later (depending only on n, g, b and $C_j, j = 1, \ldots, J$). Choose $m \gg g, m \geqslant m_0$. Let $n = h^0(E(m))$. Let P be the Hilbert polynomial of $E(m)$ in S. Let $Q = \mathrm{Quot}_X(\mathscr{O}_X^N, P)$ be the Hilbert scheme of coherent sheaves on X which are quotients of $\mathscr{O}_X^N$ and have Hilbert polynomial equal to P. There is a universal sheaf U on $Q \times X$. Let R be the subscheme of Q consisting of points q in Q corresponding to sheaves U_q which are vector bundles generically generated by sections and satisfy $H^0(U_q) \approx \mathscr{O}_X^N$ (by the Riemann Roch theorem, $H^1(U_q) = 0$ for $q \in R$). By our choice of m, R contains the subset of Q determined by $E(m)$ with $(E, \mathscr{F}, \underline{\alpha}) \in S$. It is well known that R is a nonsingular variety. Let $p_1 : R \times X \to R$ be the projection. Define $V_j = (p_1)_*(U | R \times D_j)$. Let $G(V_j)$ be the flag bundle over R of the type determined by the parabolic structure over D_j. Let $G(V)$ be the fibre product of $\{G(V_j)\}$, $j = 1, \ldots, J$ over R. We denote the total space of $G(V)$ by $\tilde{R}$. Obviously $\tilde{R}$ has the local universal property for GPBs. It is the universal space for GPBs which we wanted to construct. Let $\tilde{R}^{ss}$ (respectively $\tilde{R}^s$) denote the subset of $\tilde{R}$ corresponding to semistable (resp. stable) GPBs.

The group $PGL(N)$ acts naturally on $\mathscr{O}_X^N$ and hence on $R, \tilde{R}^s, \tilde{R}^{ss}$. We shall construct a projective variety Y with $PGL(N)$-action such that a good quotient of Y modulo

$PGL(N)$ exists. We shall give an affine injective morphism from $\tilde{R}$ to Y which is $PGL(N)$-equivariant. The existence of a good quotient of $\tilde{R}$ modulo $PGL(N)$ then follows (Proposition 3.12, [N]). Following Gieseker [G2], let us define a 'good pair' (F, φ) to be a flat family $F \to T \times X$ of vector bundles on X such that F_t is generated by its global sections at the generic point of $t \times X$ and $\varphi: \mathcal{O}_X^N \to p_*(F)$ is an isomorphism, p being the projection $T \times X \to T$. Let A denote the Jacobian of X corresponding to line bundles on X of degree equal to deg $E(m)$, $E \in S$. Let M be the Poincaré bundle on $X \times A$ and $g: X \times A \to A$ the projection. Define $Z = \mathbf{P}\,(\mathrm{Hom}(\wedge^n \mathcal{O}_A^N, g_* M)^\vee)$. Given a good pair (F, φ) one defines a morphism $T(F, \varphi): T \to Z$ as follows. Fix $t \in T$. One has a natural map $\psi: \wedge^n H^0(F_t) \to H^0(\wedge^n F_t)$ given by $\psi(s_1 \wedge \cdots \wedge s_n) = s$, $s(x) = s_1(x) \wedge \cdots \wedge s_n(x)$. Also, φ gives a map $\wedge^n \varphi: \wedge^n k^N \to \wedge^n H^0(F_t)$. Define $T(F, \varphi)(t) = \psi \circ \wedge^n \varphi$. It is easy to see that this defines a morphism [G2]. Via the action on $\mathcal{O}_A^N$, $PGL(N)$ acts naturally on Z preserving the fibres over A. Note that the fibre of Z over $L \in A$ is $\mathbf{P}(\mathrm{Hom}(\wedge^n k^N, H^0(X, L))^\vee)$. If we are given a good pair $(\underline{F}, \varphi)$ where $\underline{F} = (F, \underline{\mathscr{F}}, \underline{\alpha})$ is a family of GPBs of fixed type considered in the theorem, parametrized by T, then for every t in T, $F_t | D_j$ has flag

$$\mathscr{F}^j: F_0^j(F_t) = H^0(F_t \otimes \mathcal{O}_{D_j}) \supset F_1^j(F_t) \supset \cdots \supset F_r^j(F_t) = 0.$$

Let $e_j: H^0(F_t) \to H^0(F_t \otimes \mathcal{O}_{D_j})$ be the natural map. Via e_j (by pulling back) the flag $\mathscr{F}^j$ induces a flag on $H^0(F_t)$. Identifying $H^0(F_t)$ with k^N by φ, we get a flag (of fixed type) on k^N:

$$\mathscr{F}^j(k^N): k^N = F_0^j(k^N) \supset F_1^j(k^N) \supset \cdots \supset F_r^j(k^N),$$

with $F_r^j(k^N) =$ kernel of e_j. Let $f_i^j =$ dimension of F_i^j, $i = 0, \ldots, r$, $j = 1, \ldots, J$. Let G_i^j denote the Grassmannian of subspaces of k^N of dimension f_i^j. Let $G = \Pi G_i^j$, $i = 1, \ldots, r-1$, $j = 1, \ldots, J$. Thus $(\underline{F}, \varphi)$ determines a morphism $f: T \to G$. Define a morphism $\tilde{T}(\underline{F}, \varphi): T \to Z \times G$ by $\tilde{T}(\underline{F}, \varphi) = T(F, \varphi) \times f$. Thus we get a morphism $\tilde{T}: \tilde{R} \to Z \times G$. The space Y we wanted to define is the product $Z \times G$ and $\tilde{T}$ is the required $PGL(N)$-equivariant morphism. Let $\delta_0 = b + [n(m + 1 - g - \alpha_r \Sigma_j C_j)]$, $\delta_i = n(\alpha_{i+1} - \alpha_i)$ for $i = 1, \ldots, r-1$. Let $L_0 = \mathcal{O}_Z(1)$, $L_{ij} =$ generator of Pic G_i^j for all i, j. Let N_1 be an integer such that $N_1 \delta_i$ is an integer for all i. Let $q_Z: Z \times G \to Z$, $q_{ij}: Z \times G \to G_i^j$ be the projections. Define a line bundle L on $Z \times G$ by

$$L = (q_z)^* L^{N_1 \delta_0} \otimes (\otimes_{i=1}^{r-1} \otimes_{j=1}^{J} (q_{ij})^* L_{ij}^{N_1 \delta_i}).$$

On $Y = Z \times G$ we take the linearlization of the $PGL(N)$-action given by L. Let $Y^{ss}, (Y^s)$ denote the set of semistable (stable) points of Y.

PROPOSITION 2.12

(a) $q \in \tilde{R}^{ss} \Rightarrow \tilde{T}(q) \in Y^{ss}$.
(b) $q \in \tilde{R}^s \Rightarrow \tilde{T}(q) \in Y^s$.
(c) $q \in \tilde{R}, \tilde{T}(q) \in Y, q \notin \tilde{R}^{ss} \Rightarrow \tilde{T}(q) \notin Y^{ss}$.
(d) $q \in \tilde{R}^{ss} - \tilde{R}^s \Rightarrow \tilde{T}(q) \notin Y^s$.

Proof. Similar to 3.12 [B3].

PROPOSITION 2.13

The morphism $\tilde{T}$ is a proper injective morphism.

Proof. Similar to 3.13 [B3] or 3 [B1].

Since $\tilde{T}$ is affine and a good quotient of Y^{ss} modulo $PGL(N)$ exists (well known), it follows that a good quotient M of $\tilde{R}^{ss}$ modulo $PGL(N)$ exists (Proposition 3.12 [N]). $\tilde{R}$ is a nonsingular projective variety of dimension $n^2(g-1)+N^2+\sum_j \dim G_j$, G_j being the flag variety of flags of type $\mathscr{F}^j, j=1,\ldots,J$. Hence M is a normal projective variety of dimension $n^2(g-1)+1+\sum_j \dim G_j$. Also, if $\tilde{R}^{ss}=\tilde{R}^s$ and $\mathrm{Aut}(E,\mathscr{F},\underline{\alpha})=k$ (scalars), then M is a geometric quotient and is nonsingular. We end the proof of existence of M with this remark.

