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Moduli for principal bundles over algebraic curves: II

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Abstract. We classify principal bundles on a compact Riemann surface. A moduli space for semistable principal bundles with a reductive structure group is constructed using Mumford's geometric invariant theory.

Keywords. Principal bundles; compact Riemann surface; geometric invariant theory; reductive algebraic groups.

4. Reduction to a quotient space problem

In this section we reduce the problem of constructing coarse moduli schemes for the functors $\mathbf{F}^{\tau}_{ss}$ to one of proving the existence of a good quotient of certain universal spaces under the action of a full linear group.

We recall the definition of a good quotient ([22], Definition 1.5, p. 516).

4.1. DEFINITION

Let $\alpha: H \times T \to T$ be an action of the algebraic group H on the scheme T. A morphism $p: T \to Y$ is called a *good quotient* of T modulo H if the conditions (i), (ii) and (iii) below are satisfied.

i) p is surjective, affine and H-invariant.
ii) $p_*(O^H_T) = O_Y$, where O^H_T is the sheaf of H-invariant functions on T.
iii) If Z is a closed H-stable subset of T then $p(Z)$ is closed in Y; further if Z_1, Z_2 are two closed H-stable subsets of T such that $Z_1 \cap Z_2 = \phi$, then $p(Z_1) \cap p(Z_2) = \phi$.

If in addition the condition (iv) below is also satisfied we call $p: T \to Y$ a *geometric quotient.*

iv) $p(x_1) = p(x_2) \Leftrightarrow$ orbit of x_1 = orbit of x_2 (or equivalently, in view of (iii), all orbits are closed).

4.2. *Remark.* A good quotient is a *categorical quotient*, i.e. given any H-invariant morphism $f: T \to Z$ there is a unique morphism $\tilde{f}: Y \to Z$ such that $f = \tilde{f} \circ p$ ([[22] p. 516).

4.3. *Notation.* Let $\alpha: H \times T \to T$ be an action of the algebraic group H on the scheme T. Then for morphisms $h: S \to H$ and $t: S \to T$ we denote by $h[t]$ the composite $S \xrightarrow{h \times t} H \times T \xrightarrow{\alpha} T$. For any morphism $f: S_1 \to S_2$ we denote by $\bar{f}$ the product $f \times id_X: S_1 \times X \to S_2 \times X$.

This is the second and concluding part of the thesis of late Professor A Ramanathan; the first part was published in the previous issue.

–Editor

If E is a bundle (or more generally a scheme) over a scheme M and $m:S\rightarrow M$ is a morphism we denote by E_m the pull back m^*E. For $E'\rightarrow M\times X$ we write E'_m instead of $E'_{\bar{m}}$.

4.4. DEFINITION

Let S be a set of isomorphism classes of G-bundles on X. Let $\tilde{\mathbf{F}}^S$ be the sheaf associated to the functor $\mathbf{F}^S$: (Sch)$\rightarrow$(Sets) which associates to a scheme T the set of isomorphism classes of families of G-bundles in S parametrized by T. On morphisms $\mathbf{F}^S$ is defined to be pulling back. Let M be a scheme and H an algebraic group acting on M by $\alpha:H\times M\rightarrow M$. Let $\mathbf{H}\backslash\mathbf{M}$ be the sheaf associated to the presheaf $H\backslash M(T)=$ the quotient set $\mathrm{Hom}(T,H)\backslash\mathrm{Hom}(T,M)$. We call M a *universal space with group H for the set S* if there is an isomorphism of sheaves Φ: $\tilde{\mathbf{F}}^S\rightarrow\mathbf{H}\backslash\mathbf{M}$.

4.5. PROPOSITION

Let S^τ be the set of isomorphism classes of semistable G-bundles of topological type τ. Suppose there is a universal space M with group H for the set S^τ and Φ: $\tilde{\mathbf{F}}^{S^\tau}\rightarrow\mathbf{H}\backslash\mathbf{M}$ is the isomorphism of sheaves. Then a good quotient of M modulo H, if it exists, gives a coarse moduli scheme for the functor $\bar{\mathbf{F}}^\tau_{ss}$ (see Definitions 3.2 and 3.9 : in Part I) in a natural way.

Proof. Suppose $\pi:M\rightarrow Y$ is a good quotient of M modulo H.

Clearly $\mathbf{F}^{S^\tau}=\mathbf{F}^\tau_{ss}$ (see Definition 3.1). Therefore we have a morphism Φ: $\mathbf{F}^\tau_{ss}\rightarrow\mathbf{H}\backslash\mathbf{M}$. Let h_M, h_Y be the functors represented by M, Y respectively. The morphism $h_\pi:h_M\rightarrow h_Y$ induced by $\pi:M\rightarrow Y$ gives rise to a morphism Ψ: $\mathbf{H}\backslash\mathbf{M}\rightarrow h_Y$ because of the H-invariance of π. We claim that the morphism $\eta=\Psi\circ\Phi:\mathbf{F}^\tau_{ss}\rightarrow h_Y$ goes down to a morphism $\bar{\eta}:\bar{\mathbf{F}}^\tau_{ss}\rightarrow h_Y$ making Y the coarse moduli scheme for $\bar{\mathbf{F}}^\tau_{ss}$.

Suppose the family $\mathscr{F}\rightarrow S\times X$ in S^τ has an admissible reduction of structure group to $P=M\cdot U$. Then by Proposition 3.5 we have a family $\mathscr{F}'\rightarrow(\mathbb{C}\times S)\times X$ in S^τ such that $p_S^*(\mathscr{F})|_{\mathbb{C}^*\times S\times X}\approx\mathscr{F}'|_{\mathbb{C}^*\times S\times X'}$, where p_S: $\mathbb{C}\times S\times X\rightarrow S\times X$ is the projection and $\mathscr{F}'_0\rightarrow S\times X$ is isomorphic to $\mathscr{F}[P,M](G)\rightarrow S\times X$. Therefore $\eta_S(\mathscr{F}')$: $\mathbb{C}\times S\rightarrow Y$ coincides with $\eta_S(p_S^*(\mathscr{F}))$ on $\mathbb{C}^*\times S$ and hence on the whole of $\mathbb{C}\times S$. In particular $\eta_S(\mathscr{F})=\eta_S(\mathscr{F}'_0)$: $S\rightarrow Y$. It follows that η goes down to a morphism $\bar{\eta}$: $\bar{\mathbf{F}}^\tau_{ss}\rightarrow h_Y$.

That $\bar{\eta}:\bar{\mathbf{F}}^\tau_{ss}(\mathrm{Spec}\,\mathbb{C})\rightarrow\mathrm{Hom}(\mathrm{Spec}\,\mathbb{C},Y)$ is surjective follows from the fact that $\pi:M\rightarrow Y$ is surjective. To check injectivity we only have to show that if E_1 and E_2 are two semistable G-bundles of type τ on X (considered as a family parametrized by $\mathrm{Spec}\,\mathbb{C}$) such that $\eta(E_1)=\eta(E_2)$ then E_1 and E_2 are equivalent. Let the point $m_i\in M$ represent $\Phi_{\mathbb{C}}(E_i)$. Then by the property (iii) in the definition of a good quotient (Definition 4.1), $\bar{C}_1$ and $\bar{C}_2$, the closures of the H-orbits C_1 and C_2 of m_1 and m_2 respectively, intersect in M. Let $m\in\bar{C}_1\cap\bar{C}_2$. We take the canonical reduced scheme structures on C_i and $\bar{C}_i$, $i=1,2$. Let $[id_M]\in\mathbf{H}\backslash\mathbf{M}(M)$ be the class of the identity morphism of M. The element $\Phi_M^{-1}([id_M])\in\tilde{\mathbf{F}}^\tau_{ss}(M)$ then gives for some neighbourhood U of $m\in M$ a faithfully flat morphism $f:U'\rightarrow U$ of schemes and a family of G-bundles $\mathscr{F}\rightarrow U'\times X$ in S^τ. Let $C'_i=f^{-1}(C_i)$ and $\bar{C}'_i=f^{-1}(\bar{C}_i)$. Since f is faithfully flat, C'_i is dense open in $\bar{C}'_i$. Since Φ is a morphism it follows easily that for the family $\mathscr{F}|\bar{C}'_i\rightarrow\bar{C}'_i\times X$, $\mathscr{F}_x\approx E_i\;\forall x\in C_i$. Therefore by Proposition 3·24(i) $\mathscr{F}_{m'}$ is equivalent to $E_i\;\forall m'\in\bar{C}'_i$, and in particular for $m'\in\bar{C}'_i$ such that $f(m')=m$. This proves that E_1 and E_2 are equivalent.

To verify the condition (ii) for coarse moduli scheme (Definition 3.2) suppose Z is a scheme and $\chi:\bar{\mathscr{F}}^{\tau}_{ss}\to h_Z$ a morphism. Then it is easy to see that corresponding to $\Phi_M^{-1}([id_M])$ the morphism χ gives a morphism $g:M\to Z$ which is H-invariant. Then Y being the categorical quotient of M modulo H(Remark 3.2), g induces $\tilde{g}: Y\to Z$ such that $g=\tilde{g}\circ\pi$. If $h_g:h_Y\to h_Z$ is the corresponding morphism of functors it follows that $h_{\tilde{g}}$ is the unique morphism which satisfies $\chi = h_{\tilde{g}}\circ\tilde{\eta}$.

This proposition reduces the problem of constructing coarse moduli schemes for $\bar{\mathbf{F}}^{\tau}_{ss}$ to one of constructing suitable universal spaces and then proving the existence of quotients. To construct universal spaces for G-bundles we will start with the universal spaces for vector bundles provided by the Quot schemes ([19], § 6). These spaces have a stronger universal property (which we have formulated as a definition; see Definition 4·6 below) which is essential for our construction. By taking an embedding of G in some $GL(n,\mathbb{C})$ we will consider a G-bundle as a vector bundle (or $GL(n,\mathbb{C})$-bundle) with a reduction of structure group to G and thus will construct universal spaces for G-bundles as schemes over the universal spaces for $GL(n,\mathbb{C})$-bundles.

4.6. DEFINITION

Let $\mathscr{S}$ be a set of isomorphism classes of G-bundles on X. Let $\mathscr{E}\to T\times X$ be a family of G-bundles in $\mathscr{S}$. Suppose an algebraic group H acts on T by $\alpha:H\times T\to T$ and also on $\mathscr{E}$ as a group of G-bundle isomorphisms compatible with α, we have the commutative diagram

$$\begin{array}{ccc} H\times\mathscr{E} & \xrightarrow{\ \tilde{\alpha}\ } & \tilde{\alpha}^*(\mathscr{E}) \\ & \searrow \quad \swarrow & \\ & H\times T\times X & \end{array}$$

where $\tilde{\alpha}=\alpha\times id_X$ (cf. 4.3). We call $\mathscr{E}\to T\times X$ a *universal family with group H for the set $\mathscr{S}$* if the following conditions hold.

i) Given any family of G-bundles $\mathscr{F}\to S\times X$ in $\mathscr{S}$ and a point $s_0\in S$ there exists an open neighbourhood U of s_0 in S and a morphism $t:U\to T$ such that $\mathscr{F}|_{U\times X}\approx\mathscr{E}_t$ (cf. § 4.3 for notation).

ii) Given two morphisms $t_1,t_2:S\to T$ and an isomorphism $\varphi:\mathscr{E}_{t_1}\approx\mathscr{E}_{t_2}$ there exists a *unique* morphism $h:S\to H$ such that $t_2=h[t_1]$ and $\varphi=(h\times t_1)^*$ (α) (noting that $(h\times t_1)^*(H\times\mathscr{E})=\mathscr{E}_{t_1}$ and, since $t_2=h[t_1]$, $(h\times t_1)^*(\tilde{\alpha}^*\mathscr{E})=\mathscr{E}_{t_2}$).

4.7. *Remark.* The condition (ii) in particular implies that the isotropy group H_x at $x\in T$ is precisely the automorphism group of $\mathscr{E}_x$.

4.7.1. *Remark.* If $\mathscr{E}\to T\times X$ is a universal family with group H for $\mathscr{S}$ it is clear that T is a universal space with group H for $\mathscr{S}$.

4.8. Let A, A' be two algebraic groups and $\rho:A'\to A$ a homomorphism. Let $\mathscr{E}\to T\times X$ be a family of A bundles. Let $\Gamma(\rho,\mathscr{E})$:(Sch/T)$\to$(Sets) be the functor defined by

$\Gamma(\rho, \mathscr{E})(t:S \to T)$ = the set of isomorphism classes of pairs $(\mathscr{E}', \varphi)$ where $\mathscr{E}' \to S \times X$ is an A' bundle and φ: $\rho_* \mathscr{E}' \to \mathscr{E}_t$ is an isomorphism of A bundles. The pair $(\mathscr{E}'_1, \varphi_1)$ is isomorphic to the pair $(\mathscr{E}'_2, \varphi_2)$ if there is an A' bundle isomorphism $\psi: \mathscr{E}'_1 \overset{\approx}{\to} \mathscr{E}'_2$ such that the diagram

$$\begin{array}{ccc} \rho_* \mathscr{E}'_1 & \xrightarrow{\rho_* \psi} & \rho_* \mathscr{E}'_2 \\ & {\scriptstyle \varphi_1} \searrow \quad \swarrow {\scriptstyle \varphi_2} & \\ & \mathscr{E}_t & \end{array}$$

commutes. Note that if ρ is injective such a ψ, if it exists, is unique for, since A' acts faithfully on A $\rho_* \psi = \varphi_2^{-1} \circ \varphi_1$ uniquely determines ψ. On morphisms $\Gamma(\rho, \mathscr{E})$ is defined as pulling back.