PROPOSITION 2.14

Let h denote the canonical morphism from $\tilde{R}^{ss}$ onto the quotient M. Let $\mathscr{E}$ denote the universal family of GPBs on $\tilde{R}^{ss}\times X$. Then for $p, q\in\tilde{R}^{ss}$, one has $h(p)=h(q)$ if and only if $\mathrm{gr}(\mathscr{E}_p)\approx\mathrm{gr}(\mathscr{E}_q)$.

Proof. Similar to 3.15, [B3].

3. Interesting special cases of the moduli spaces of GPBs

3A *α-stability, α-semistability of QPBs*

Notation 3.1. In this section we study QPBs $(E,\mathscr{F}), \mathscr{F}=(\mathscr{F}^1,\ldots,\mathscr{F}^J)$ with $\mathscr{F}^j: F_0^j(E)\supset F_1^j(E)\supset 0$. Let $\dim F_1^j(E)=a_j, a=\sum_j a_j, j=1,\ldots,J$. If $J=1$, then we may often denote $(E,\mathscr{F})$ by $(E,F_1(E))$. The results of this section are also needed later for applications.

DEFINITION 3.2

Let α be a real number with $0\leqslant\alpha\leqslant 1$. A QPB$(E,\mathscr{F})$ with $\mathscr{F}^j: F_0^j(E)\supset F_1^j(E)\supset 0$ is called α-semistable (respectively α-stable) if for any proper subbundle F of E with induced quasiparabolic structure, one has

$$\frac{\deg F+\sum_j\alpha\dim F_1^j(F)}{\operatorname{rank} F}\leqslant(<)\frac{\deg E+\sum_j\alpha\dim F_1^j(E)}{\operatorname{rank} E}.$$

Note that for $0\leqslant\alpha<1$, α-semistability (or α-stability) is the same as semistability (or stability) of the GPB$(E,\mathscr{F},\underline{\alpha})$ with $\alpha^j=(0,\alpha)$ for all j.

PROPOSITION 3.3

Let $(E,\mathscr{F})$ be a QPB with $\mathscr{F}^j: F_0^j(E)\supset F_1^j(E)\supset 0$, $a=\sum_j\dim F_1^j(E)$.

(1) *Suppose that $1-1/[a(n-1)]<\alpha<1$. Then if $(E,\mathscr{F})$ is α-semistable, it is also 1-semistable. If $(E,\mathscr{F})$ is 1-stable, it is also α-stable.*
(2) *Suppose that rank and degree of E are coprime and a is an integral multiple of rank E. Then $(E,\mathscr{F})$ is 1-stable if and only if it is 1-semistable.*
(3) *If the conditions of* (1) *and* (2) *are satisfied then α-stability is equivalent to α-semistability and the moduli space M is nonsingular.*

Proof. (1) Let F be a proper subbundle of E of rank r with induced (quasi)parabolic structure. Define

$$B(F)=n\deg(F)-r\deg(E), A(F)=ra-n\textstyle\sum_j\dim F_1^j(F).$$

The condition for α-stability (α-semistability) can then be written as $B(F) < (\leqslant) \alpha A(F)$, for all subbundles F of E. Define $\delta = 1 - \alpha$. If $A(F) \geqslant 0$ then $B(F) \leqslant \alpha A(F) \Rightarrow B(F) \leqslant A(F)$ since $\alpha \leqslant 1$. If $A(F) < 0$, then $[B(F) \leqslant A(F) - \delta A(F)] \Rightarrow B(F) \leqslant A(F)$ if and only if $-\delta A(F) < 1$. Since $\sum \dim F_1^j(F) \leqslant a$, one has $-A(F) \leqslant a(n-r)$. Thus $-A(F) \leqslant a(n-1)$ for any F. Hence if $\delta < 1/a(n-1)$, for $0 < 1 - \delta = \alpha < 1$, α-semistability implies 1-semistability. Suppose now $(E, \mathscr{F})$ is 1-stable. For $A(F) \leqslant 0$, if $B(F) < A(F)$, then $B(F) < \alpha A(F)$ as $\alpha > 0$. For $A(F) > 0$, $B(F) < A(F) \Rightarrow B(F) < \alpha A(F) = A(F) - \delta A(F)$ if and only if $\delta A(F) < 1$. Since $\dim F_1^j(F) \geqslant 0, A(F) \leqslant ra \leqslant a(n-1)$ for all F. Thus for $1/\delta > a(n-1)$, $0 < \alpha = 1 - \delta < 1$, 1-stability implies α-stability.

(2) Proof as in the vector bundle case.

(3) The first assertion is clear from (1) and (2). The second assertion follows from Theorem 1.

Theorem 2. *Let $M(n,d)$ denote the moduli space of stable GPBs $(E, \mathscr{F}, \underline{\alpha})$ of rank n and degree d on a nonsingular curve X of genus g satisfying the following conditions.*

(1) $\mathscr{F} = (\mathscr{F}^1, \ldots, \mathscr{F}^J)$, $\mathscr{F}^j : F_0^j(E) \supset F_1^j(E) \supset 0$, $\dim F_1^j(E) = a_j$ *is a fixed integer depending on j*, $a = \sum_j \dim F_1^j(E)$.

(2) $\underline{\alpha} = (\underline{\alpha}^1, \ldots, \underline{\alpha}^J)$ *with* $\underline{\alpha}^j = (0, \alpha)$ *for all* j, $1 - 1/a(n-1) < \alpha < 1$.

(3) *The rank n and degree d are coprime and $a = \sum_j a_j$ is an integral multiple of n.*

Then $M(n,d)$ is a fine moduli space.

Proof. Similar to 3.16 [B3].

3B *Operations on QPBs*

In this section we assume that $D_j = x_j + z_j$, for all j and the QPBs are of the type defined in 3.1. We denote such a QPB by $(M, F_1^j(M))$.

3.4. *Direct sum and tensor product of two QPBs.* The direct sum of two QPBs $(M, F_1^j(M))$ and $(N, F_1^j(N))$ is the QPB $(M \oplus N, F_1^j(M) \oplus F_1^j(N))$. The tensor product of the two QPBs is the QPB$(M \otimes N, F_1^j(M \otimes N))$ where $F_1^j(M \otimes N)$ is the image of $F_1^j(M) \otimes F_1^j(N)$ under the projection map $(M_{x_j} \oplus M_{z_j}) \otimes (N_{x_j} \oplus N_{z_j}) \to (M \otimes N)_{x_j} \oplus (M \otimes N)_{z_j}$. We remark that if the projections from $F_1^j(M)$ (respectively $F_1^j(N)$) to M_{x_j} and M_{z_j} (resp. N_{x_j} and N_{z_j}) are isomorphisms then a similar statement is true for $F_1^j(M \otimes N)$.

3.5. Let T denote an operation on vector spaces such that for $V_1 \subset V_2$, $T(V_1) \subset T(V_2)$ and for any two vector spaces V_1, V_2 there is a map pr: $T(V_1 \oplus V_2) \to T(V_1) \oplus T(V_2)$. For example, $T(V) = \mathrm{End}(V)$, $\otimes^m(V)$, $S^m(V)$, $\Lambda^m(V)$ etc. In these examples $T(V_1) \oplus T(V_2)$ is a direct summand of $T(V_1 \oplus V_2)$ (characteristic zero), hence there is a canonical projection map pr. T which induces a corresponding operation on vector bundles which we again denote by T. We want to extend T to QPBs. Note that pr $T(F_1^j(E)) \subset T(E_{x_j}) \oplus T(E_{z_j})$. For a QPB $(E, F_1^j(E)$, define $T(E, F_1^j(E)) = (T(E), \mathrm{pr}(T(F_1^j(E)))$.

3.6. In particular when $T = \Lambda^n$, the top exterior product, then $T(E, F_1^j(E))$ is a rank 1 QBP called the determinant. Let $(E, \mathscr{F}, \underline{\alpha})$ be a GPB with $E, \mathscr{F}$ as in 3.1. Then we define the determinant of $(E, \mathscr{F}, \underline{\alpha})$, denoted by $\det(E, \mathscr{F}, \underline{\alpha})$, to be the rank one GPB (det $E, \mathscr{F}_0, \underline{\alpha})$.

PROPOSITION 3.7

Let p_j and q_j be the projections $F_1^j(E) \to E_{x_j}$ and $F_1^j(E) \to E_{z_j}$ respectively. Let M' be the subset of the moduli space $M(n,d)$ corresponding to GPBs satisfying the condition that at least one of p_j or q_j is an isomorphism for every $j = 1, \ldots, J$. Then $\det : M' \to M(1,d)$ defined by $(E, \mathscr{F}, \underline{\alpha}) \mapsto \det(E, \mathscr{F}, \underline{\alpha})$ is a morphism.

Proof. Let $(\mathscr{E}, \mathscr{F}, \underline{\alpha}) \to T \times X$ be a (flat) family of GPBs. It suffices to show that we can globalize the construction in 3.6 to $(\mathscr{E}, \mathscr{F}, \underline{\alpha})$ replacing $(E, \mathscr{F}, \underline{\alpha})$. By definition, $\mathscr{F}$ gives a rank n subbundle $F_1^j(\mathscr{E})$ of $\mathscr{E}_j = p_{T_*}(\mathscr{E} | T \times D_j)$ for all j. Let $\wedge^n \mathscr{E} = \mathscr{L}$. We have

$$\begin{aligned}(p_T)_*(\mathscr{L} | T \times D_j) &= (p_T)_*(\mathscr{L} | T \times x_j) \oplus (p_T)_*(\mathscr{L} | T \times z_j)\\ &= \wedge^n(p_{T_*}(\mathscr{E} | T \times x_j)) \oplus \wedge^n(P_{T_*}(\mathscr{E} | T \times z_j)).\end{aligned}$$

Let $p: \wedge^n \mathscr{E}_j \to \wedge^n(p_{T_*}(\mathscr{E} | T \times x_j)) \oplus \wedge^n(p_{T_*}(\mathscr{E} | T \times z_j))$ be the natural projection. The rank n bundle $F_1^j(\mathscr{E})$ determines a rank 1 subbundle F^j of $\wedge^n \mathscr{E}_j$. Then $p(F^j)$ is a rank 1 subbundle of $(p_T)_*(\mathscr{L} | T \times D_j)$ giving the parabolic structure over D_j on $\mathscr{L}$. Thus we get a family $(\mathscr{L}, \mathscr{F}_0, \underline{\alpha})$ of rank one GPBs on $T \times X$.

Notation 3.8. Fix a GPB$\bar{L} = (L, \mathscr{F}_0, \underline{\alpha})$ of rank 1 and degree d. Let $M'_{\bar{L}}$ denote the subset of M' consisting of GPBs $(E, \mathscr{F}, \underline{\alpha})$ such that det $(E, \mathscr{F}, \underline{\alpha}) = (L, \mathscr{F}_0, \underline{\alpha})$. Let $M_{\bar{L}}$ denote the closure of $M'_{\bar{L}}$ in $M(n,d)$. Notice that $M'_{\bar{L}}$ is closed in M'.

3.9. Let V_1, V_2 be two vector spaces of dimension n. Let G be the Grassmannian of n-dimensional subspaces of $V_1 \oplus V_2$. Let $e_1, \ldots, e_n$ be a basis of V_1 and let $e_{n+1}, \ldots, e_{2n}$ be a basis of V_2. G is embedded in $\mathbf{P}(\wedge^n(V_1 \oplus V_2))$ by the Plucker embedding. Let $\{P_{i_1, \ldots, i_n}\}$, $1 \leqslant i_1 < \cdots < i_n \leqslant 2n$ be the Plucker coordinates. Let H be the hyperplane defined by $a.P_{1\ldots n} - b.P_{n+1\ldots(2n)} = 0$ (a, b both nonzero and fixed). The following Lemma seems to be known.

Lemma 3.10. *$G \cap H$ is nonsingular.*

Remark 3.11. The result of the lemma does not hold if one of a or b is zero as can be seen by taking $n = 2$. In that case $G \cap H$ becomes a cone with base a nonsingular quadric in $\mathbf{P}^3$.

Theorem 3. *Suppose that $\bar{L} = (L, \mathscr{F}_0, \underline{\alpha})$ is such that $p_j(F_1^j(L)) \neq 0, q_j(F_1^j(L)) \neq 0$ for all j. Then one has the following.*

(1) *$M_{\bar{L}}$ is normal.*
(2) *If $(n,d) = 1, \underline{\alpha} = (0, \alpha)$ with $1 - 1/nJ(n-1) < \alpha < 1$, then $M_{\bar{L}}$ is nonsingular.*

Proof. Let $(b_j, a_j), b_j \in L_{x_j}, a_j \in L_{z_j}$ be a generator of $F_1^j(L), j = 1, \ldots, J$. By our assumption, a_j and b_j are both nonzero for all j. Consider a GPB$(E, \mathscr{F}, \underline{\alpha})$ (as in 3.1). Let $V_{1j}, = E_{xj}, V_{2j} = E_{zj}, G_j = \mathrm{Gr}(n, V_{1j} \oplus V_{2j}) \approx G$. We identify $F_1^j(E)$ with the element of

G_j determined by it. Then a semistable $(E, \mathscr{F}, \underline{\alpha})$ corresponds to an element of $M'_{\bar{L}}$ if and only if for all j, $F_1^j(E)$ belongs to the subset of G_j defined by

$$\{a_j P_{1\ldots n} - b_j P_{\{(n+1)\ldots(2n)\}} = 0,\ P_{(n+1)\ldots(2n)} \neq 0,\ P_{1\ldots n} \neq 0\}.$$

The closure of this set is the hyperplane section $G_j \cap H_j$ of G_j; where H_j is defined by $a_j P_{1\ldots n} - b_j P_{(n+1)\ldots(2n)} = 0$. Thus a semistable GPB$(E, \mathscr{F}, \underline{\alpha})$ corresponds to an element of $M_{\bar{L}}$ if and only if $F_1^j(E) \in G_j \cap H_j$ for all j. Let $R, \tilde{R}$ be the spaces defined in the proof of Theorem 1. $\tilde{R}$ is a bundle over R with fibres $\Pi_j G$, a J-fold product of G. Let R_L denote the subset of R corresponding to vector bundles E with fixed determinant L. R_L is known to be nonsingular. Let $\tilde{R}_{\bar{L}}$ be the fibre bundle over R_L with fibres $\Pi_j G \cap H_j$, which is a subbundle of $\tilde{R}|_{R_L}$. By Lemma 3.10, $\tilde{R}_{\bar{L}}$ is nonsingular. $M_{\bar{L}}$ is the quotient of $\tilde{R}_{\bar{L}}$ by $PGL(N)$ (in the sense of geometric invariant theory), hence $M_{\bar{L}}$ is normal. If the conditions of (2) are satisfied, then by Proposition 3.3 $M_{\bar{L}}$ is a geometric quotient and hence is nonsingular.

3C *Moduli spaces for rank* 2

3.12. Throughout this subsection, we assume that $J = 1, D = x + z, r(E) = 2, 0 < \alpha < 1$. Let (e_1, e_2) and (e_3, e_4) denote bases of E_x and E_z respctively, these will be chosen suitably in different cases. Let G_r denote the Grassmannian of 2-dimensional subspaces of $V = E_x \oplus E_z, G_r \subset \mathbf{P}(\wedge^2 V)$. Any element in $\wedge^2 V$ can be written in the form $X_1 e_1 \wedge e_2 + Y_1 e_3 \wedge e_4 + X_2 e_1 \wedge e_4 + Y_2 e_2 \wedge e_3 + X_3 e_3 \wedge e_1 + Y_3 e_2 \wedge e_4$. G_r is defined by $X_1 Y_1 + X_2 Y_2 + X_3 Y_3 = 0$. $F_1(E)$ defines a point in G_r. The subset of G_r corresponding to stable (resp. semistable) QPS $(E, F_1(E))$ will be denoted by G_r^s (resp. G_r^{ss}). Let H denote the hyperplane $hX_1 - Y_1 = 0, h \neq 0$.