Let τ be a topological A' bundle on X. Let $\Gamma^\tau(\rho, \mathscr{E})$ be the subfunctor of $\Gamma(\rho, \mathscr{E})$ defined by

$$\Gamma^\tau(\rho, \mathscr{E})(S) = \left\{ (\mathscr{E}', \varphi) \in \Gamma(\rho, \mathscr{E})(S) \;\middle|\; \begin{array}{l} \mathscr{E}_S \text{ is topologically} \\ \text{isomorphic to } \tau \,\forall s \in S. \end{array} \right\}$$

We then have the following lemma (cf. [17], Proposition 9, § 3·5, p. 18).

4.8.1. *Lemma. If $\rho: A' \to A$ is injective the functor $\Gamma(\rho, \mathscr{E})$ is representable by a T-scheme $T'' \to T$ of locally finite type and a universal pair $(\mathscr{U}, u) \in \Gamma(\rho, \mathscr{E})(T'')$. The functor $\Gamma^\tau(\rho, \mathscr{E})$ is representable by an algebraic subscheme T' of T'' and the restriction of $(\mathscr{U}, u)$ to T'.*

Proof. Since ρ is injective we identify A' with its image in A. Let $\Gamma = \Gamma(\rho, \mathscr{E})$.

Let Γ':(Sch/T)$\to$(Sets) be the functor such that $\Gamma'(f:S \to T) = \mathrm{Hom}_{S \times X}(S \times X, f^*(\mathscr{E}/A')) = \mathrm{Hom}_{T \times X}(S \times X, \mathscr{E}/A')$. We define a morphism of functors $\Phi: \Gamma' \to \Gamma$ as follows.

Let $\sigma \in \Gamma'(S) = \mathrm{Hom}_{S \times X}(S \times X, \mathscr{E}_f/A')$. Define $\Phi_S(\sigma) = (\sigma^* \mathscr{E}_f, \varphi_\sigma)$ where $\varphi_\sigma: \rho_* \sigma^* \mathscr{E}_f \to \mathscr{E}_f$ is induced by $((S \times X)_{\mathscr{E}_{f/A'}} \times \mathscr{E}_f) \times A \to \mathscr{E}_f$, $(s, x, e, a) \mapsto e.a$ where $s \in S$, $x \in X$, $e \in \mathscr{E}_f$ and $a \in A$.

We can also define an inverse morphism $\Psi: \Gamma \to \Gamma'$. Let $(\mathscr{E}', \varphi) \in \Gamma(f:S \to T)$. Then the fiber bundle associated to $\rho_* \mathscr{E}'$ with fiber A/A' is canonically isomorphic to the fiber bundle associated to $\mathscr{E}$ with fiber A/A'. Since A' leaves the coset (A') of A/A' invariant we have a canonical section σ of $(\rho_* \mathscr{E}')/A'$. Using φ this gives a section, again denoted by σ, of $\mathscr{E}_f/A'$. Define $\Psi_S((\mathscr{E}', \varphi)) = \sigma$.

It is easy to check that $\Phi \circ \Psi = id_\Gamma$ and $\Psi \circ \Phi = id_{\Gamma'}$. Thus the functors Γ and Γ' are isomorphic. We shall show that the functor Γ' is representable using the results of ([TDTE, IV]).

By Chevalley's semi-invariants theorem ([2], Theorem 5.1, p. 161) there is a representation of A on a vector space V with a line $l \subset V$ such that A' is the stabilizer of l in A. Let χ^{-1} be the character by which A' acts on l. Then the line bundle L on A/A' associated to the A'-bundle $A \to A/A'$ is the ample line bundle corresponding to the embedding of A/A' in $\mathbb{P}(V)$. Then the line bundle $\mathscr{L}$ on $\mathscr{E}/A'$ associated to the A'-bundle $\mathscr{E} \to \mathscr{E}/A'$ by the character χ is relatively ample for the morphism $\mathscr{E}/A' \to T \times X$, for it corresponds to the embedding $\mathscr{E}/A' \subsetneq \mathbb{P}(\mathscr{E}(V))$ induced by $A/A' \subsetneq \mathbb{P}(V)$. Therefore, we see that $\mathscr{E}/A'$ is quasi-projective over $T \times X$ and hence $\mathscr{E}/A' \to T$ is also quasi-projective. Therefore it follows from ([TDTE, IV] §4, C. pp. 19–20) that Γ' is representable by a scheme T'' $(= \Pi_{T\times X/T}(\mathscr{E}/A')/T \times X$, in the notation of ([TDTE, II], C. n° 2, pp. 12, 13)) of locally finite type. In fact T'' is an open subscheme of $\mathrm{Hilb}_{(\mathscr{E}/A')/T}$ whose closed points correspond to subschemes of $\mathscr{E}/A'$ which map isomorphically onto $t \times X$, for some $t \times T$, under the projection $\mathscr{E}/A' \to T \times X$ (*loc. cit.*). Therefore if a section $\sigma : t \times X \to \mathscr{E}/A'$ in $\Gamma'(t \subsetneq T)$ is such that the A'-bundle $\sigma^*(\mathscr{E}_t)$ on $t \times X \simeq X$ is of topological type τ then the Hilbert polynomial of the subscheme $\sigma(X) \simeq X$ of $\mathscr{E}_t/A'$ corresponding to the section σ (with respect to the ample line bundle $\mathscr{L}_t$ on $\mathscr{E}_t/A'$) is determined by τ since the restriction of $\mathscr{L}_t$ to $\sigma(X) \simeq X$ is topologically isomorphic to $\chi_*(\tau)$ so that its degree depends only on τ. Since subschemes with a fixed Hilbert polynomial are represented by an algebraic subscheme of Hilb ([TDTE, IV] pp. 17, 20) it follows that Γ^τ is represented by an algebraic subscheme T' of T''.

4.8.2. *Remark.* If A' and A are reductive groups then A/A' is affine and we can take a representation of A in V such that the character χ is trivial, so that A/A' is embedded in V itself. In this case, therefore, it follows that T'' is itself already algebraic.

4.8.3. *Remark.* If P is a parabolic subgroup of G and $\mathscr{E} \to Y$ is a G-bundle on a scheme Y then $\mathscr{E}/P \to Y$ is a projective morphism. Since G/P is projective, this follows as in the proof of the lemma above (taking $A = G$ and $A' = P$).

4.9 If ρ is not an injection the functor $\Gamma(\rho, \mathscr{E})$ may not be a sheaf. Let $\tilde{\Gamma}(\rho, \mathscr{E})$ be the sheaf associated to the functor $\Gamma(\rho, \mathscr{E})$. The following lemma shows that we can construct a universal *space* for A'-bundles starting from a universal *family* $\mathscr{E}$ of A-bundles when $\tilde{\Gamma}(\rho, \mathscr{E})$ is representable and if $\Gamma(\rho, \mathscr{E})$ itself is representable, then we can actually construct a universal *family* for A'-bundles. So taking an embedding $G \subsetneq GL(n, \mathbb{C})$ and starting with a universal family for vector bundles we can construct a universal family for G-bundles. But then to prove the existence of coarse moduli scheme for $\tilde{\mathbf{F}}^\tau_{ss}$ we have to prove the existence of a good quotient of the parameter scheme. For this it is convenient to take the adjoint representation. The existence of a good quotient reduces to proving that a certain morphism is proper and if we take the adjoint representation this follows from the (local) rigidity of the Lie algebra structure of a semisimple Lie algebra (see §5 below). But the adjoint representation is not faithful and hence we construct universal families in two steps, first from vector bundles to G/Z-bundles and then from G/Z-bundles to G-bundles. This involves the representability of the functor $\Gamma(\rho, \mathscr{E})$ where (essentially) ρ is the projection $G \to G/Z$. But this functor is not a sheaf (e.g. $\mathbb{C}^* \to 1 = \mathbb{C}^*/\mathbb{C}^*$, cf. ([TDTE, V, §1])) and we are forced to take the associated sheaf $\tilde{\Gamma}(\rho, \mathscr{E})$ which we can prove to be representable by identifying it with a suitable Picard functor (Lemma 4.15.1) If $\tilde{\Gamma}(\rho, \mathscr{E})$ alone is representable we can construct only a universal *space* for G-bundles even starting from a universal *family* for G/Z-bundles. But by Proposition 4.5 this is enough to prove the existence of a coarse moduli scheme for $\tilde{\mathbf{F}}^\tau_{ss}$.

4.10. *Lemma. Suppose the family $\mathscr{E} \to T \times X$ is a universal family with group H for a set $\mathscr{S}$ of A-bundles. Also suppose that the sheaf $\tilde{\Gamma}(\rho, \mathscr{E})$ is representable by a scheme M.*

(i) *The group H can be made to act on M in a natural way and M with this action of H then becomes a universal space with group H for the set $\mathscr{S}'$ of A'-bundles which give A-bundles in $\mathscr{S}$ on extending the structure group by $\rho: A' \to A$.*

(ii) *Moreover if $\rho: A' \to A$ is an injection so that $\Gamma(\rho, \mathscr{E})$ itself is representable and we have a universal pair $(\mathscr{U}, u) \in \Gamma(\rho, \mathscr{E})(M)$* (Lemma 4.8.1.) *the group H can be made to act in a natural way, on $\mathscr{U}$ as a group of A'-bundle isomorphisms compatible with its action on M* ((i) *above). With this action of H, $\mathscr{U}$ then becomes a universal family with group H for $\mathscr{S}'$.*

Proof. (i) To give the action of H on M we describe the action of $\mathrm{Hom}(S, H)$ on $\mathrm{Hom}(S, M)$ for any scheme S. Let $h \in \mathrm{Hom}(S, H)$ and $m \in \mathrm{Hom}(S, M)$. Let $\pi: M \to T$ be the structural morphism and $t = \pi \circ m$. Since M represents the sheaf $\tilde{\Gamma}(\rho, \mathscr{E})$ corresponding to the morphism m we have an open covering $\{U_i\}$ of S, faithfully flat morphisms $f_i: U_i' \to U_i$, A'-bundles $\mathscr{E}_i' \to U_i' \times X$ and A-bundles isomorphisms $\varphi_i: \rho_* \mathscr{E}_i' \to \overline{(t \circ f_i)}^* \mathscr{E}$. We define $h[m]$ to be the morphism from S to M corresponding to the element $(\mathscr{E}_i', \tilde{\alpha}_{h \times t} \circ \varphi_i)$ in $\tilde{\Gamma}(\rho, \mathscr{E})(S)$ where $\tilde{\alpha}_{h \times t} = \overline{(h \times t)}^*(\tilde{\alpha})$ and $\tilde{\alpha}: H \times \mathscr{E} \to \tilde{\alpha}^* \mathscr{E}$ gives the action of H on $\mathscr{E}$ (Definition 4.6). Then it is easy to see that we have indeed an action of H on M and that $h[t] = \pi \circ h[m]$. To prove that M is a universal space let $\mathscr{E}' \to S \times X \in \mathbf{F}^{\mathscr{S}'}(S)$ i.e. a family of A'-bundles in $\mathscr{S}'$. By extending the structure group by ρ we get a family of A-bundles $\rho_* \mathscr{E}'$ in $\mathscr{S}$. Since T is universal this gives an open covering $\{U_i\}$ of S and morphisms $f_i: U_i \to T$ such that $\mathscr{E}_{f_i} \stackrel{\approx}{\to} \rho_* \mathscr{E}'|_{U_i \times X}$. This then gives morphisms $f_i': U_i \to M$. Using condition (ii) of Definition 4.6 satisfied by $\mathscr{E} \to T \times X$ these f_i' are seen to define an element of $\mathbf{H} \backslash \mathbf{M}(S)$. It is easy to check that by associating this element of $\mathbf{H} \backslash \mathbf{M}(S)$ to $\mathscr{E}' \in \mathbf{F}^{\mathscr{S}'}(S)$ we have an isomorphism of sheaves $\tilde{\mathbf{F}}^{\mathscr{S}'} \to \mathbf{H} \backslash \mathbf{M}$ (see proof of (ii) below, locally, in the faithfully flat topology, the arguments run on the same lines).

(ii) In this case we can define $h[m]$ as the morphisms from S to M corresponding to the pair $(\mathscr{U}_m, \tilde{\alpha}_{h \times t} \circ u_m)$. Therefore by definition the pair $(\mathscr{U}_m, \tilde{\alpha}_{h \times t} \circ u_m)$ is isomorphic to $(\mathscr{U}_m, u_{h[m]})$, and hence there is an isomorphism (which is unique since ρ is injective, cf. § 4.8) $\tilde{\beta}_{h \times t}: \mathscr{U}_m \to \mathscr{U}_{h[m]}$ making the diagram

$$\begin{array}{ccc} \rho_* \mathscr{U}_m & \xrightarrow{\rho_* \tilde{\beta}_{h \times m}} & \rho_* \mathscr{U}_{h[m]} \\ {\scriptstyle u_m} \downarrow & & \downarrow {\scriptstyle u_{h[m]}} \\ \mathscr{E}_t & \xrightarrow[\tilde{\alpha}_{h \times t}]{} & \mathscr{E}_{h[t]} \end{array}$$

commutative. Taking $S = H \times M$ and h, m to be the projections $p_H: H \times M \to H$, $p_M: H \times M \to M$ respectively, in the above we get the action $p_H[p_M] = \beta: H \times M \to M$ of H on M and $\tilde{\beta}_{p_H \times p_M} = \tilde{\beta}: H \times \mathscr{U} \to \tilde{\beta}^* \mathscr{U}$ which gives the action of H on $\mathscr{U}$.