Lemma 3.13. *Let the assumptions be as above.*

(i) *A QPB$(E, F_1(E))$ of degree* 1 *is α-stable* (= 1*-stable*) *for* $1/2 < \alpha < 1$ *if and only if one of the conditions* (a), (b) *is satisfied,* (a) *E is a stable vector bundle and* $F_1(E) \neq M_x \oplus M_z$ *for any line subbundle M of E of degree zero.* (b) *E has a subbundle* M_1 *of degree* 1 *with* $E/M_1 = M$, $\deg(M_1) = 1$, $\deg(M) = 0$ *and* $F_1(E) \cap ((M_1)_x \oplus (M_1)_z) = 0$, $F_1(E) \neq L_x \oplus L_z$ *for any line subbundle L of E isomorphic to M.*
(ii) *A QPB$(E, F_1(E))$ of degree zero is α-semistable for* $0 < \alpha < 1$ *if and only if E is a semistable vector bundle and for any line subbundle L of E of degree* 0, $F_1(E) \neq L_x \oplus L_z$. *Further, it is α-stable if and only if it satisfies the additional condition* $F_1(E) \cap (L_x \oplus L_z) = 0$ *for L as above.*

Proof. This follows from straightforward computations.

PROPOSITION 3.14

Assumptions as in 3.12. *Assume further that degree* $E = 1, g = 1, 1/2 < \alpha < 1$. (1) *The open subset of* $M_{\bar{L}}$ *corresponding to QPBs with underlying vector bundle E stable is isomorphic to* $G_r \cap H - X$. (2) *The closed subset of* $M_{\bar{L}}$ *corresponding to QPBs with E not stable is a fibration over X with fibres isomorphic to* $\mathbf{P}^1$.

Proof. (1) On the elliptic curve X there is a unique vector bundle E of rank 2, degree 1 with a fixed determinant. By Lemma 3.13(i)(a), $(G_r \cap H)^{ss} = G_r \cap H - \operatorname{Pic} X$. Since $X \approx \operatorname{Pic} X$, the result follows.

(2) By 3.13, $E = M_1 \oplus M$. Fix $M \in \mathrm{Pic}^0 X$, since det E is fixed, this fixes E too. Choose nonzero elements $e_1 \in (M_1)_x, e_2 \in M_x, e_3 \in (M_1)_z, e_4 \in M_z$. Any automorphism of E is of the form $f = \begin{bmatrix} \lambda & vs \\ 0 & \mu \end{bmatrix}$ with $\lambda \in k^* = \mathrm{Aut}\, M_1, \mu \in k^* = \mathrm{Aut}\, M, v \in k, s \in \Gamma(M^* \otimes M_1) - \{0\}$. There are two cases depending on zeroes of s.

Case (i). Assume $s(x)$ and $s(y)$ are both nonzero. By suitable choice of the basis elements, one can have $(G_r \cap H)^{ss} = \{Y_3 \neq 0\} - \{X_1 = 0 = X_2 - Y_2\}$ (by 3.13) and $f(e_1) = \lambda e_1, f(e_2) = \mu e_2 + v e_1, f(e_3) = \lambda e_3, f(e_4) = \mu e_4 + v e_3$. Then Aut (E) acts on $\mathbf{P}^5$ by

$$f(X_1, Y_1, X_2, Y_2, X_3, Y_3) = (\lambda\mu X_1, \lambda\mu Y_1, \lambda\mu X_2 + \mu v Y_3, \lambda\mu Y_2 + \mu v Y_3, \lambda^2 X_3 - \lambda v X_2 - \lambda v Y_2 - v^2 Y_3, \mu^2 Y_3).$$

For the normal subgroup G_a defined by $\lambda = \mu = 1$ acting on the cone $C(H) \approx k^5$ (coordinates $(X_1, X_2, Y_2, X_3, Y_3)$) the ring of invariants is generated by $X_1, X_2 - Y_2, Y_3, U = X_2 Y_2 + X_3 Y_3$. The affine cone $C(G_r \cap H)$ is given by $U = -hX_1^2$. Hence the quotient of $(G_r \cap H)^{ss}$ by G_a is $(\mathbf{A}^2 - 0) \subset \mathbf{P}^2$, it is given by the map $(X_1, X_2, Y_2, X_3, Y_3) \mapsto (X_1, X_2 - Y_2, Y_3)$. On this quotient the induced action of $G_m = P(\mathrm{Aut}\, E)/G_a$ is given by multiplication of coordinates by $1, 1, t$ respectively $(t = \mu\lambda^{-1})$. The quotient is $\mathbf{P}^1$, it is given by mapping to $(X_1, X_2 - Y_2)$.

Case (ii). Suppose $s(x) = 0, s(z) \neq 0$. By 3.13, $(G_r \cap H)^{ss} = \{Y_3 \neq 0\} - \{X_1 = 0 = X_2 = X_3\}$. As in case (ii), one has $f(e_1) = \lambda e_1, f(e_2) = \mu e_2, f(e_3) = \lambda e_3, f(e_4) = \mu e_4 + v e_3$. The action of Aut E on $\mathbf{P}^5$ is given by

$$f(X_1, Y_1, X_2, Y_2, X_3, Y_3) = (\lambda\mu X_1, \lambda\mu Y_1, \lambda\mu X_2 \lambda\mu Y_2 + \mu v Y_3, \lambda^2 X_3 - \lambda v X_2, \mu^2 Y_3).$$

The normal subgroup G_a acts on $H \approx k^5$ by

$$v(X_1, X_2, Y_2, X_3, Y_3) = (X_1, X_2, Y_2 + vY_3, X_3 - vX_2, Y_3).$$

The ring of invariants is generated by X_1, X_2, Y_3 and $U = X_2 Y_2 + X_3 Y_3, G_r \cap H$ is defined by $U = -hX_1^2$. The quotient of $(G_r \cap H)^{ss}$ by G_a is $(\mathbf{A}^2 - 0) \subset \mathbf{P}^2$, it is given by projection to (X_1, X_2, Y_3) coordinates, $\mu\lambda^{-1} = t \in G_m = P$ (Aut $E)/G_a$ acts on it by $t(X_1, X_2, Y_3) = (X_1, X_2, tY_3)$. The (required) quotient is $\mathbf{P}^1$ given by projection to coordinates (X_1, X_2).

Remark 3.15. The above calculations indicate that $M_{\bar{L}}$ is obtained by blowing up an elliptic curve (isomorphic to X) in a nonsingular quadric ($G_r \cap H$ above) in $\mathbf{P}^4$.

PROPOSITION 3.16

With the notations of 3.12, *assume that* $g = 1, 0 < \alpha < 1$ *and the determinant of* E *is trivial. Then* $M_{\bar{L}}$ *is a* $\mathbf{P}^2$*-bundle over* $\mathbf{P}^1$.

Proof. Lemma 3.13 (ii) implies that either (a) $E = M \oplus M^{-1}, M \in \mathrm{Pic}^0 X$ or (b) E comes in a nontrivial extension $0 \to M_1 \xrightarrow{g} E \xrightarrow{h} M_2 \to 0, M_1 = M_2 = M \in \mathrm{Pic}^0 X, M^2 = \mathcal{O}$. Up to isomorphism there are four vector bundles of type (b) corresponding to four roots of $\mathcal{O}$. The vector bundles of type (a) are parametrized by $(\mathrm{Pic}^0 X)/(\mathbf{Z}/2) \approx \mathbf{P}^1$.

(a) If $M = M^{-1}$, then $(E, F_1(E))$ is equivalent to a GPB with E of type (b). Therefore, we may assume that $M \neq M^{-1}$. Let e_1, e_2, e_3, e_4 be basis of $M_x, M_x^{-1}, M_z, M_z^{-1}$ respectively. Since any line subbundle of E of degree zero is either M or M^{-1}, by 3.13(ii) the only nonsemistable points in $G_r \cap H$ are $(0,0,0,0,1,0)$ and $(0,0,0,0,0,1)$. Stable points are given by $X_3 Y_3 \neq 0$. $P(\mathrm{Aut}\, E) = P(G_m \times G_m) = G_m . t \in G_m$ acts by $t(X_1, Y_1, X_2, Y_2, X_3, Y_3) = (X_1, Y_1, X_2, Y_2, tX_3, t^{-1} Y_3)$. Projections to (X_1, X_2, Y_2) coordinates show that the quotient of $G_r \cap H$ under G_m is $\mathbf{P}^2$. The semistable but nonstable GPBs correspond to the quadric $hX_1^2 = X_2 Y_2$ in $\mathbf{P}^2$.
(b) Let e_1, e_2, e_3, e_4 be the basis elements of $(M_1)_x, (M_2)_x, (M_1)_z, (M_2)_z$ respectively. Any automorphism of E is of the form $\lambda Id + \mu g \circ h . \mathbf{P}(\mathrm{Aut}\, E) \approx G_a$. Taking $\mu\lambda^{-1} = t \in G_a$, it acts by $te_1 = e_1, te_3 = e_3, te_2 = e_2 + te_1, te_4 = e_4 + te_3$. Hence one has $t(X_1, Y_1, X_2, Y_2, X_3, Y_3) = (X_1, Y_1, Y_2 + tY_3, X_2 + tY_3, X_3 - t(X_2 + Y_2) - t^2 Y_3, Y_3)$. The ring of invariants for G_a-action on k^6 is generated by $X_1, Y_1, U_1 = X_2 - Y_2, Y_3$ and $U_2 = X_2 Y_2 + X_3 Y_3$. Hence the quotient of the cone $C(G_r \cap H)^{ss}$ is the affine quadric $hX_1^2 = U_2$ in $(k^3 - \{0\}) \times k$, where the latter k has U_2 as coordinate, while coordinates in k^3 are X_1, Y_3, U_1. Note that the nonsemistable points for G_a are $\{X_1 = Y_3 = U_1 = 0\}$. The quotient of $C(G_r \cap H)^{ss}$ by scalar multiplication is $\mathbf{P}^2$, given by projection to (X_1, Y_3, U_1) coordinates. The nonstable GPBs correspond to $Y_3 = 0$ in this $\mathbf{P}^2$. It is not difficult to see that in case $E = M \oplus M, M^2 = \mathcal{O}$, there are no stable GPBs and the semistable GPBs give a $\mathbf{P}^1$ (in the moduli space $M_{\bar{L}}$), which is the same as $\{Y_3 = 0\}$ in $\mathbf{P}^2$ above.

These calculations show that there is a surjective map $h: M_{\bar{L}} \to \mathbf{P}^1$ with fibres $\mathbf{P}^2$ and over $\mathbf{P}^1 - \{4 \text{ points}\}$ this fibration is locally trivial. The result now follows from Tsen's theorem (p. 108, case (d), [M].

4. Applications to curves with nodes and ordinary cusps

4A *Preliminaries*

Let Y be an integral projective curve over an algebraically closed field k. Let $\pi: X \to Y$ be the normalization map. Let (A, m) be the local ring at a singular point y of Y. We assume that Y has only nodes and ordinary cusps as singularities.

PROPOSITION 4.1

Let F be a torsionfree A-module of rank n. Then $F \approx rA \oplus (n-r)m$ (rA denotes the direct sum of r copies of A).

Proof. We assume that y is an ordinary cusp, the nodal case being proved in Proposition 2, Part 8 [S]. By Corollary 6.2 [B], every indecomposable torsion-free A-module is isomorphic to an ideal. Any ideal is isomorphic either to A or m [1.4[D]]. The result follows by induction on rank.

PROPOSITION 4.2

Let A be the local ring at a node. Let m_1, m_2 be the two maximum ideals of the semilocal ring $\bar{A}$, $k_i = \bar{A}/m_i, i = 1, 2$; $k_1 = k_2 = k$. Let $p: \bar{A} \to k_1 \oplus k_2$ be the canonical surjection, $q = \oplus_n p, n > 0$. Let V be a subspace of $k_1^n \oplus k_2^n$ of dimension n. Let $p_i: V \to k_i^n$ be the projection, a_i = dimension of the kernel of p_i. Then $F = q^{-1}(V) \approx (n - a_1 - a_2) A \oplus (a_1 + a_2) m$.

Proof. We assert that there is an automorphism of $\bar{A}^n$ such that the induced automorphism h_1 of $(\bar{A}/m)^n$ maps V onto the subspace $V_1 = k_2^{a_1} \oplus D^{n-a_1-a_2} \oplus k_1^{a_2} \subset (k_1 \oplus k_2)^{a_1} \oplus (k_1 \oplus k_2)^{n-a_1-a_2} \oplus (k_1 \oplus k_2)^{a_2}$, D is the diagonal of $k_1 \oplus k_2$. Since $p^{-1}D = A, p^{-1}k_i \approx m$, one has $p^{-1}(V_1) \approx (n-a_1-a_2)A \oplus (a_1+a_2)m$, hence the result.

We now prove our assertion. Let K_i, I_i denote respectively the kernel and image of $p_i, i=1,2$. Write $V = K_1 \oplus W \oplus K_2$. Let $e_1,\ldots,e_{a_1}$ (resp. $e_{n-a_2+1},\ldots,e_n$) be a basis of K_1 (resp. K_2). Let $e_{a_1+1},\ldots,e_{n-a_2}$ be a basis of W. For $e_i \in W$, write $e_i = e_{i,1} + e_{i,2}, e_{i,j}$ being the component in $k_j^n, j=1,2$. Complete the basis $e_1,\ldots,e_{a_1}, e_{a_1+1,2},\ldots,e_{n-a_2,2}$ of I_2 to a basis $\{e_{i,2}\}, i=1,\ldots,n$ of k_2^n where $e_{i,2} = e_i, i \leqslant a_1$. Similarly choose a basis $\{e_{i,1}\}$ of k_1^n extending a basis of I_1 with $e_{i,1} = e_i, i = n-a_2+1,\ldots,n$. Let $b_i = e_{i,1} + e_{i,2}$ for all i. Since q is a surjection there exist $M_{j,i}$ in $\bar{A}, i, j = 1,\ldots,n$, such that $q(M_{1,i},\ldots,M_{n,i}) = b_i$ for all i. The matrix $M = (M_{ji})$ is the matrix of an endomorphism f of $\bar{A}^n$ which induces an endomorphism h_1 of $(\bar{A}/m)^n$. h_1 maps the canonical basis of $(\bar{A}/m)^n$ to the basis $\{b_i\}$. M modulo m is the matrix of the base change. Hence the determinant of M modulo m is a unit and therefore det M is a unit. Thus f is an automorphism, so is h_1. With respect to the basis $\{b_i\}$, $h_1(V_1) = V$.

4.3. We now assume that A is the local ring at an ordinary cusp y. Then $(\bar{A}, m_x)$ is a local ring. One has $A \cap m_x = m = m_x^2, m^{-1} = \bar{A}$. There is a canonical A-splitting of the exact sequence $0 \to m_x/m \to \bar{A}/m \to \bar{A}/m_x \to 0$ as follows. $\bar{A}$ is a k-algebra, we consider $\bar{A}/m_x = k$ embedded in $\bar{A}$. If $f(x)$ denotes the element of k determined by f, then $f - f(x) \in m_x$. This induces a map $s: \bar{A}/m \to m_x/m$. It is A-linear (as $m_x \cap A = m$) but not $\bar{A}$-linear. Using the splitting given by s we write $\bar{A}/m = k_1 \oplus k_2, k_1 = \bar{A}/m_x, k_2 = m_x/m$, $k_1 \approx k_2 \approx k$.