The condition (i) of Definition 4.6 for $\mathscr{U}$ follows immediately from the universal properties of M and $\mathscr{E}$. To check condition (ii) of Definition 4.6, let $m_1, m_2: S \to M$ and $\varphi: \mathscr{U}_{m_1} \to \mathscr{U}_{m_2}$ an isomorphism be given. Let $t_1 = \pi \circ m_1$ and $t^2 = \pi \circ m^2$. Define φ' by the commutative diagram:

$$\begin{array}{ccc} p_* \mathscr{U}_{m_1} & \xrightarrow{p_*\varphi} & p_* \mathscr{U}_{m_2} \\ \Big\downarrow {\scriptstyle u_{m_1}} & & \Big\downarrow {\scriptstyle u_{m_2}} \\ \mathscr{E}_{t_1} & \xrightarrow[\varphi']{} & \mathscr{E}_{t_2} \end{array} \qquad (*)$$

By the universal property of $\mathscr{E}$ there is a unique morphism $h: S \to H$ such that $t_2 = h[t_1]$ and $\varphi' = \tilde{\alpha}_{h \times t_1}$. Now m_2 is defined by the pair $(\mathscr{U}_{m_2}, u_{m_2})$ and $h[m_1]$ by the pair $(\mathscr{U}_{m_1}, \tilde{\alpha}_{h\times t_1} \circ u_{m_1})$. But from the diagram (*) and the observation above, it follows that $\varphi: \mathscr{U}_{m_1} \to \mathscr{U}_{m_2}$ gives an isomorphism of the pairs $(\mathscr{U}_{m_1}, \tilde{\alpha}_{h\times t_1} \circ u_{m_1})$ and $(\mathscr{U}_{m_2}, u_{m_2})$. Therefore $m_2 = h[m_1]$, as required to be shown.

4.11. We shall now construct universal families for semistable vector bundles. This is essentially contained in ([19], § 6; [20], § 3). We have made it a little more explicit to suit our purposes.

Let $\mathscr{S}^{r,d}$ be the set of isomorphism classes of semistable vector bundles of rank r and degree d. Let L be an ample line bundle on X of degree d_0. Since $\mathscr{S}^{r,d}$ is bounded i.e. there is a family of vector bundles on X parametrized by an algebraic scheme in which every element of $\mathscr{S}^{r,d}$ occurs ([19], Proposition 3.2, p. 307), we can find an integer m_0 such that for any $m \geqslant m_0$ and every $V \in \mathscr{S}^{r',d}$, $r' \leqslant r$, $H^i(X, V \otimes L^m) = 0$ for $i > 0$ and $H^0(X, V \otimes L^m)$ generates $V \otimes L^m$. Then $H^0(X, V \otimes L^m)$ has same rank, say n, for all $V \in \mathscr{S}^{r,d}$.

4.11.1. Let $P \subset GL(n, \mathbb{C})$ be the parabolic subgroup defined as the stabilizer of the subspace $\mathbb{C}^{n-r} \subset \mathbb{C}^n$. The decomposition $\mathbb{C}^n = \mathbb{C}^{n-r} \oplus \mathbb{C}^r$ gives a homomorphism $P \to GL(r, \mathbb{C})$. Given a P-bundle on a scheme Y the representations $P \subsetneq GL(n, \mathbb{C})$ and $P \to GL(r, \mathbb{C})$ give rise to vector bundles V_1 and V_2 (of ranks n and r respectively) on Y and there is a surjective homomorphism $V_1 \to V_2$ induced by the P-equivariant projection $\mathbb{C}^n \to \mathbb{C}^r$. Conversely given two vector bundles V_1 and V_2 of rank n and r respectively and a surjective homomorphism $V_1 \to V_2 \to 0$ we can construct a P-bundle as follows. For any open set $U \subset Y$ and a faithfully flat morphism $f: U' \to U$ associate the set of all vector bundle isomorphisms φ_1, φ_2 making the diagram

$$\begin{array}{ccccc} U' \times \mathbb{C}^n & \longrightarrow & U' \times \mathbb{C}^r & \longrightarrow & 0 \\ \Big\downarrow {\scriptstyle \varphi_1} & & \Big\downarrow {\scriptstyle \varphi_2} & & \\ f^* V_1 & \longrightarrow & f^* V_2 & \longrightarrow & 0 \end{array}$$

commutative. Thus we get a sheaf $\mathscr{F}$ for the faithfully flat topology and the trivial group scheme $Y \times P$ over Y given by P acts on $\mathscr{F}$ by $(\varphi_1, \varphi_2)p = (\varphi_1 \circ p, \varphi_2 \circ p)$, $p \in P$. Then $\mathscr{F}$ is a principal homogeneous space under $Y \times P$ and by descent ([SGA, 3] exposé XXIV) it is representable by a P-bundle over Y. Thus we can consider a P-bundle as a surjective homomorphism $V_1 \to V_2 \to 0$ of vector bundles. Then an isomorphism of the P-bundles $V_1 \to V_2 \to 0$ and $V'_1 \to V'_2 \to 0$ is given by a commutative diagram

$$\begin{array}{ccccc} V_1 & \longrightarrow & V_2 & \longrightarrow & 0 \\ \downarrow \approx & & \downarrow \approx & & \\ V'_1 & \longrightarrow & V'_2 & \longrightarrow & 0 \end{array}$$

Since $GL(n,\mathbb{C}) \to GL(n,\mathbb{C})/P$ is locally trivial ([3], Theorem 4.13, p. 90) any P-bundle is in fact locally trivial ([17], Theorem 2, §4.3, p. 1–24).

4.11.2. The trivial bundle $I_n = X \times \mathbb{C}^n \to X$ thought of as a family of vector bundles on X parametrized by a point, with the group $GL(n,\mathbb{C})$ acting in the natural way on I_n gives a universal family with group $GL(n,\mathbb{C})$ for the singleton set $\{I_n\}$. (Condition (ii) of Definition 4.6 is obviously satisfied and condition (i) follows from ([11], Lecture 7, p. 51, (ii) and (iii)).) So applying Lemma 4.8.1 with $A' = P \subset GL(n,\mathbb{C}) = A$ and fixing the topological type of the P-bundle such that we get by the extension of structure group $P \to GL(r,\mathbb{C})$ vector bundles of degree d (since the topological type of a vector bundle on X is determined by its degree and rank and any extension splits topologically this condition fixes the topological type of the P-bundle; cf. 4.11.1), we get a universal family of P-bundles $\mathcal{U} \to M \times X$ with group $GL(n,\mathbb{C})$ for the set of P-bundles of the fixed topological type which give I_n on extending the structure group by $P \to GL(n,\mathbb{C})$, parametrized by an algebraic scheme M. Let $\mathcal{O}$ be the $GL(r,\mathbb{C})$-bundle obtained from $\mathcal{U}$ by the extension of structure group $P \to GL(r,\mathbb{C})$.

4.11.3. By our constructions in §4.8 it follows that M is an open subscheme of the locally finite type scheme $\mathrm{Hom}(X, G_{n,r})$ which represents the functor Γ', $\Gamma'(S) = \mathrm{Hom}(S \times X, G_{n,r})$ where $G_{n,r} = GL(n,\mathbb{C})/P$ is the Grassmannian of r-dimensional quotients of $\mathbb{C}^n$. Then $\mathcal{U}$ is the pull back of the P-bundle $GL(n,\mathbb{C}) \to GL(n,\mathbb{C})/P$ by the universal section in $\mathrm{Hom}(M \times X, G_{n,r})$ and $\mathcal{O}$ is the pull back of the $GL(r,\mathbb{C})$-bundle* $Q \to G_{n,r}$ obtained from $GL(n,\mathbb{C}) \to G_{n,r}$ by the extension of structure group $P \to GL(r,\mathbb{C})$. The P-bundle $\mathcal{U}$ corresponds to the surjection $I_n \to \mathcal{O} \to 0$ which is the pull back of the surjection $G_{n,r} \times \mathbb{C}^n \to Q \to 0$ induced by $\mathbb{C}^n \to \mathbb{C}^r$. (Note that $GL(n,\mathbb{C}) \to G_{n,r}$ becomes trivial when we extend the structure group by $P \subsetneq GL(n,\mathbb{C})$.)

4.11.4. Let

$$R = \left\{ q \in M \;\middle|\; \begin{array}{l} \mathcal{O}_q \text{ is semistable and the canonical map} \\ I_n \to H^0(X, \mathcal{O}_q) \text{ is an isomorphism} \end{array} \right\}.$$

It follows from the semi-continuity theorem and the fact that the points corresponding to semistable bundles form an open subset of the parameter scheme in any family of vector bundles ([19], Corollary 7.2, p. 332) that R is an open subset of M. We take R with the open subscheme structure induced from M. Clearly R is stable under the action of $GL(n,\mathbb{C})$. We denote the restriction of $\mathcal{O}$ to $R \times X$ also by $\mathcal{O}$. Let $\mathcal{O}(-m) = \mathcal{O} \otimes p_X^*(L^{-m})$ where $p_X : R \times X \to X$ is the projection. Let $GL(n,\mathbb{C})$ act on $\mathcal{O}(-m)$ by its action on $\mathcal{O}$ and the trivial action on $p_X^*(L^{-m})$.

*Editor's Note: For the definition of Q see Introduction, in part I. It is the principal $GL(r)$ bundle associated to the universal quotient bundle on $G_{n,r}$.

4.11.5. PROPOSITION

The family $\mathcal{O}(-m) \to R \times X$ with this action of $GL(n, \mathbb{C})$ gives a universal family with group $GL(n, \mathbb{C})$ for the set $\mathcal{S}^{r,d}$ of semistable vector bundles of rank r and degree d.

Proof. Let $\mathcal{F} \to S \times X$ be a family in $\mathcal{S}^{r,d}$ and $s_0 \in S$. The direct image $p_{S_*}(\mathcal{F}(m))$ is locally free ([11], Lecture 7, p. 51). Choose a trivialization $P_{S_*}(\mathcal{F}(m))|_U \approx U \times \mathbb{C}^n$ in a neighbourhood U of s_0. Then we have a surjection $p_S^*(U \times \mathbb{C}^n) = I_n \to \mathcal{F}(m)|_{U \times X} \to 0$. This gives a reduction of structure group of the trivial $GL(n, \mathbb{C})$ bundle I_n to P (cf. 4.11.1). Then by the universal property of the P-bundle $\mathcal{U}$ on $R \times X$ we get a morphism $f: U' \to R$, $s_0 \in U' \subset U$, inducing the P-bundle $I^n \to \mathcal{F}(m)|_{U'} \to 0$. Extending the structure group by $P \to GL(r, \mathbb{C})$ and tensoring by L^{-m} we get $\bar{f}(\mathcal{O}(-m)) \approx \mathcal{F}|_{U' \times X}$.

To check condition (ii) of Definition 4.6, let $r_1, r_2: S \to R$ and $\varphi: \mathcal{O}_{r_1}(-m) \to \mathcal{O}_{r_2}(-m)$ be given. Let $\varphi': \mathcal{O}_{r_1} \to \mathcal{O}_{r_2}$ be the isomorphism induced by φ.

Then φ' induces an isomorphism $\varphi'': p_{S_*}(\mathcal{O}_{r_1}) \xrightarrow{\approx} p_{S_*}(\mathcal{O}_{r_2})$. But by commutativity under base change (since $H^1(X, \mathcal{O}_q) = 0 \forall q \in R$([11], Lecture 7, p. 51)), $p_{S_*}(\mathcal{O}_{r_i})$ is canonically isomorphic to $r_i^*(p_{R_*}\mathcal{O})$, $i = 1, 2$. Since $p_{R_*}\mathcal{O}$ is canonically isomorphic to the trivial bundle we have the commutative diagram

$$\begin{array}{ccccc} r^*{}_1(I_n) & \longrightarrow & \mathcal{O}_{r_1} & \longrightarrow & 0 \\ \downarrow \varphi'' & & \downarrow \varphi' & & \\ r^*{}_2(I_n) & \longrightarrow & \mathcal{O}_{r_2} & \longrightarrow & 0 \end{array}$$

Now use the universal property for the P-bundle $\mathcal{U}$(cf, 4.11.1).

4.11.6. *Remark.* If $\mathcal{U}$ is any universal family of vector bundles with group H for a set $\mathcal{S}$ of vector bundles then $\mathcal{U} \otimes L_1$ is a universal family with group $H(H$ acting trivially on $L_1)$ for the set $\mathcal{S} \otimes L_1 = \{V \otimes L_1 | V \in \mathcal{S}\}$, L_1 being any line bundle. Therefore it follows from Proposition 4.11.5 that $\mathcal{O}(q)$ is a universal family for $\mathcal{S}^{r, d + rd_0(m+q)}$ the set of isomorphism classes of semistable vector bundles of rank r and degree $d + rd_0(m+q)$, $q \in \mathbb{Z}$.

4.11.7. *Remark.* It is easy to see that the scheme R above is the same as scheme R^{ss} which Seshadri constructs in ([20], § 3; [19], § 6).

4.12. PROPOSITION

The set of isomorphism classes of semistable G-bundles of a fixed topological type τ is bounded, i.e. there exists a family $\mathcal{E} \to M \times X$ of G-bundles such that given any semistable G-bundle E of type τ there is an $m \in M$ such that $E \approx \mathcal{E}_m$.

Proof. Let $\rho: G \to GL(V)$ be a faithful representation. Let $V = V_1 \oplus \cdots \oplus V_q$ be a decomposition into irreducible subspaces. Let r_i be the rank of V_i and d_i be the degree of the vector bundle $\rho_{i_*}(\tau)$ where $\rho_i: G \to GL(V_i)$ gives the action of G on V_i. Let $u_i \to M_i \times X$ be the universal family for the set $\mathcal{S}^{r_i,d_i}$ and $U = U_1 \underset{X}{\times} \cdots \underset{X}{\times} U_r \to (M_1 \times \cdots \times M_r) \times X$. Let $\mathscr{E} \to M \times X$ represent the functor $\Gamma(\rho', U)$ (§§4.8, 4.8.1), where $\rho' = \rho_1 \times \cdots \times \rho_r$: $G \to GL(V_1) \times \cdots \times GL(V_r)$; use Proposition 3.17 to see that this ρ satisfies what we require.