Lemma 4.4. *Let $p: \bar{A} \to k_1 \oplus k_2$ be the canonical map. Let V be a one dimensional subspace of $k_1 \oplus k_2$, $p_i: V \to k_i$ projections, $F = p^{-1}(V)$. Then one has*

(1) *If $V = k_2$ then $F = m_x$.*
(2) *If p_1 is nonzero then $F \approx A$.*

Proof. It is easy to check (1) and that if $V = k_1$, $F = A$. For (2) it suffices to show that there is a unit $b \in \bar{A}$ such that multiplication by b induces a linear automorphism h of $k_1 \oplus k_2$ with $h(V) = k_1$. Let t be a uniformizing parameter in $\bar{A}$, $m_x = t\bar{A}$, $m = t^2\bar{A}$. For $f \in \bar{A}$, $f = f_0 + f_1 t$ mod m, $f_0, f_1 \in k$ i.e. $p(f) = (f_0, f_1) \in k_1 \oplus k_2$. Then $h(f_0, f_1) = p(bf) = (b_0 f_0, f_0 b_1 + f_1 b_0)$. Choose b with $b_0 = 1, b_1 = -v_1/v_0$ where (v_0, v_1) is a generator of V.

PROPOSITION 4.5

With A as above let V, q, p_i, F be as in 4.2. Let a be the rank of p_1. Then $F \approx aA \oplus (n-a)m$.

Proof. As in Proposition 4.2 we can find $f, h_1, V_1 = h_1(V)$ (use M mod m_x is a unit). Thus we may assume that $V = V_1$. Consider the automorphism of $\bar{A}^n$ whose matrix is a diagonal matrix with first a_1 (diagonal) entries 1, next $n - a_1 - a_2$ entries $b \in \bar{A}$ and the last a_2 entries 1. Choose b with $p(b) = (1, -1) \in \bar{A}/m$. It follows from the proof of Lemma 4.4 that the induced automorphism h_2 of $(\bar{A}/m)^n$ is identity on the first a_1 and last a_2 factors and maps each D in the middle $n - a_1 - a_2$ factors onto k_1. Thus $h_2(V_1) = k_2^{a_1} \oplus k_1^{n-a_1} = k_2^{n-a} \oplus k_1^a$, hence $F \approx q^{-1}(V_2) = (n-a)m \oplus aA$.

4B *Relation between torsionfree sheaves and QPBs*

Notation 4.6. Let $y_1,\ldots,y_J$ be the singular points of Y. Define divisors $D_j=\pi^{-1}(y_j)$, $D_j=x_j+z_j$ if y_j is a node, $D_j=2x_j$ if y_j is an ordinary cusp. Let Q denote the set of isomorphism classes of $QPBs(E,\mathscr{F})$ of rank n, degree d on X with $\mathscr{F}^j:F_0^j(E)\supset F_1^j(E)\supset 0$, dim $F_1^j(E)=n$ for all j. If y_j is a node let p_j,q_j be the projections from $F_1^j(E)$ to E_{x_j},E_{z_j} respectively and a_j,b_j be the dimensions of their kernels. If y_j is a cusp let p_j,a_j be defined as in 4.5. For $\bar r=(r_1,\ldots,r_J)$, $0\leqslant r_j\leqslant n$, define $Q_{\bar r}=\{(E,\mathscr{F})|a_j+b_j=n-r_j$ if y_j is a node and $a_j=n-r_j$ if y_j is a cusp$\}$. Let (A_j,m_j) be the local ring at y_j. Let S denote the set of isomorphism classes of torsion-free sheaves of rank n and degree d on Y. Let $S_{\bar r}=\{F\in S|$ stalk $F_{y_j}\approx r_jA_j\oplus(n-r_j)m_j\}$. Then S (resp. Q) is a disjoint union of $S_{\bar r}$ (resp. $Q_{\bar r}$), $0\leqslant r_j\leqslant n$. Let $Q_n=Q_{(n,\ldots,n)}$, $S_n=S_{(n,\ldots,n)}$. The latter is the set of locally free sheaves in S.

PROPOSITION 4.7

There exists a map $f:Q\to S$ with the following properties.

(1) $f(Q_{\bar r})=S_{\bar r}$.
(2) $f|_{Q_n}:Q_n\to S_n$ *is a bijection.*
(3) $(E,\mathscr{F})$ is 1-stable (resp. 1-semistable) *if and only if its image F under f is a stable (resp. semistable) torsionfree sheaf.*

Proof. Let $(E,\mathscr{F})\in Q$. By 4.2, 4.3, $\pi_*(E)\otimes k(y_j)=(k_1\oplus k_2)^n=H^0(E|_{D_j})$. Then $f(E,\mathscr{F})=F$ is defined by the exact sequence $0\to F\to\pi_*(E)\to\oplus_j(\pi_*E\otimes k(y_j))/F_1^j\ (E)\to 0$. Since $\chi(\mathcal{O}_Y)=\chi(\mathcal{O}_X)-J$, $\deg\cdot F=\chi(F)-\text{rank}\ F\cdot\chi(\mathcal{O}_Y)$, $\deg.E=\chi(E)-n\chi\ (\mathcal{O}_X)$, F and E have the same degree.

(1) In view of Propositions 4.2, 4.5 we have only to show that for $F\in Q_{\bar r}$, there is $(E,\mathscr{F})$ mapping to F under f. Let $E_0=\pi^*\ F/\text{torsion}$. E is given by an extension $0\to E_0\to E\to\oplus_jT_j\to 0$, where $T_j=k(x_j)^{a_j}\oplus k(z_j)^{b_j}$, $a_j+b_j=n-r$ if y_j is a node and $T_j=k(x_j)^{n-r_j}$ if y_j is a cusp. The composite of sheaf inclusions $F\to\pi_*(E_0)\to\pi_*(E)$ induces a linear map $F\otimes k(y_j)\to\pi_*E\otimes k(y_j)$. Define $F_1^j(E)$ to be the image of this linear map.
(2) The inverse f^{-1} of $f|_{S_n}$ is defined as follows. For $F\in S_n$, define $E=\pi^*F$, $F_1^j(E)=F\otimes k(y_j)\subset(F\otimes\pi_*\mathcal{O}_X)\otimes k(y_j)=(\pi_*(E)\otimes k(y_j))$. Since the above inclusion is induced by $\mathcal{O}_Y\subset\mathcal{O}_X$ and $\mathcal{O}_Y$ maps onto k_1 (resp. onto each of k_1,k_2) in $k_1\oplus k_2$ if y_j is a cusp (resp. a node), it follows that p_j (resp. each of p_j,q_j) is of maximum rank for all j.
(3) Similar to 4.2 [B3].

Theorem 4. *Let $M=M(n,d)$ denote the moduli space of semistable GPBs of rank n, degree d on X satisfying 4.6 and weights $(0,\alpha)$, $1-1/[nJ(n-1)]<\alpha<1$. Let $U=U(n,d)$ denote the moduli space of semistable torsionfree sheaves of rank n, degree d on Y.*

(1) *The map f (see 4.7) induces a morphism $f:M\to U$.*
(2) $f|_{(M_n)^s}:(M_n)^s\to(U_n)^s$ *is an isomorphism, where the superscript s denotes stable points. In particular f is birational.*
(3) *f is surjective.*
(4) *If $(n,d)=1$, then $M(n,d)$ is a desingularization of $U(n,d)$.*

Proof. Note first that semistable GPBs are also 1-semistable QPBs and hence map to semistable torsionfree sheaves under f. (1) and (2) now follow since the above constructions globalize easily to families of bundles. (4) follows from Proposition 3.3. (3) We show that the image of f contains the set U^s of stable points in U. If F is a stable torsionfree sheaf on Y sheaf then by 4.7 there exists a 1-stable QPB and hence a stable GPB mapping to F. Since Y has only planar singularities, U is irreducible [R] and U^s is an open dense subset. Since f is proper if follows that f is a surjection.

Theorem 5. *Assume that Y has only J nodes as singularities. Let L be a fixed line bundle on Y. Let U_n^L denote the closed subset of U_n corresponding to vector bundles with fixed determinant L. Let U_L be the closure of U_n^L in U. Let $\bar{L} = f_1^{-1}(L)$, f_1 being the map f in* case $n = 1$. *Let $M_{\bar{L}}$ be as in Theorem 3. Then f induces a birational surjective morphism $M_{\bar{L}} \to U_L$. If $(n, d) = 1$, then M_L is a desingularization of U_L.*

Proof. First note that $\det \circ f = f \circ \det$ where the latter f is the map f in case $n = 1$. Hence $f(M'_{\underline{L}}) = U_n^L$, the subset corresponding to vector bundles with fixed determinant L. Since f is proper it follows that $f(M_{\underline{L}}) = U_L$. The rest of the assertions follow from Theorems 3 and 4.

Remark 4.8. *Relation with singular intersection of quadrics.* Assume that $g = 1$, $J = 1$, Y is a hyperelliptic curve with Weierstrass points w_0, w_1, w_2, w_3, w_4; w_0 being the unique node of Y. The desingularization X is an elliptic curve. Let L be a line bundle on Y of degree 5. The linear system $|L|$ gives an embedding of Y in $\mathbf{P}^3$. The linear system $|\pi^* L|$ gives an embedding of X in $\mathbf{P}^4$. The inclusion $H^0(L) \to H^0(\pi_* \pi^* L) = H^0(\pi^* L)$ induces a projection from $\mathbf{P}^4$ to $\mathbf{P}^3$ mapping X onto Y (isomorphically outside w_0). There exists a Cartier divisor W_0 of degree two supported at w_0. Let $W = W_0 + \sum_{i>0} w_i$. On $\mathbf{P}^5 = \mathbf{P}((L \otimes \mathcal{O}_W)^*)$ there is a singular pencil of quadrics of the form $Q_1 = X_1 X_2 + X_3^2 + \; X_4^2 + X_5^2 + X_6^2$, $Q_2 = X_1^2 + 2a_1 X_1 X_2 + a_3 X_3^2 + a_4 X_4^2 + a_5 X_5^2 + a_6 X_6^2$, a_i being distinct scalars [B6]. $Q = Q_1 \cap Q_2$ is a 3-fold with a unique singular point q. Q is a normalization of $U_L(2, 1)$ and is bijective with it (Main theorem, [NE]). The blow up of Q at q is isomorphic to the blow up of a nonsingular quadric Q_0 in $\mathbf{P}^4$ along X. Q_0 is the base of the unique quadric cone in the singular pencil with vertex q and $X = Q_0 \cap \{X_1 = X_2 = 0\}$. The latter is isomorphic to $M_{\underline{L}}$, $\underline{L}$ being the generalized parabolic line bundle on X corresponding to L on Y (Remark 3.15). The injective evaluation map $H^0(L) \to L \otimes \mathcal{O}_W$ induces a projection to $\mathbf{P}^3$ mapping Q to a surface containing Y. The space of maximum isotropic spaces of one system for Q_1 is isomorphic to $\mathbf{P}^3$. There is a bijective morphism from this space to $U_{\mathcal{O}}(2, 0)$ which is an isomorphism outside the singular set (*a* $\mathbf{P}^1$) corresponding to nonlocally free sheaves, while the singular set is isomorphic to $U_{\mathcal{O}}^X(2, 0)$. $M_{\underline{L}}(2, 0)$ is a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$ (see 3.16). The latter is isomorphic to the blow up of $\mathbf{P}^3$ along a line.

Generalizations of these results to hyperelliptic curves of higher genera are possible [B6].

4.9 *Variation of* α. Let $M(\alpha; n, d)$ (respectively $M_{\underline{L}}(\alpha; n, d)$) denote the moduli spaces $M(n, d)$ (resp. $M_{\underline{L}}(n, d)$) for weights $(0, \alpha)$.

(A) $\alpha = 0$. In this case the semistability, stability of a QPB is the same as that of the underlying vector bundle. Hence $M(0; n, d) \approx U_X(n, d) \times \mathrm{Gr}(n, 2n)$.

(B) $n = 2$, $d = 1, 0 < \alpha \leqslant 1$.

(1) Let $0<\alpha<1/2$. Then α-semistability is equivalent to the stability of the underlying bundle and hence there is a surjective morphism from $M(\alpha;2,1)\to U_X(2,1)$. α-stability coincides with α-semistability but it does not imply 1-stability. Consequently there is only a rational map $f{:}M(\alpha;2,1)\to U_Y(2,1)$. $M(\alpha;2,1)$. is nonsingular.
(2) $\alpha=\frac{1}{2}$. The α-stability is equivalent to the stability of the underlying bundle. But α-semistability does not imply stability of the underlying bundle, so h is only a rational map. It is defined on α-stable bundles and is surjective. α-semistability does not imply 1-semistability, but 1-stability implies α-semistability. Hence f is also a rational surjective map. As α-semistability does not imply α-stability, $M(\frac{1}{2};2,1)$ could be singular.
(3) Let $1/2<\alpha<1$. The undering bundle of a semistable QPB can be non-stable. Hence there is only a rational map $h{:}M(\alpha;2,1)\to U_X(2,1)$. However the morphism f is surjective and birational. $M(\alpha;2,1)$ is nonsingular (Theorem 4).
(4) Let $\alpha=1$. Then $M(1;2,1)$ is not nonsingular since it includes sheaves which have torsion over D_j, s [RN]. The maps f, h have the same properties as in case (2). Under the assumptions of 3.8, one can see that $M_L(1;2,1)$ is a blow up (possibly a double blow up) of $\mathbf{P}^3$ along Y, where Y is embedded in $\mathbf{P}^3$ by the linear system $|L|$, $\deg L=5$. $M_L(\alpha;2,1)$ for case (3) has been described in 4.8.

(C) $n=2$, $d=0$, $0<\alpha\leqslant 1$.

(1) Let $0<\alpha<1$. Then α-stability implies α-semistability, the converse is not true. Also α-semistability implies 1-semistability. Hence there is a morphism $f{:}M(\alpha;2,0)\to U_Y(2,0)$, it is surjective and birational. Since the underlying bundle is semistable, there is a surjective morphism $h{:}M(\alpha;2,0)\to U_X(2,0)$.
(2) Let $\alpha=1$. Then the morphism f is as in (1) above, but h is only a rational map, the underlying bundle of a 1-semistable QPB may not be semistable.

5. Generalized parabolic orthogonal bundles

5.1. Let the base field be algebraically closed and of characteristic different from 2. For simplicity of exposition, we assume that Y has a single node y_0 as its only singularity. Let $\pi^{-1}(y_0)=x_1+x_2$. For a vector bundle E, we denote the rank and degree of E by $r(E)$ and $d(E)$ respectively. We indentify an orthogonal bundle with a pair (E,q) where E is a vector bundle and q a nondegenerate quadratic form on E (with values in the trivial line bundle $\mathcal{O}$). For a closed point x, let q_x denote the induced quadratic form on the fibre E_x. Let $q_1=\frac{1}{2}(q_{x_1}\oplus q_{x_2})$, $q_2=q_{x_1}\oplus(-q_{x_2})$.

Lemma 5.2. *Let $\sigma{:}E_{x_1}\approx E_{x_2}$ be a quadratic isomorphism (i.e. preserving q_{x_1}, q_{x_2}). Then the following holds.*
(a) The graph Γ_σ of σ is isotropic for q_2.
(b) $p_i{:}\Gamma_\sigma\to E_{x_i}$, $(i=1,2)$ is an isotropy for q_1 on Γ_σ and q_{x_i} on E_{x_i}(i.e. $q_{x_i}(p_i(v))=q_i(v)$ for $v\in\Gamma_\sigma$). In particular, q_1 is nondegenerate on Γ_σ.

Proof. Easy.

Remark 5.3. Γ_σ is in fact a maximum isotropic space for q. The space S of maximum isotropic spaces for a nondegenerate quadratic form Q on a vector space of dimension $2n$ has two components, each being a smooth variety of dimension $n(n-1)/2=\dim$

$O(n)$. The above lemma means that S may be regarded as a compactification of $O(n)$. This suggests the following.