4.13. Let Aut $\mathcal{G}'$ be the group of Lie algebra automorphisms of $\mathcal{G}'$. Let dim $\mathcal{G}' = r$. The group Aut $\mathcal{G}'$ may not be connected and Ad $G = G/Z$ is the connected component of identity of Aut $\mathcal{G}'$. Let $\rho: \mathbb{C}^* \times \text{Aut}\,\mathcal{G}' \hookrightarrow GL(\mathcal{G}')$ be the natural inclusion. Note that, in $GL(\mathcal{G}')$, $\mathbb{C}^* \cap \text{Aut}\,\mathcal{G}'$ is trivial. Applying Lemma 4.8.1. for this ρ and the universal family $\mathcal{O} \to R \times X$ for $\mathcal{S}^{r,rmd_0}$ to get a universal family $\mathscr{E}' \to R_1' \times X$ with group $GL(n, \mathbb{C})$ for $\mathbb{C}^* \times \text{Aut}\mathcal{G}'$-bundles which under the extension of structure group ρ give semistable vector bundles of degree rmd_0. Since any Aut$\mathcal{G}'$-bundle gives a vector bundle of degree zero on extension of structure group by $\text{Aut}\mathcal{G}' \hookrightarrow GL(\mathcal{G}')$ (Aut$\mathcal{G}'$ is contained in the orthogonal group corresponding to the Killing form) it follows that for the extension of structure group $\mathbb{C}^* \times \text{Aut}\mathcal{G}' \to \mathbb{C}^*$, $\mathscr{E}'$ gives a family of line bundles $\mathscr{E}'(\mathbb{C}^*)$ of degree md_0 on X. Let J^{md_0} be the Jacobian of X of line bundles of degree md_0. Then by the universal property of J^{md_0} we have a morphism $R_1' \to J^{md_0}$ corresponding to $\mathscr{E}'(\mathbb{C}^*) \to R_1' \times X$. Let R_1 be the fiber over $L^m \in J^{md_0}$ under this morphism. Restricting $\mathscr{E}'$ to $R_1 \times X$ and extending the structure group by $\mathbb{C}^* \times \text{Aut}\mathcal{G}' \to \text{Aut}\mathcal{G}'$ we get an Aut$\mathcal{G}'$-bundle $\mathscr{E}_1 \to R_1 \times X$. The action of $GL(n, \mathbb{C})$ on $\mathscr{E}' \to R_1' \times X$ gives an action of $GL(n, \mathbb{C})$ on $\mathscr{E}_1 \to R_1 \times X$.

4.13.1. PROPOSITION

The Aut$\mathcal{G}'$*-bundle* $\mathscr{E}_1 \to R_1 \times X$, *with the natural action of* $GL(n, \mathbb{C})$, *constructed above is a universal family with group* $GL(n, \mathbb{C})$ *for the set of isomorphism classes of* Aut$\mathcal{G}'$*-bundles which give semistable vector bundles of degree zero on extension of structure group by the inclusion* $\text{Aut}\mathcal{G}' \hookrightarrow GL(\mathcal{G}')$. *Moreover the parametrizing scheme* R_1 *is non-singular.*

Proof. The universal property of $\mathscr{E}_1$ is clear from the above discussion. The non singularity of R_1 is proved in the following two lemmas (4.13.3 and 4.13.4).

4.13.2. *Remark.* The elements of the tensor space $\mathscr{H} = \mathcal{G}'^* \otimes \mathcal{G}'^* \otimes \mathcal{G}' = \text{Hom}(\mathcal{G}' \otimes \mathcal{G}', \mathcal{G}')$ give algebra structures on $\mathcal{G}'$. Those elements of $\mathscr{H}$ which give algebra structures which satisfy the Jacobi identity and skew symmetry form a closed sub-variety of $\mathscr{H}$ and give Lie algebra structures on $\mathcal{G}'$. The points of the variety $Y = GL(\mathcal{G}')/\mathbb{C}^* \times \text{Aut}\mathcal{G}' \subset \mathbb{P}(\mathscr{H})$ then give Lie algebra structures on $\mathcal{G}'$, determined up to a scalar and isomorphic to the original Lie algebra structure of $\mathcal{G}'$. If $0 \neq h \in \mathscr{H}$ is such that $\{h\} \in \bar{Y}$ and the Lie algebra structure of $\mathcal{G}'$ given by h is semisimple then $\{h\} \in Y$ ([16], Corollary 4.3, p. 514). This fact will be crucial for us in proving the existence of quotient space in the next section (cf. proof of Lemma 5.6) and is the reason why we have chosen the adjoint representation for constructing universal families.

Let $r_1 \in R_1 \subset R_1'$ and $r \in R$ its image. Then the $\mathbb{C}^* \times \text{Aut}\mathcal{G}'$-bundle $\mathscr{E}'_{r_1}$ gives the vector bundle $\mathcal{O}_r$ under the extension of structure group $\mathbb{C}^* \times \text{Aut}\mathcal{G}' \to GL(\mathcal{G}')$ and the Aut$\mathcal{G}'$-bundle $\mathscr{E}_{r_1}$ gives the vector bundle $\mathcal{O}_r(-m)$ under $\text{Aut}\mathcal{G}' \hookrightarrow GL(\mathcal{G}')$. If we take the embedding $Y \subset \mathbb{P}(\mathscr{H})$, r_1 gives a section $r_1: X \to Q_r(Y) \subset \mathbb{P}(\mathcal{O}_r^* \otimes \mathcal{O}_r^* \otimes \mathcal{O}_r)$ such that $r_1^*(\Lambda) = L^{-m}$ where Λ is the tautological line bundle on $\mathbb{P}(\mathcal{O}_r^* \otimes \mathcal{O}_r^* \otimes \mathcal{O}_r)$

corresponding to the vector bundle $\mathcal{O}_r^* \otimes \mathcal{O}_r^* \otimes \mathcal{O}_r$. Since $\mathbb{P}(\mathcal{O}_r^* \otimes \mathcal{O}_r^* \otimes \mathcal{O}_r) = \mathbb{P}(\mathcal{O}_r(-m)^* \otimes \mathcal{O}_r(-m)^* \otimes \mathcal{O}_r(-m))$, r_1 also gives a section $r_1: X \to \mathbb{P}(\mathcal{O}_r^* \otimes \mathcal{O}_r^* \otimes \mathcal{O}_r \otimes L^m)$, and $r_1(\Lambda') = L^{-m} \otimes L^m = 1$, where $\Lambda' = \Lambda \otimes p_X^* L^m$. Therefore r_1 gives a section s, determined up to a scalar, in $H^0(X, \mathcal{O}_r(-m)^* \otimes \mathcal{O}_r(-m)^* \otimes \mathcal{O}_r(-m))$ such that $s(x) \neq 0 \; \forall x \in X$ and $s(x)$ is a Lie algebra structure on the fiber of $\mathcal{O}_r(-m)$ over x, isomorphic to the natural Lie algebra structure of $\mathscr{G}'$.

Since Y is embedded in the *projective* space $\mathbb{P}(\mathscr{H})$ it is easy to see that the scalars in $GL(n, \mathbb{C})$ act trivially on R_1. This is a reason why we have constructed a universal family for $\mathrm{Aut}\,\mathscr{G}'$-bundles first working with $\mathbb{C}^* \times \mathrm{Aut}\,\mathscr{G}'$ instead of directly using the inclusion $\mathrm{Aut}\,\mathscr{G}' \hookrightarrow GL(\mathscr{G}')$.

4.13.3. *Lemma. The schemes R and R_1' are nonsingular and* $\dim R = n^2 + r^2(g-1)$ *and* $\dim R_1' = n^2 + (r+1)(g-1)$.

Proof. We use the notation of §4.11.3. Let $Y = GL(\mathscr{G}')/\mathbb{C}^* \times \mathrm{Aut}\,\mathscr{G}'$ and $Q(Y)$ be the fiber bundle with fiber Y associated to Q considered as a $GL(\mathscr{G}')(= GL(r, \mathbb{C}))$-bundle. The scheme R is an open subscheme of $\mathrm{Hom}(X, G_{n,r})$ and R_1 is an open subscheme of $\mathrm{Hom}(X, Q(Y))$. The morphism $R_1 \to R$ is induced by $\varphi: Q(Y) \to G_{n,r}$. By associating a morphism $f: X \to Q(Y)$ to its graph Γ_f in $X \times Q(Y)$, $\mathrm{Hom}(X, Q(Y))$ becomes an open subscheme of $\mathrm{Hilb}(X \times Q(Y))$ ([TDTE, IV], §4, pp. 19, 20). The graph $\Gamma_f \approx X$ of $f \in R_1'$ is a nonsingular complete subvariety of the non-singular variety $X \times Q(Y)$. Therefore the obstruction to the smoothness of the Hilbert scheme at Γ_f is an element of $H^1(\Gamma_f, N_{\Gamma_f})$ where N_{Γ_f} is the normal bundle of Γ_f in $X \times Q(Y)$ ([TDTE, IV], §5; cf. also [9]). Identifying Γ_f with X by the projection $X \times Q(Y) \to X$, $N_{\Gamma_f} \approx f^*(T_q)$ where T_q is the tangent bundle of $Q(Y)$. We shall show that $H^1(X, f^*(T_q)) = 0$ from which it will follow that R_1' is nonsingular.

We have the following diagram of vector bundles on X, which is commutative and exact (in the obvious sense)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & (\varphi f)^*(\mathrm{Ad}\,Q) & \longrightarrow & f^*(T'_q) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & (\varphi f)^*(M) & \longrightarrow & f^*(T_q) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (\varphi f)^*(T_g) & \xrightarrow{=} & (\varphi f)^*(T_g) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{1}$$

where T_q (respectively T_g) is the tangent bundle of $Q(Y)$ (resp. $G_{n,r}$) and T'_q is the tangent bundle along fibers of $\varphi: Q(Y) \to G_{n,r}$, the canonical map.

The first column is the pull back by φf of the Atiyah exact sequence of $Q \to G_{n,r}$. The horizontal arrows from the first to the second column are induced by the differential of the projection $Q \to Q(Y) = Q/\mathbb{C}^* \times \mathrm{Aut}\,\mathscr{G}'$.

Let $\varphi f \in R$ correspond to the quotient

$$0 \to H_f \to I_n \to F_f \to 0, \tag{2}$$

i.e. $F^*(Q)=F_f$. Then $(\varphi f)(T_g)=H_f^*\otimes F_f$ ([TDTE] IV, § 5).

It is easy to check the following identifications:

$$\begin{array}{ccccccccc} 0 \to & (\varphi f)^*(\mathrm{Ad}\,Q) & \longrightarrow & (\varphi f)^*(M) & \longrightarrow & (\varphi f)^*(T_g) & \to 0 \\ & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & \\ 0 \to & F^*{}_f \otimes F_f & \longrightarrow & I_n \otimes F_f & \longrightarrow & H^*{}_f \otimes F_f & \to 0 \end{array} \tag{3}$$

where the second row is obtained from (2) by dualizing and tensoring by F_f.

Also, K is the adjoint bundle of the $\mathbb{C}^*\times \mathrm{Aut}\,\mathscr{G}'$ bundle obtained from the reduction of structure group of F_f to $\mathbb{C}^*\times\mathrm{Aut}\,\mathscr{G}'$ corresponding to f.

From the cohomology exact sequence of the bottom row of (3) we get $H^1(X,(\varphi f)^*M)=H^1(X,I_n\otimes F_f)=n$-copies of $H^1(X,F_f)$. But $F_f\in\mathscr{S}^{r,rmd_0}$ and hence $H^1(X,F_f)=0$. Therefore $H^1(X,(\varphi f)^*M)=0$. Also $H^1(X,H_f^*\otimes F_f)=0$ and therefore $H^1(X,(\varphi f)^*(T_g))=0$. This proves that R is nonsingular (for the same reason that $H^1(X,f^*(T_g))=0$ proves R_1' is nonsingular).

From diagram (1), taking cohomology, we have

$$H^1((\varphi f)^*M)\to H^1(f^*T_q)\to 0.$$

(All cohomologies over X). This proves that $H^1(X,f^*T_q)=0$ as claimed.

By ([TDTE, IV], § 5) the Zariski tangent space to R (resp. R_1') at φf (resp. f) can be canonically identified with $H^0(X,H_f^*\otimes F_f)$ (resp. $H^0(X,f^*T_q)$). From the bottom row of (3) we get that $\deg H_f^*\otimes F_f=n\deg F_f=n\cdot rmd_0$. Applying Riemann–Roch: $\dim H^0(X,H_f^*\otimes F_f)=n\cdot rmd_0+r(n-r)(1-g)$. Since $n=\dim H^0(X,F_f)=rmd_0+r(1-g)$ we have $\dim R=\dim H^0(X,H_f^*\otimes F_f)=n^2+r^2(g-1)$.

From the exact sequence

$$0\to H^0(f^*T_q')\to H^0(f^*T_q)\to H^0(f^*T_g)\to H^1(f^*T_q')\to 0$$

(all cohomologies over X) we get

$$\dim H^0(f^*T_q)=\dim H^0(f^*T_g)+\dim H^0(f^*T_q')-\dim H^1(f^*T_q').$$

We apply Riemann–Roch to f^*T_q', noting that $\deg f^*T_q'=0$ (since in the first row of (1) both K and $(\varphi f)^*\mathrm{Ad}\,Q$ being adjoint bundles have degree zero) and $\mathrm{rk}\,f^*T_q'=r^2-r-1(=\dim Y)$, to get $\dim R_1'=\dim H^0(X,f^*T_q)=n^2+r^2(g-1)+(r^2-r-1)(1-g)=n^2+(r+1)(g-1)$.

4.13.4. *Lemma. The scheme* R_1 *is nonsingular and* $\dim R_1=n^2+(r+1)(g-1)-g$.