DEFINITION 5.4

A generalized quasiparabolic orthogonal bundle (orthogonal QPB in short) on X is an orthogonal bundle (E, q) of rank n together with an n-dimensional vector subspace $F_1(E)$ of $E_{x_1} \oplus E_{x_2}$ which is isotropic for $q_2 = q_{x_1} \oplus (-q_{x_2})$.

For a subbundle N of E define $F_1(N) = F_1(E) \cap (N_{x_1} \oplus N_{x_2})$ and $f_1(N) = \dim F_1(N)$.

DEFINITION 5.5

Let α be a real number, $\alpha \in [0, 1]$. An orthogonal QPB $(E, F_1(E), q)$ is α-stable (resp. α-semistable) if for every isotropic proper subbundle N of E, one has $(d(N) + \alpha f_1(N))/r(N) <$ (resp. $\leqslant$) α.

Remark 5.6. If one considers special orthogonal QPBs, one should take the underlying bundle an $SO(n)$-bundle and $F_1(E)$ belonging to the unique component of S (see 5.3) which contains the graphs of isomorphisms $\sigma: E_{x_1} \to E_{x_2}$ which preserve the $SO(n)$-structure modulo a maximum parabolic subgroup. Recall that an $SO(n)$-bundle can be identified with a vector bundle E of rank n with a nondegenerate quadratic form q and a given trivialization of $\Lambda^n E$ whose square is the trivialization of $(\Lambda^n E)^{\otimes 2}$ given by $\Lambda^n q$.

Example 5.7. Generalized parabolic $SO(2)$-bundles on X are in bijective correspondence with generalized parabolic line bundles on X (recall that the latter give a desingularization of the compactified Jacobian of Y).

Proof. Since $SO(2)_k = k^*$, every $SO(2)$-bundle is of the form $E = L \oplus L^{-1}$, $L \in \mathrm{Pic}^0(X)$ with the natural quadratic map $q = L \oplus L^{-1} \to L \otimes L^{-1} = \mathcal{O}$. L, L^{-1} are isotropic subbundles. Each system S of lines in the quadric q_2 in $\mathbf{P}^3 = \mathbf{P}(E_{x_1} \oplus E_{x_2})$ is isomorphic to $\mathbf{P}^1$. We can choose nonzero elements $e_1 \in L_{x_1}$, $e_2 \in L_{x_2}$, $f_1 \in L_{x_1}^{-1}$, $f_2 \in L_{x_2}^{-1}$ such that $B_{x_i}(e_i, f_i) = 1$, $i = 1, 2$; where B_{x_i} is the bilinear form associated to q_{x_i}. With respect to (ordered) bases (e_1, f_1) of E_{x_1} and (e_2, f_2) of E_{x_2}, any isomorphism $\sigma: F_{x1} \to F_{x2}$ preserving the $SO(2)$-structures is of the form $\sigma(e_1) = ae_2, \sigma(f_1) = a^{-1} f_2$, $a \in k^*$. So $\Gamma_\sigma =$ span of $\{e_1 + ae_2, f_1 + a^{-1} f_2\}$. Thus

$$(E, F_1(E)) = (L, F_1(L)) \oplus (L^{-1}, F_1(L^{-1})),$$

where $F_1(L)$ is spanned by $\lambda e_1 + \mu e_2$ and $F_1(L^{-1})$ by $\mu f_1 + \lambda f_2$, $(\lambda, \mu) \in \mathbf{P}^1$.

DEFINITION 5.8

An orthogonal sheaf on Y is a pair (F, q_F) where F is a torsionfree sheaf on Y and Q_F is a nondegenerate quadratic form on F with values in $\mathcal{O}_Y$. An orthogonal sheaf (F, q_F) is semistable (resp. stable) if for every nonzero proper (totally) isotropic subsheaf N of F, $d(N)/r(N) \leqslant$ (resp. $<$) 0.

DEFINITION 5.9

An isomorphism of orthogonal sheaves is a sheaf isomorphism which preserves the quadratic forms. In case of orthogonal QPBs we also demand (in addition) that the quasiparabolic structures $F_1(E)$ should be preserved.

PROPOSITION 5.10

(a) *There is a map f from the set of isomorphism classes of orthogonal QPBs on X to the set of isomorphism classes of orthogonal sheaves on Y. Let $f(E, F_1(E), q) = (F, q_F)$.*
(b) *If p_i, $i = 1, 2$ are both isomorphisms, then (F, q_F) is an orthogonal bundle. f gives a bijection between orthogonal QPBs on X with p_1, p_2 isomorphisms and orthogonal bundles on Y.*
(c) *$(E, F_1(E), q)$ is 1-semistable (resp. 1-stable) if and only if (F, q_F) is semistable (resp. stable).*

Proof. (a) To a QPB $(E, F_1(E))$ one can associate a torsionfree sheaf F given by the exact sequence $0 \to F \to \pi_* E \to \pi_* E \otimes k(y_0)/\pi_* F_1(E) \to 0$. The quadratic form q on E induces one on $\pi_* E$ and on F. The quadratic form q_F on F is nondegenerate outside y_0 and *a priori* has values in $\pi_* \mathcal{O}_X$. Consider $q_{x_1} \oplus q_{x_2}$ as a form on $E_{x_1} \oplus E_{x_2}$ with values $k(x_1) \oplus k(x_2)$. Since $F_1(E)$ is isotropic for q_2, one sees that $q_{x_1} \oplus q_{x_2}$ maps $F_1(E)$ into $k(y_0)$ contained diagonally in $k(x_1) \oplus k(x_2)$. This means that the form q_F on F has values in $\mathcal{O}_Y \subset \pi \mathcal{O}_X$ and (F, q_F) is an orthogonal sheaf (5.8).
(b) F is locally free if and only if p_i are both isomorphisms. Moreover, $E = \pi^* F$ and hence gets a nondegenerate quadratic form with values in $\mathcal{O}_X$. Since the correspondence $(E, F_1(E)) \mapsto F$, is bijective for F locally free (4.7) the result follows.
(c) This can be checked similarly in 4.2 [B3]. One has only to notice that a subsheaf is totally isotropic if and only if it is generically totally isotropic.

PROPOSITION 5.11

An orthogonal QPB is α-semistable if and only if the underlying QPB is so.

Proof. We only have to check that if $(E, F_1(E), q)$ is α-semistable, then $(E, F_1(E))$ is α-semistable. Let F be a subbundle of E. We may assume F is nonisotropic. By the proof of Proposition 4.2 [RS] we have an exact sequence $0 \to N \to F \oplus F^\perp \to N^\perp \to 0$ where N is the isotropic subbundle generated by $F \cap F^\perp$, $\perp$ denoting orthogonal complement. Also, $d(F) = d(F^\perp) = d(N)$.
Case (i) When $N = 0$. Then $E = F \oplus F^\perp$, $d(F) = 0$. Since $q|\, F$ is nondegenerate so is $q_2 = q_{x_1} \oplus (-q_{x_2})$. Since $F_1(F) \subset F_1(E)$ is isotropic for q_2, $f_1(F) \leqslant r(F)$. Thus $(d(F) + \alpha f_1(F))/r \leqslant \alpha$.
Case (ii) When $N \neq 0$. We need to show $d(F) + \alpha(f_1(F) - r(F)) \leqslant 0$. Since N is isotropic, orthogonal α-semistability implies $d(N) + \alpha(f_1(N) - r(N)) \leqslant 0$. Since $d(F) = d(N)$, it suffices to check that $(*) f_1(F) - r(F) \leqslant f_1(N) - r(N)$. Now F/N is a vector bundle of degree 0 with induced nondegenerate quadratic form $\bar{q}$. The image of $F_1(F)$ in $(F/N)_{x_1} \oplus (F/N)_{x_2}$ is isomorphic to $F_1(F)/F_1(N)$ and is an isotropic subspace for $\bar{q}_{x_1} \oplus (-\bar{q}_{x_2})$. Hence $\dim F_1(F)/F_1(N) \leqslant r(F/N)$, i.e., $f_1(F) - r(F) \leqslant f_1(N) - r(N)$. This finishes the proof.

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