Proof. Since R_1' and J^{md_0} are non-singular and R_1 is the fiber over $L^m\in J^{md_0}$, it is enough to check that the differential of $\psi:R_1'\to J^{md_0}$ is surjective at any point $r\in R_1$. Let $t:\mathrm{Spec}\,\mathbb{C}[\varepsilon]\to J^{md_0}$ be a tangent to J^{md_0} at L^m. We have to show that we can lift this morphism to R_1' such that the unique closed point of $\mathrm{Spec}\,\mathbb{C}[\varepsilon]$ goes to r. For this we use the universal properties of R and R_1'. Let $r\in R_1'$ correspond to the $\mathbb{C}^*\times\mathrm{Aut}\,\mathscr{G}'$-bundle $M\times E$ where M is a line bundle on X and $E\to X$ is an $\mathrm{Aut}\,\mathscr{G}'$-bundle. Let $\mathscr{P}\to J^{md_0}\times X$ be the Poincaré bundle. Consider the $\mathbb{C}^*\times\mathrm{Aut}\,\mathscr{G}'$-bundle $\bar{t}^*\mathscr{P}\times_X E_\varepsilon\to\mathrm{Spec}\,\mathbb{C}[\varepsilon]\times X$ where $E_\varepsilon=\mathrm{Spec}\,\mathbb{C}[\varepsilon]\times E$. Then $H^1(X,(\bar{t}^*(\mathscr{P})\otimes E_\varepsilon(\mathscr{G}'))_o)=H^1(X,L^m\otimes E(\mathscr{G}'))=0$, where o denotes the closed point of $\mathrm{Spec}\,\mathbb{C}[\varepsilon]$. Therefore the direct image

$p_{\varepsilon *}(\bar{t}^* \mathscr{P} \underset{X}{\times} E_\varepsilon(\mathscr{G}')) = \mathscr{I}$ on Spec $\mathbb{C}[\varepsilon]$ is locally free and hence free and the natural map $p_\varepsilon^*(\mathscr{I}) \to \bar{t}^*(\mathscr{P}) \otimes E_\varepsilon(\mathscr{G}')$ is surjective ([11], pp. 51–52). If we choose a trivialization of $\mathscr{I}$ we get a morphism Spec $\mathbb{C}[\varepsilon] \to R$ by the universal property of R, and a lift of this to R_1' by the universal property of R_1'. It is easy to see that for a suitably chosen trivialization of $\mathscr{I}$ the latter gives the required lift of Spec $\mathbb{C}[\varepsilon] \to J^{md_0}$ to R_1'. Since $\dim R_1 = \dim R_1' - \dim J^{md_0}$ we get, by lemma 4.13.3, $\dim R_1 = n^2 + (r+1)(g-1) - g$.

4.14. PROPOSITION

Let $\mathscr{E}_2 \to R_2 \times X$ be the Ad G*-bundle representing the functor $\Gamma(\rho, \mathscr{E}_1)$ where ρ:*Ad $G \to$ Aut $\mathscr{G}'$ *is the inclusion and $\mathscr{E}_1 \to R_1 \times X$ is the universal* Aut $\mathscr{G}'$*-bundle constructed in the preceding article. Then $\mathscr{E}_2$ is a universal family with group $GL(n, \mathbb{C})$ (for the action of $GL(n, \mathbb{C})$ provided by Lemma* 4.10) *for the set of all isomorphism classes of semistable* Ad G-bundles.

The natural morphism $\pi_2 : R_2 \to R_1$ is étale and finite and hence R_2 is nonsingular.

Proof. Since Ad G is semisimple there are only finitely many isomorphism classes of topological Ad G-bundles (π_1(Ad G) is finite, cf. [14], § 5). Therefore, by Lemma 3.8.1, R_2 is a scheme of finite type. The universal property for $\mathscr{E}_2$ follows from Lemma 4.10. We note that $\mathscr{E}_2$ is universal for all semistable Ad G-bundles because by Corollary 3.18 for any Ad G-bundle E, $E(\mathscr{G}')$ is semistable if and only if E is semistable. (Note that $E(\mathscr{G}')$ is semistable if and only if $E(\mathscr{G})$ is semistable.)

Since Ad G is the connected component of identity of Aut $\mathscr{G}'$ the morphism $GL(\mathscr{G}')/\text{Ad}\, G \to GL(\mathscr{G}')/\text{Aut}\, \mathscr{G}'$ is an étale covering. Therefore it follows that $\mathscr{E}_1/\text{Ad}\, G \to R_1 \times X$ is an étale covering. Taking $Y = R_1 \times X$, $\tilde{Y} = \mathscr{E}_1/\text{Ad}\, G$ and $T = R_1$ in the following lemma we see that $\pi_2 : R_2 \to R_1$ is étale and finite.

4.14.1. *Lemma. Let $Y \to T$ and $\tilde{Y} \to T$ be schemes over a scheme T and $\tilde{Y} \to Y$ a T-morphism which is an étale (surjective) covering. Suppose Y is flat and projective over T and $\tilde{Y}$ is quasi-projective over T so that the functor π:(locally noetherian schemes/T)$\to$(Sets) defined by*

$$\pi(f : S \to T) = \text{Hom}_{S \underset{T}{\times} Y}(S \underset{T}{\times} Y, S \underset{T}{\times} \tilde{Y}) = \text{Hom}_T(S \underset{T}{\times} Y, \tilde{Y})$$

is representable by a locally finite type T-scheme $\pi : \Pi \to T$ ([TDTE, IV], § 4). *Assume also that $Y \to T$ is faithfully flat. Then π is étale and finite.*

Proof. To prove $\pi : \Pi \to T$ is étale, let A be a scheme, A_0 a subscheme of A defined by a nilpotent ideal and

$$\begin{array}{ccc} A_0 & \longrightarrow & \Pi \\ \big\downarrow & \nearrow & \big\downarrow \\ A & \longrightarrow & T \end{array}$$

a commutative diagram. We only have to show that the morphism corresponding to the broken arrow exists uniquely ([EGA IV], Definitions 17.1.1, 17.3.1). Consider the commutative diagram

$$\begin{array}{ccc} A_0 \underset{T}{\times} Y & \longrightarrow & \tilde{Y} \\ \cap\big\downarrow & \nearrow & \big\downarrow \\ A \underset{T}{\times} Y & \longrightarrow & Y \end{array}$$

where the morphism $A_0 \times Y \to \tilde{Y}$ is induced by $A_0 \to \Pi$. Since $A_0 \underset{T}{\times} Y$ in $A \underset{T}{\times} Y$ is also defined by a nilpotent ideal and $\tilde{Y} \to Y$ is étale the broken arrow in the diagram can be realized uniquely and hence by the universal property of Π the broken arrow in the preceding diagram can also be realized.

It follows from ([SGA I], Exposé I, § 5, Corollary 5.3) that $\pi : \Pi \to T$ is quasi finite (i.e. has finite fibers). Since a proper and quasi finite morphism is finite ([EGA III], Corollary 4.4.11) it now suffices to show that π is proper. We use the valuation criterion for this.

Let A be a discrete valuation ring over $\mathbb{C}$ with residue field $\mathbb{C}$ and quotient field K. Suppose we are given morphisms

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \Pi \\ \cap\big\downarrow & \nearrow & \big\downarrow \\ \operatorname{Spec} A & \longrightarrow & T \end{array}$$

Again by the universal property of Π this gives

$$\begin{array}{ccc} \operatorname{Spec} K \underset{T}{\times} Y & \longrightarrow & \tilde{Y} \\ \cap\big\downarrow & \nearrow & \big\downarrow \\ \operatorname{Spec} A \underset{T}{\times} Y & \longrightarrow & Y \end{array}$$

By taking the base change of the étale covering $\tilde{Y} \to Y$ by $\operatorname{Spec} A \underset{T}{\times} Y \to Y$ we get an étale covering over $\operatorname{Spec} A \underset{T}{\times} Y$ which, by the above diagram, has a section over the open subset $\operatorname{Spec} K \underset{T}{\times} Y$ which is dense ($Y \to T$ being faithfully flat) and hence is trivial. This proves that the broken arrow in this diagram, and hence in the preceding diagram, can be realized.

4.15. Let G' be the commutator subgroup $[G, G]$ of G. Let τ be a topological G-bundle on X. The group G/G' is a torus isomorphic to $\mathbb{C}^{*q}$. Therefore fixing an isomorphism $G/G' \approx \mathbb{C}^q$, a G/G'-bundle can be considered as a q-tuple of line bundles. Let $d_1, \ldots, d_q$ be the degrees of the topological line bundles $L_1, \ldots, L_q$ corresponding to the G/G'-bundle obtained from τ by the extension of structure group $G \to G/G'$. Let $J_\tau = J^{d_1} \times \cdots \times J^{d_q}$ where J^{d_i} is the Jacobian of X of line bundles of degree d_i. Let $U_i \to J^{d_i} \times X$ be the Poincaré bundle and $U_\tau \to J_\tau \times X$ be $(id_{J_\tau} \times \Delta_X)^*$ $(U_1 \times \cdots \times U_r)$ where $\Delta_X : X \to X \times \cdots \times X$ is the diagonal embedding and $U_1 \times \cdots \times U_q \to J_\tau \times X \times \cdots \times X$ is the external product of $U_i \to J_i \times X$. Then U_τ is a G/G'-bundle and making G/G' act trivially on J_τ and in the natural way on U_τ (since G/G' is abelian this action gives G/G'-bundle automorphisms) it is easy to see, using the universal property of the Jacobian, that U_τ is a universal family with group G/G' for the set of isomorphism classes of G/G'-bundles of the topological type determined by the degrees $d_1, \ldots, d_q$. The $G/G' \times G/Z$-bundle $\mathscr{E}'_2 = U_\tau \underset{X}{\times} \mathscr{E}_2 \to J_\tau \times R_2 \times X$ is then a universal family with group $G/G' \times GL(n, \mathbb{C})$ for semistable $G/G' \times G/Z$- bundles of suitable topological type, in the obvious way.

4.15.1. *Lemma. The sheaf* $\tilde{\Gamma}^\tau(\rho, \mathscr{E}'_2)$ *corresponding to the natural projection* $\rho: G \to G/G' \times G/Z$ *and the family* $\mathscr{E}'_2 \to R'_2 \times X$, $R'_2 = J_\tau \times R_2$, (§§ 4.8, 4.9) *is representable by a scheme* R_3, *étale and finite over* R'_2.

Proof. We will prove the lemma by identifying $\tilde{\Gamma}(\rho, \mathscr{E}'_2)$ with a product of Picard functors and using the representability theorems for Picard functors. The idea is simply to generalize the fact that given a projective bundle $P \to X$ to give a vector bundle $V \to X$ such that $\mathbb{P}(V) = P$ is equivalent to giving a tautological line bundle on P. For this purpose we construct an algebraic group H which is related to Ad G similar to the way in which $GL(n, \mathbb{C})$ is related to $PGL(n, \mathbb{C}) = \operatorname{Ad} SL(n, \mathbb{C})$.

Let T be a maximal torus of $G' = [G, G]$. Then $Z \cdot T$ is a maximal torus for G and $Z \cap T$ is the finite group $Z' = Z[G']$. Let $\chi_1, \ldots, \chi_l$ be a set of characters of $Z \cdot T$ such that the homomorphism $t \mapsto (\chi_1(t), \ldots, \chi_l(t))$ from $Z \cdot T$ to $\mathbb{C}^{*l}$ is injective on Z'. Let F be the finite subgroup $\{(\chi_1(t), \ldots, \chi_l(t), t^{-1}) \mid t \in Z'\}$ of $\mathbb{C}^{*l} \times G$. Define H to be the quotient group $(\mathbb{C}^{*l} \times G)/F$. Let A be the quotient group $(\mathbb{C}^{*l} \times Z')/F$. Then we have the diagram which is commutative and exact.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & \rho & & & \\
1 & \to & Z' & \longrightarrow & G & \longrightarrow & G/G' \times G/Z & \to & 1 \\
 & & \downarrow & & \downarrow & \rho' & \| & & \\
1 & \to & A & \longrightarrow & H & \longrightarrow & G/G' \times G/Z & \to & 1 \\
 & & \alpha\downarrow & & \downarrow & & & & \\
 & & A/Z' & \xrightarrow{\approx} & H/G & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{1}$$

Note that both A and A/Z' are isomorphic to $\mathbb{C}^{*l}$ since Z' is a finite subgroup of A and we can choose isomorphisms such that we have the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z' & \longrightarrow & A & \longrightarrow & A/Z' & \longrightarrow & 1 \\
 & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \\
1 & \longrightarrow & C_{\underline{n}} & \longrightarrow & \mathbb{C}^{*l} & \xrightarrow[\times \underline{n}]{} & \mathbb{C}^{*l} & \longrightarrow & 1
\end{array} \tag{2}$$

where $\underline{n} = (n_1, \ldots, n_l)$, $n_i \in \mathbb{N}$,

$$C_{\underline{n}} = \{(z_1, \ldots, z_l) \in \mathbb{C}^{*l} \mid z_i^{n_i} = 1, \forall i\}$$

and $\times\, \underline{n}$ is the homomorphism $(z_1, \ldots, z_l) \mapsto (z_1^{n_1}, \ldots, z_l^{n_l})$.

Let B be a Borel subgroup of G containing $Z\cdot T$. The characters $\chi_1,\ldots,\chi_l$ extend to B and give a homomorphism $B\to A\approx(\mathbb{C}^{*l}\times Z')/F$ by sending $b\in B$ to $(\chi_1(b),\ldots,\chi_l(b),1)\cdot F$. Clearly H is a connected reductive algebraic group and $B_H=(\mathbb{C}^{*l}\times B)/F$ is a Borel sub-group of H. Again the characters $\chi_1,\ldots,\chi_l$ extend to B_H by defining $\chi_1((z_1,\ldots,z_l,b)\cdot F)\approx z_1\cdot\chi_i(b)$ and we have a homomorphism $B_H\to A$ defined by $(z_1,\ldots,z_l,b)\cdot F\to(z_1\chi_1(b),\ldots,z_l\chi_l(b),1)\cdot F$. We then have the commutative diagram

$$\begin{array}{ccc} & B & \\ \swarrow & & \searrow \\ B_H & \longrightarrow & A \\ \downarrow & & \downarrow \\ H/G & \xrightarrow[\approx]{} & A/Z' \end{array} \tag{3}$$

The image $\bar{B}=B/Z$ of $\bar{G}=G/Z$ is a Borel subgroup of $\bar{G}$ and $G/B\to\bar{G}/\bar{B}$ is an isomorphism. The projection $\mathbb{C}^{*l}\times G\to G$ induces an isomorphism $H/B_H\to G/B$ which is the inverse of the isomorphism $G/B\to H/B_H$ induced by the inclusion $G\hookrightarrow H$. We then have the commutative diagram

$$\begin{array}{ccccc} H & \hookleftarrow & G & \longrightarrow & \bar{G} \\ \downarrow & & \downarrow & & \downarrow \\ H/B_H & \approx & G/B & \approx & \bar{G}/\bar{B} \end{array} \tag{4}$$

Let $M\to G/B$ be the A-bundle obtained from the B-bundle $G\to G/B$ by the extension of structure group $B\to A$. Note that M is also the A-bundle obtained from the B_H-bundle $H\to H/B_H$ by the extension of structure group $B_H\to A$ as follows from the diagrams (3) and (4). The group H operates on G/G' (by left multiplication through ρ') and on $G/B\approx H/B_H$. Since $M\to G/B\approx H/B_H$ is a bundle associated to $H\to H/B_H$, H operates on M also, compatibly with its action on H/B_H. Further H is precisely the automorphism group of the 'structure' consisting of $G/G'\times G/B$ and M, i.e., to put it more precisely, given an isomorphism $\varphi:G/G'\times G/B\to G/G'\times G/B$ induced by an element of G and an A-bundle isomorphism $\varphi_M:M\to M$ over φ_2, where $\varphi_2:G/B\to G/B$ is given by φ, then there exists an unique $h\in H$ whose action gives φ and φ_M. The existence of such an h is clear and for uniqueness note that since $\cap_{g\in G}gBg^{-1}=Z$, only Z' acts trivially on $G/G'\times G/B$ and the action of Z' on M is faithful since it is given by the characters $\chi_1,\ldots,\chi_l$.

Let $\tilde{\mathbf{P}}_M$ be the sheaf associated to the functor $\mathbf{P}_M:(\mathrm{Sch}/R_2')\to(\mathrm{Sets})$ which associates to $f:S\to R_2'$ the set of isomorphism classes of A-bundles on $\bar{f}^*(\mathscr{E}_2/B)$, where $\bar{f}=f\times id_X:S\times X\to R_2'\times X$, such that for every point $(s,x)\in S\times X$, and for any trivialization $\varphi_{(s,x)}:\mathscr{E}_{2,(s,x)}/B\xrightarrow{\approx}G/B$ there exists an isomorphism $\tilde{\varphi}_{(s,x)}$ of A-bundles over $\varphi_{(s,x)}$:

$$\begin{array}{ccc}
\mathscr{E}_{(s,x)} & \xrightarrow{\tilde{\varphi}_{(s,x)}} & M \\
\downarrow & & \downarrow \\
\mathscr{E}_2(s,x)/B & \xrightarrow[\varphi_{(s,x)}]{} & G/B
\end{array}$$

We define $\mathbf{P}_M$ on morphisms to be pull back. We shall show that $\tilde{\Gamma}(\rho', \mathscr{E}'_2)$ is isomorphic to $\tilde{\mathbf{P}}_M$.

Let $\Lambda \in \mathbf{P}_M(f: S \to R'_2)$. Consider the functor F:(Sch/$S \times X$)$\to$(Sets) which associates to $f': S' \to S \times X$ the set of pairs (φ, φ_M) where φ is an isomorphism of the trivial $(G/G' \times G/Z)$-bundle on S' with $f'^*(\bar{f}^*(\mathscr{E}'_2))$ and φ_M is an isomorphism of A-bundles such that

$$\begin{array}{ccc}
S' \times M & \xrightarrow{\varphi_M} & f'^*(\Lambda) \\
\downarrow & & \downarrow \\
S' \times G/B & \xrightarrow[\varphi]{} & f'^*(\bar{f}^*(\mathscr{E}'_2/B))
\end{array}$$

commutes. On morphisms $\mathbf{F}$ is defined to be pull back. Let $\mathbf{H} = S \times X \times H$ be the constant group scheme over $S \times X$ (considered as a functor) defined by H. Since H is the automorphism group of the structure consisting of $G/G' \times G/B$ and $M \to G/B$, as explained above, it is easy to see that $\mathbf{F}$ is a principal homogeneous space under $\mathbf{H}$. Since $\mathbf{H}$ is an affine algebraic group scheme over $S \times X$ it follows by descent ([SGA 3], exposé XXIV) that $\mathbf{F}$ is representable. Thus we get an H-bundle $\mathscr{E}_\Lambda$ on $S' \times X$ and, from the construction, there is a natural isomorphism $\varphi: \rho'_* \mathscr{E}_\Lambda \to \bar{f}^*(\mathscr{E}'_2)$. We define the morphism $\Phi: \tilde{\mathbf{P}}_M \to \tilde{\Gamma}(\rho', \mathscr{E}'_2)$ by setting $\Phi_S(\Lambda) = (\mathscr{E}_\Lambda, \varphi)$.

We now construct an inverse $\Psi: \tilde{\Gamma}(\rho', \mathscr{E}'_2) \to \tilde{\mathbf{P}}_M$ for Φ. Let $(\mathscr{E}, \varphi) \in \tilde{\Gamma}(\rho', \mathscr{E}'_2)$ $(f: \hat{S} \to R'_2)$. Thus $\mathscr{E} \to S \times X$ is an H-bundle and $\varphi: \rho'_* \mathscr{E} \to \bar{f}^*(\mathscr{E}_2)$ is an isomorphism. Now φ induces an isomorphism $\mathscr{E}/B_H \approx \bar{f}^*(\mathscr{E}_2(G/B))$ (see diagrams (3) and (4)). Note that $\mathscr{E}'_2(G/B) = \mathscr{E}'_1/B$. Define $\Lambda \to \bar{f}(\mathscr{E}_2/B)$ to be the A-bundle obtained from the B_H-bundle $\mathscr{E} \to \mathscr{E}/B_H$ by the extension of structure group $B_H \to A$ (see diagram (3)). Define $\Psi_S((\mathscr{E}, \varphi)) = \Lambda$. Then it is straightforward to check that Ψ is a morphism inverse to Φ.

Let $\tilde{\mathbf{P}}$ be the sheaf associated to the functor $\mathbf{P}$:(Sch/R'_2)$\to$(Sets) which associates to $f: S \to R'_2$ the set of isomorphism classes of A-bundles on $\bar{f}^*(\mathscr{E}'_2(G/B))$. Since $A \approx \mathbb{C}^{*l}$ (diagram (2)) an A-bundle is just an l-tuple of line bundles and hence $\mathbf{P} \approx \mathbf{Pic} \times \cdots \times \mathbf{Pic}$ the l-fold product of the relative Picard functor of $\mathscr{E}'_2(G/B)/R'_2$ ([TDTE, V], §1). Since the morphism $\mathscr{E}'_2(G/B) = \mathscr{E}_2/B \to R'_2$ is projective (Remark 4.8.3), flat and smooth with irreducible fibers, **Pic** is representable by a locally finite type scheme P over R'_2 ([TDTE V], Theorem 3.1; [TDTE VI], Theorem 4.1, Corollary 4.2) and hence so is $\tilde{\mathbf{P}}$. Clearly $\mathbf{P}_M$ is a subfunctor of $\mathbf{P}$.

Let $\tilde{\mathbf{F}}_1$ (resp. $\tilde{\mathbf{F}}_2$) be the sheaf associated to the functor $\mathbf{F}_1$ (resp. $\mathbf{F}_2$):(Sch R'_2)$\to$(Sets) which associates to $f: S \to R'_2$ the set of isomorphism classes of H G-bundles on

$S \times X$ (resp. on $\bar{f}^*(\mathscr{E}'_2(G/B))$). The morphism $\mathscr{E}'_2(G/B) \to R'_2 \times X$ induces a morphism $\mathbf{F}_1 \to \mathbf{F}_2$. Since $\mathscr{E}'_2(G/B) \to R'_2 \times X$ is projective and flat with irreducible fibers by ([TDTE, V], Proposition 2.1 and [EGA, III], Corollary 7.8.8) it follows that $\mathbf{F}_1 \to \mathbf{F}_2$ is a monomorphism.

The exact sequence $1 \to G \to H \to H/G \to 1$ gives morphisms $\Gamma(\rho, \mathscr{E}'_2) \to \Gamma(\rho', \mathscr{E}'_2) \to \mathbf{F}_1$ and an exact sequence (part of which is)

$$H^0(S \times X, H/G) \to H^1(S \times X, G) \to H^1(S \times X, H) \to H^1(S \times X, H/G)$$

([17], §3.6, Propositions 11 and 12; see also [SGA I], exposé XI, §4). Since $H = Z[H] \cdot G$ it follows that $H^0(S \times X, H/G)$ operates trivially on $H^1(S \times X, G)$. Therefore $H^1(S \times X, G) \to H^1(S \times X, H)$ is an injection. This means that $\Gamma(\rho, \mathscr{E}_2) \to \Gamma(\rho', \mathscr{E}_2)$ is a monomorphism.

We use the isomorphism $H/G \approx A/Z'$ of diagram (1) and the isomorphism $A/Z' \approx \mathbb{C}^{*l}$ of diagram (2) to get an isomorphism of $\mathbf{F}_2$ with $\mathbf{P}$. We have the commutative diagram

$$\begin{array}{ccccc} & & & & \mathbf{F}_1 \\ & & & & \downarrow \\ \Gamma(\rho, \mathscr{E}'_2) & \hookrightarrow & \Gamma(\rho', \mathscr{E}'_2) & \longrightarrow & \mathbf{F}_2 \\ & & \downarrow & & \downarrow \\ \mathbf{P}_M & \hookrightarrow & \mathbf{P} & \longrightarrow & \mathbf{P} \end{array}$$

where $\mathbf{P} \to \mathbf{P}$ is induced by $\times \underline{n}$: of diagram (2).

It follows from ([TDTE VI], Theorem 2.5) that the morphism $\mathbf{P} \to \mathbf{P}$ corresponding to $\mathbf{P} \to \mathbf{P}$ is étale. Corresponding to the trivial H/G-bundle on $R'_2 \times X$ we get a morphism $R'_2 \to \mathbf{P}$. Let R'_3 be the fiber product

$$\begin{array}{ccc} R'_3 & \longrightarrow & \mathbf{P} \\ \downarrow & & \downarrow \\ R'_2 & \longrightarrow & \mathbf{P} \end{array}$$

Then it follows from ([TDTE VI], Corollary 4.2 and [EGA II], Corollary 5.4.3) that R'_3 is proper over R'_2. Also R'_3 is étale over R'_2.

Using the fact that an H-bundle $\mathscr{E}$ comes from a G-bundle if and only if $\mathscr{E}(H/G)$ is trivial, it is straight forward to check that R'_3 represents the sheaf $\tilde{\Gamma}(\rho, \mathscr{E}'_2)$. Let R_3 be the subscheme of R'_3 corresponding to G-bundles of type τ. Then clearly $R_3 \to R'_2$ is again étale and finite (since topological type is a discrete invariant).

4.15.2. PROPOSITION

The scheme R_3 with the action of $GL(n, \mathbb{C}) \subset G/G' \times GL(n, \mathbb{C})$ (given by Lemma 4.10) is a universal space with group $GL(n, \mathbb{C})$ for semistable G-bundles of topological type τ.

The $GL(n, \mathbb{C})$-equivariant morphism $R_3 \to R'_2$ is étale and finite and hence R_3 is nonsingular.

Proof. This follows immediately from the preceding lemma and Lemma 4.10 noting that G/G' acts trivially on R'_2 and hence trivially on R_3.

5. Existence of a quotient space

5.1. *Lemma. Suppose a reductive algebraic group H acts on the schemes Y and Z. If $f: Y \to Z$ is an affine H-equivariant morphism and Z has a good quotient $p: Z \to \bar{Z}$ modulo H then Y has a good quotient $q: Y \to \bar{Y}$ modulo H and the induced morphism $\bar{f}: \bar{Y} \to \bar{Z}$ is affine.*

If moreover f is finite then $\bar{f}$ is also finite. When f is finite and $p: Z \to \bar{Z}$ is a geometric quotient then $q: Y \to \bar{Y}$ is also a geometric quotient.

Proof. Let $\{U_i\}$ be a covering of $\bar{Z}$ by affine open sets. Let $U'_i = (p \circ f)^{-1}(U_i)$. Then $\{U'_i\}$ is a covering of Y by affine H-invariant open sets. Since U'_i is affine there exists a good quotient $q_i: U'_i \to \bar{U}'_i$ of U'_i modulo H, with $\bar{U}'_i$ affine ([22], Theorem 1.1(A)). We shall now give a patching up data for $\{\bar{U}'_i\}$. Let $\bar{f}_i: \bar{U}'_i \to U_i$ be the morphism induced by f. Let $\bar{U}'_{ij} = \bar{f}_i^{-1}(U_i \cap U_j)$. Then $q_i^{-1}(\bar{U}'_{ij}) = (p \circ f)^{-1}(U_i \cap U_j) = U'_i \cap U'_j$. Since, being a good quotient is local with respect to the base ([20], § 3, Property 1, p. 356) $q_i: U'_i \cap U'_j \to \bar{U}'_{ij}$ is a good quotient of $U'_i \cap U'_j$ modulo H. Interchanging i and j, $q_j: U'_i \cap U'_j \to \bar{U}'_{ji}$ is also a good quotient of $U'_i \cap U'_j$ modulo H. A good quotient, being a categorical quotient, is unique. Therefore we have natural isomorphisms $h_{ij}: \bar{U}'_{ij} \to \bar{U}'_{ji}$. The h_{ij}'s satisfy the cocycle condition and we can patch up the $\bar{U}'_i$ by h_{ij} along $\bar{U}'_{ij}$ to get a prescheme $\bar{Y}$. The q_i patch up to give a morphism $q: Y \to \bar{Y}$ and the $\bar{f}_i$ a morphism $\bar{f}: \bar{Y} \to \bar{Z}$. Clearly $\bar{f}$ is affine and $\bar{Z}$ being a (separated) scheme $\bar{Y}$ is also a (separated) scheme ([EGA II], Proposition 1.2.4). Again since being a good quotient is local with respect to the base $q: Y \to \bar{Y}$ is a good quotient.

To show that $\bar{f}$ is finite if f is finite we can assume that Y and Z are affine. So let $Y = \operatorname{Spec} A$ and $Z = \operatorname{Spec} B$ and f be given by the homomorphism $f: B \to A$ making A into a finite B-module. Then $\bar{Z} = \operatorname{Spec} B^H$ and $\bar{Y} = \operatorname{Spec} A^H$ ([22], Theorem 1.1(A)) and $\bar{f}$ is given by $B^H \to A^H$ the restriction of f (where A^H, B^H are the rings of invariants under H). We have to show that A^H is a finite B^H-module. Since A is a finite B-module there exists $a_1, \ldots, a_r \in A^H$ such that any $a \in A^H$ can be written as $a = \Sigma_{i=1}^r f(b_i) a_i$ with $b_i \in B$. Applying Reynold's operator P on both sides we get $a = \Sigma f(P(b_i)) \cdot a_i$ by Reynold's identity and functoriality (cf. [10], Chapter 1, Theorem 1.19). Since $P(b_i) \in B^H$ this proves that A^H is a finite B^H-module. The last assertion of the lemma is easily verified.

5.2. It follows immediately from the preceding lemma and the results of section 4 that to prove the existence of a good quotient for R_3 modulo $GL(n, \mathbb{C})$ it is enough to prove the existence of a good quotient of R_1 modulo $GL(n, \mathbb{C})$ (or equivalently, modulo $SL(n, \mathbb{C})$ since the scalars act trivially on R_1. See Remark 4.13.2). We shall prove this by using Mumford's theory of stable and semistable points for actions of reductive groups ([10]).

5.3. Let $ad \in \mathscr{G}'^* \otimes \mathscr{G}'^* \otimes \mathscr{G}' = \operatorname{Hom}(\mathscr{G}' \otimes \mathscr{G}', \mathscr{G}') = \mathscr{H}$ be the tensor which gives the Lie algebra structure of $\mathscr{G}'$, i.e. for $x, y \in \mathscr{G}'$, $ad(x, y) = [x, y]$. For the natural action of $GL(\mathscr{G}')$ on the tensor space $\mathscr{H}$ the stabilizer of the line (ad) generated by ad is $\mathbb{C}^* \times \operatorname{Aut} \mathscr{G}'$, where $\operatorname{Aut} \mathscr{G}'$ is the group of Lie algebra automorphisms of $\mathscr{G}'$. Therefore

$GL(\mathscr{G}')/\mathbb{C}^* \times \operatorname{Aut}\mathscr{G}' = Y$ gets embedded as a locally closed subscheme of $\mathbb{P}(\mathscr{H})$. Let $\bar{Y}$ be the closure of Y in $\mathbb{P}(\mathscr{H})$. We take on $\bar{Y}$ the canonical reduced subscheme structure.

Now by our constructions we have the 'universal morphisms' $f: R \times X \to G_{n,r}$ and $f_1: R_1 \times X \to Q(Y)$ such that

$$\begin{array}{ccc} R_1 \times X & \xrightarrow{f_1} & Q(Y) \\ \downarrow & & \downarrow \\ R \times X & \xrightarrow[f]{} & G_{n,r} \end{array}$$

commutes.

Now the morphisms $Y \subset \bar{Y} \subset \mathbb{P}(\mathscr{H})$ give rise to the morphisms $Q(Y) \subset Q(\bar{Y}) \subset Q(\mathbb{P}(\mathscr{H}))$. Since $Y \subset \bar{Y}$ is an open immersion $Q(Y) \subset Q(\bar{Y})$ is an open immersion and since $\bar{Y} \subset \mathbb{P}(\mathscr{H})$ is a closed immersion $Q(\bar{Y}) \subset \mathbb{P}(Q(\mathscr{H}))$ is also a closed immersion (§2.4). Let $\mathbb{R} = Q(\mathscr{H}) = Q^* \otimes Q^* \otimes Q$. Let Λ be the relatively ample line bundle $O_{\mathbb{P}(\mathbb{R})}(1)$ on $\mathbb{P}(\mathbb{R})$. Let Λ denote also its restriction to $Q(\bar{Y})$. Consider the functor $\Gamma: (\mathrm{Sch}/R) \to (\mathrm{Sets})$ defined by

$$\Gamma(S \to R) = \{\sigma \in \operatorname{Hom}_{G_{n,r}}(S \times X, Q(\bar{Y})) \mid \sigma^*(\Lambda)|_{s \times X} \approx L^{-m}, s \in S\}.$$

This is representable by an algebraic scheme S_1 over R. Let $g_1: S_1 \times X \to Q(\bar{Y})$ be the universal section. It is easy to see that the functor represented by R_1 is a subfunctor of Γ (Remark 4.13.2). Since $Q(Y)$ is open in $Q(\bar{Y})$ it follows that $R_1 \subset S_1$ is an open immersion. We have the commutative diagram

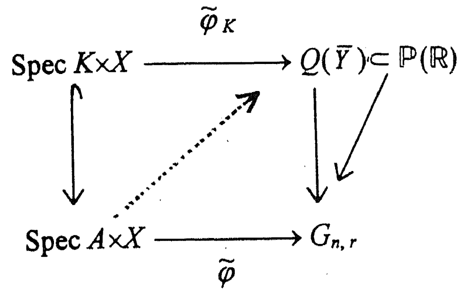
$$\begin{array}{ccc} R_1 \times X & \xrightarrow{f_1} & Q(Y) \\ \downarrow & g_1 & \downarrow \\ S_1 \times X & \longrightarrow & Q(\bar{Y}) \\ \downarrow & f & \downarrow \\ R \times X & \longrightarrow & G_{n,r} \end{array}$$

5.3.1. *Lemma. The morphism $S_1 \to R$ is proper.*

Proof. We make use of the valuative criterion for properness. Let A be a discrete valuation ring over $\mathbb{C}$ with residue field $\mathbb{C}$ and quotient field K. Suppose we have morphisms

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\varphi_K} & S_1 \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} A & \xrightarrow[\varphi]{} & R \end{array} \qquad (1)$$

By the universal properties of R and R_1 (recall that R and R_1 represent the corresponding functors of sections over the category of locally noetherian schemes ([TDTE IV])). We have



(2)

On Spec $K \times X$, we have the exact sequence

$$0 \to \tilde{\varphi}_K^*(\Lambda) \to \tilde{\varphi}_K^*(\mathbb{R}) \to \tilde{\varphi}_K^*(\mathbb{R}/\Lambda) = V_K \to 0.$$

We can extend the quotient vector bundle V_K to Spec $A \times X$ as a quotient coherent sheaf V of $\tilde{\varphi}^*(\mathbb{R})$ flat over Spec A(cf. [EGA IV], Proposition 2.8.1). Let V_0 be the restriction of V to Spec $\mathbb{C} \times X \simeq X$ corresponding to the closed point of Spec A. We then have the surjection $\tilde{\varphi}_0^*(\mathbb{R}) \to V_0 \to 0$ of coherent sheaves on X. Note that $\tilde{\varphi}_0^*(\mathbb{R})$ is a semistable vector bundle (by Proposition 3.17 since $\tilde{\varphi}_0^*(Q)$ is semistable) of degree $-r^3 md_0$ and therefore $\mu(\tilde{\varphi}_0^*(\mathbb{R})) = -md_0$. From the definition of S_1 and the flatness of V over Spec A it follows that deg $V_0 = -r^3 md_0 - md_0$. Since X is a curve we can write $V_0 = V'_0 \oplus T$ where V'_0 is locally free and T is torsion. Since $\tilde{\varphi}_0^*(\mathbb{R})$ is semistable, V'_0 being a quotient $\mu(V'_0) \geqslant -\mathrm{md}_0$. Therefore deg $V'_0 \geqslant -(r^3+1)md_0$. On the other hand T being a torsion sheaf deg $T \geqslant 0$ and since deg $V_0 = -(r^3+1)md_0$ we have deg $V'_0 \leqslant -(r^3+1)md_0$. This shows that deg $V'_0 = -(r^3+1)\cdot md_0$. Therefore $T = 0$ and $V'_0 = V_0$. Hence V_0 is locally free. Then V is a vector bundle on Spec $A \times X$ (cf. [19], Lemma 6.17). By the universal property of $\mathbb{P}(\mathbb{R}), V$ determines a morphism Spec $A \times X \to \mathbb{P}(\mathbb{R})$ (cf. [EGA II], Theorem 4.2.4; note that in our notation $\mathbb{P}$ stands for the dual of what $\mathbb{P}$ stands for in this reference). But since $Q(\bar{Y})$ is closed in $\mathbb{P}(\mathbb{R})$ this morphism goes into $Q(\bar{Y})$ proving that the broken arrow in diagram (2) can be realized. This immediately implies that the broken arrow in diagram (1) can be realized.

5.4. We now recall briefly some definitions and results from Mumford's 'geometric invariant theory' ([10]; see also [22]). Let a reductive group H act on a projective algebraic scheme Y. Let $\Lambda \to Y$ be an ample line bundle on Y and H act on Λ also as a group of line bundle isomorphisms compatible with its action on Y i.e. Λ has a H-linearization ([10], Chapter I, § 3). Then a point $y \in T$ is called *semistable* if for some $m > 0$ there is a H-invariant section $s \in H^0(Y, \Lambda^m)$ such that $s(y) \neq 0$. If moreover every orbit of H in $Y_s = \{x \in Y \mid s(x) \neq 0\}$ is closed and of the same dimension as H, y is called a *(properly) stable* point ([10], Chapter I, § 4).

5.4.1. The set Y_{ss}(resp. Y_s) of semistable (resp. stable) points is open in Y and there exists a good quotient $p: Y_{ss} \to \bar{Y}_{ss}$ modulo H such that $\bar{Y}_{ss}$ is projective.
There is an open subscheme $\bar{Y}_s \subset \bar{Y}_{ss}$ such that $p: p^{-1}(\bar{Y}_s) \to \bar{Y}_s$ is a 'geometric quotient' ([10], Chapter I, Theorem 1.10 and the remark on p. 40; [22], Theorem 1.1(B)).

5.4.2. Let $\lambda: \mathbb{C}^* \to G$ be a 1-PS (i.e. 1-parameter subgroup) and $y \in Y$. Then $\lim_{t \to 0} \lambda(t) y = y_0$ exists (since Y is projective) and y_0 is fixed by λ. Let $t \mapsto t^r$ be the character by which λ operates on Λ_{y_0}. Then we define $\mu^\wedge(y, \lambda) = -r$. (In this definition Λ can be an arbitrary line bundle not necessarily ample.) Then a point $y \in Y$ is semistable (resp. stable) if and only if $\mu^{\Lambda}(y,\lambda)\geq 0$ (resp. >0) for every 1-PS λ of G ([10], Chapter 2, § 1, Theorem 2.1).

5.4.3. Let H act on the projective algebraic scheme Y' and $f:Y'\to Y$ an H-equivariant morphism. Then $\mu^{f^*\Lambda}(y',\lambda)=\mu^{\Lambda}(f(y'),\lambda), y'\in Y'$ ([10], p.49).

5.4.4. Given a 1-PS λ of G there is a parabolic subgroup $P(\lambda)\supset\lambda$ such that $\mu^{\Lambda}(y,\lambda)=\mu^{\Lambda}(y,g\lambda g^{-1})$ for every $g\in P(\lambda)$. This $P(\lambda)$ depends only on λ and not on the scheme Y or Λ ([10], Chapter 2, § 2, [22], Lemma 3.1, Proposition 3.1).

5.5. Now by the usual 'diagonal argument' we can choose an N-tuple $(x_1,\ldots,x_N)$ of points in X for N sufficiently large such that the morphism $R_1\to Q(\bar{Y})^N=Q(\bar{Y})\times\cdots\times Q(\bar{Y})$, N factors, given by evaluating f_1 at the points x_i, i.e. $R_1\ni r\to(f_1(r,x_1),\ldots,f_1(r,x_N))$, is an injection ([19], p. 326). It is proved in ([20], § 3, Lemma 2) that we can choose the N-tuple $(x_1,\ldots,x_N)$ such that under the morphism $R\to G^N_{n,r}=Z$ (defined by $R\ni r\to(f(r,x_1),\ldots,f(r,x_N))$) R goes into the open subscheme of semistable points Z_{ss} of Z for the action of $SL(n,\mathbb{C})$ (and the natural polarization of the Grassmannian) and moreover such that the morphism $R\to Z_{ss}$ is proper.

We have the commutative diagram

$$\begin{array}{ccc}
R_1 & \longrightarrow & Q(Y)^N \\
\downarrow & & \downarrow \\
S_1 & & Q(\bar{Y})^N\subset\mathbb{P}(\mathbb{R})^N \\
\downarrow & & \downarrow \\
R & \longrightarrow & G^N_{n,r}
\end{array}$$

We shall give a suitable ample line bundle on $Q(\bar{Y})^N$ and prove that R_1 goes into the semistable points of $Q(\bar{Y})^N$ for the natural action of $SL(n,\mathbb{C})$ on $Q(\bar{Y})^N$.

Let M be the very ample line bundle on the Grassmannian $G_{n,r}$ corresponding to the natural embedding of $G_{n,r}$ in $\mathbb{P}(\Lambda^{n-r}\mathbb{C}^n)$. There is a natural $SL(n,\mathbb{C})$ linearization of M given by the action of $SL(n,\mathbb{C})$ on $\Lambda^{n-r}\mathbb{C}^n$. As defined in § 5.3, let Λ be the tautological line bundle on $\mathbb{P}(\mathbb{R})$ corresponding to the vector bundle $\mathbb{R}$. Then Λ is relatively ample for the morphism $p:Q(\bar{Y})\to G_{n,r}$. Therefore the line bundle $p^*(M)^a\otimes\Lambda^b$ is ample on $Q(\bar{Y})$ for $a\gg b$ ([EGA II] § 4.6, [22], § 5).

The action of $SL(n,\mathbb{C})$ on $G_{n,r}$ has a natural lift to an action on the bundle Q and hence on the associated bundle $\mathbb{R}=Q^*\otimes Q^*\otimes Q$. This gives an $SL(n,\mathbb{C})$-linearization of Λ. By ([22], Proposition 5.1) we can choose the positive integers a,b with b/a sufficiently small such that $p(Q(\bar{Y})^N_{ss})\subset Z_{ss}$ where $Q(\bar{Y})^N_{ss}$ is the set of semistable points of $Q(\bar{Y})^N$ for the action of $SL(n,\mathbb{C})$ and the ample line bundle on $Q(\bar{Y})^N$ which is the product of the line bundles $p^*(M)^a\otimes\Lambda^b$ on each factor $Q(\bar{Y})$.

5.5.1. *Lemma. The point* $(ad)\in\mathbb{P}(\mathscr{G}'^*\otimes\mathscr{G}'^*\otimes\mathscr{G}')$ (cf. § 5.3) *is semistable for the natural action of* $SL(\mathscr{G}')$ *and the line bundle* $O(1)$ *on* $\mathbb{P}(\mathscr{G}'^*\otimes\mathscr{G}'^*\otimes\mathscr{G}')$.

Proof. Let $\varphi:(\mathscr{G}'^*\otimes\mathscr{G}'^*\otimes\mathscr{G}')=\mathrm{Hom}(\mathscr{G}',\mathrm{End}\,\mathscr{G}')\to\mathscr{G}'^*\otimes\mathscr{G}'^*=(\mathscr{G}'\otimes\mathscr{G}')^*$ be defined by $\varphi(f)(x\otimes y)=\mathrm{trace}(f(x)\circ f(y))$, $f\in\mathrm{Hom}(\mathscr{G}',\mathrm{End}\,\mathscr{G}')$ and $x,y\in\mathscr{G}'$. Then φ is an $SL(\mathscr{G}')$-equivariant morphism. Choose an arbitrary linear space isomorphism of $\mathscr{G}'^*$

with $\mathscr{G}'$. Then we get an isomorphism $\mathscr{G}'^* \otimes \mathscr{G}'^* \approx \mathrm{End}(\mathscr{G}')$. Let $\det: \mathscr{G}'^* \otimes \mathscr{G}'^* \approx \mathrm{End}(\mathscr{G}') \to \mathbb{C}$ be the determinant map. Then $\det \circ \varphi$ is an $SL(\mathscr{G}')$-invariant polynomial on $\mathscr{G}'^* \otimes \mathscr{G}'^* \otimes \mathscr{G}'$ and $(\det \circ \varphi)(ad)$ is the determinant of the Killing form of $\mathscr{G}'$ and hence is non-zero, $\mathscr{G}'$ being semisimple.

5.5.2. *Lemma. The point* $(x_1, \ldots, x_N) \in Q(Y)^N \subset Q(\bar{Y})^N$ *is semistable if and only if* $(p(x_1), \ldots, p(x_N))$ *is semistable in* $G_{n,r}^N$.

Proof. Let $(p(x_1), \ldots, p(x_N)) = (y_1, \ldots, y_N)$. If $(x_1, \ldots, x_N)$ is semistable then by our choice of $p^*(M)^a \otimes \Lambda^b$, $(y_1, \ldots, y_N)$ is semistable ([22] Proposition 5.1).

Suppose $(y_1, \ldots, y_N)$ is semistable. Let λ be a 1-PS of $SL(n, \mathbb{C})$. Let $P(\lambda)$ be the canonical parabolic subgroup associated to λ (§ 5.4.4). Let $x \in Q(\bar{Y})$ and $y = p(x) \in G_{n,r}$. Then y is an r-dimensional quotient of $\mathbb{C}^n$. Let P_y be the maximal parabolic subgroup of $SL(n, \mathbb{C})$ which is the stabilizer of $y \in G_{n,r}$. By Bruhat's lemma $P_y \cap P(\lambda)$ contains a maximal torus T of $SL(n, \mathbb{C})$. Since $\lambda \subset P(\lambda)$ there is a $g \in P(\lambda)$ such that $g\lambda g^{-1} \subset T$. Then

$$\mu(x, \lambda) = \mu(x, g\lambda g^{-1}), \tag{1}$$

where μ stands for $\mu p^{*(M)^a \otimes \Lambda^b}$ (§ 5.4.4). Since $T \subset P_y$ we can choose a basis $e_1, \ldots, e_r, e_{r+1}, \ldots, e_n$ for $\mathbb{C}^n$ such that the images $\bar{e}_i$, of e_i in y, $i = 1, \ldots, r$ form a basis for y and such that T becomes the group of diagonal matices with respect to $e_1, \ldots, e_n$. Now

$$\mu(x, \lambda) = a\mu^M(y, \lambda) + b\mu^\Lambda(x, \lambda) = a\mu^M(y, g\lambda g^{-1}) + b\mu^\Lambda(x, g\lambda g^{-1}). \tag{2}$$

We shall first calculate $\mu^\Lambda(x, \lambda) = \mu^\Lambda(x, \lambda_g)$ where $\lambda_g = g\lambda g^{-1}$. Note that

$$\mu^\Lambda(x, \lambda_g^r) = r\mu^\Lambda(x, \lambda_g) \tag{3}$$

([22], Proposition 2.1). Let $\lambda_g(z) = z^{a_i} e_i$, $z \in \mathbb{C}^*$, $s_i \in \mathbb{Z}$, $i = 1, \ldots, r$. Define the 1-PS λ' of the center of $GL(y)$ by $\lambda'(z)\bar{e}_i = z^s \bar{e}_i$, $z \in \mathbb{C}^*$, $s = s_1 + \cdots + s_r$.

Since λ_g leaves y invariant it follows easily from definitions (cf. § 5.4.3) that for calculating $\mu^\Lambda(x, \lambda_g)$ we can restrict our attention to y or equivalently the subspace generated by $e_1, \ldots, e_r$. For the action on the quotient space y we have $\lambda_g^r = \lambda' \cdot \lambda''$ where $\lambda''(z)e_i = z^{rs_i - s} e_i$, $z \in \mathbb{C}^*$, $i = 1, \ldots, r$. Note that $\lambda'' \subset SL(y)$. Since λ' is in the center of $GL(y)$ it is easy to see that

$$\mu^\Lambda(x, \lambda_g^r) = \mu^\Lambda(x, \lambda') + \mu^\Lambda(x, \lambda'') \tag{4}$$

and that

$$\mu^\Lambda(x, \lambda') = s. \tag{5}$$

Moreover we get from ([10], Chap. 4, § 4, eq.($*$), p. 67) that

$$\mu^M(y, \lambda_g) = s. \tag{6}$$

Therefore

$$\mu^\Lambda(x, \lambda) = \frac{1}{r}\{\mu^M(y, \lambda_g) + \mu^\Lambda(x, \lambda'')\}$$

$$= \frac{1}{r}\{\mu^M(y, \lambda) + \mu^\Lambda(x, \lambda'')\}$$

with λ'' a 1-PS of $SL(y)$, by (1), (4), (5) and (6). This gives

$$\begin{aligned}\mu(x,\lambda) &= a\mu^{M}(y,\lambda) + b\mu^{\wedge}(x,\lambda)\\ &= \left(a+\frac{b}{r}\right)\mu^{M}(y,\lambda) + \frac{b}{r}\mu^{\wedge}(x,\lambda'').\end{aligned} \tag{7}$$

Writing x_i for x in (7) and summing through i we get

$$\begin{aligned}\mu(x_1,\ldots,x_N),\lambda = \left(a+\frac{b}{r}\right)\mu^{M}((y_1,\ldots,y_N),\lambda)\\ +\frac{b}{r}\cdot\sum_{i=1}^{N}\mu^{\wedge}(x_i,\lambda_i'').\end{aligned} \tag{8}$$

Since $(y_1,\ldots,y_N)$ was assumed to be semistable $\mu^{M}((y_1,\ldots,y_N),\lambda)\geqslant 0$. Since $x_i\in Q(Y)$ it follows from Lemma 5.5.1 that $\mu^{\wedge}(x_i,\lambda_i'')\geqslant 0\ \forall i$ (cf. 5.4.3). Therefore $\mu((x_1,\ldots,x_N),\lambda)\geqslant 0$ which proves $(x_1,\ldots,x_N)$ is semistable (§ 5.4.2).

5.5.3. *Lemma. Under the morphism* $R_1\to Q(Y)^N\subset Q(\bar{Y})^N$ R_1 *maps into the open subscheme* $Q(\bar{Y})^N_{ss}$ *and hence* $R_1\to Q(\bar{Y})^N$ *factors as* $R_1\to Q(\bar{Y})^N_{ss}\hookrightarrow Q(\bar{Y})^N$.

Proof. Since under the morphism $R\to G_{n,r}=Z$, R maps into Z_{ss} this follows immediately from the preceding lemma.

5.6. *Lemma. The injective morphism* $R_1\to Q(\bar{Y})^N_{ss}$ *is proper.*

Proof. We use the valuation criterion. Let A be a discrete valuation ring over $\mathbb{C}$ with residue field $\mathbb{C}$ and quotient field K. Suppose we are given

$$\begin{array}{ccc}\operatorname{Spec}K & \longrightarrow & R_1\\ \big\downarrow & \nearrow{\scriptstyle ?} & \big\downarrow\\ \operatorname{Spec}A & \longrightarrow & Q(\bar{Y})^N_{ss}\end{array} \tag{1}$$

We have to complete the broken arrow.

From (1) and the diagram in § 5.5, using the facts that $p(Q(\bar{Y})^N_{ss})\subset Z_{ss}$ and the morphism $R\to Z_{ss}$ and $S_1\to R$ are proper ([20], § 3, Lemma 2 and Lemma 5.3.1) it follows that we can get a lift $\operatorname{Spec}A\to S_1$ giving the commutative diagram

$$\begin{array}{ccccc}\operatorname{Spec}K & & \longrightarrow & & R_1\\ \big\downarrow & \searrow & & \swarrow & \big\downarrow\\ & & S_1 & & \\ & \nearrow & & \searrow & \\ \operatorname{Spec}A & & \longrightarrow & & Q(\bar{Y})^N_{ss}\end{array}$$

We will be through if we show that the closed point of $\operatorname{Spec}A$ maps into R_1 under this morphism $\operatorname{Spec}A\to S_1\supset R_1$.

Let $V=\mathcal{O}_0(-m)$ be the vector bundle (semistable of degree zero) corresponding to the image of the closed point of $\operatorname{Spec}A$ in R under the composite $\operatorname{Spec}A\to S_1\to R$. The image of the closed point of Spec A in S_1 then gives a section of $Q_0(\bar{Y}) \subset \mathbb{P}(V^* \otimes V^* \otimes V) \to X$ which actually comes from a section $s \in H^0(X, V^* \otimes V^* \otimes V)$ with $s(x) \neq 0\ \forall x \in X$. This section s, since its image is in $Q_0(\bar{Y})$, gives a Lie algebra structure on the fibers of V (cf. Remark 4.13.2). Let $\bar{s} \in H^0(X, V^* \otimes V^*)$ correspond to the Killing form of the Lie algebra structure given by s. We shall show that $\bar{s}$ is nondegenerate on all fibers.

Suppose $\bar{s}$ is not nondegenerate on all fibers. Then the homomorphism $V \to V^*$ induced by $\bar{s}$ has a non-trivial kernel sheaf. Since both V and V^* are semistable vector bundles of degree zero the kernel is actually a subbundle V_1, semistable and of degree zero ([19], Proposition 3.1). Then V_1 is a solvable ideal in V, i.e. the fibers of V_1 are solvable ideals in the fibers of V ([18], Chapter VI, proof of Theorem 2.1). Again since $V_1 \otimes V_1$ and V_1 are semistable vector bundles of degree zero the image $[V_1, V_1]$ of the morphism $V_1 \otimes V_1 \to V_1, x \otimes y \mapsto [x, y]$, given by the Lie bracket operation, is a subbundle V_2 of V_1 of degree zero and semistable. Similarly $V_3 = [V_2, V_2]$ etc. are all semistable vector bundles of degree zero. Since V_1 is solvable we arrive after a certain stage at a non-zero subbundle V', of degree zero and semistable, which is an abelian ideal in V.

The inclusion $V' \otimes L^m \hookrightarrow V \otimes L^m$ induces $W = H^0(X, V' \otimes L^m) \hookrightarrow H^0(X, V \otimes L^m) = I_n$. Let W' be a supplement for W in I_n so that $I_n = W \oplus W'$. Let λ be the 1-PS of $SL(n, \mathbb{C})$ which acts on W by the character $\lambda(z) = z^{\mathrm{rk} W'}$ and on W' by the character $\lambda(z) = z^{-\mathrm{rk} W}$, $z \in \mathbb{C}^*$. Let $(x_1, \ldots, x_N)$ be the image of the closed point of Spec A in $Q(\bar{Y})^N_{\mathrm{ss}}$ and $(y_1, \ldots, y_N) = (p(x_1), \ldots, p(x_N))$. We shall now compute $\mu((x_1, \ldots, x_N), \lambda)$ and show that it is < 0 contradicting semistability of $(x_1, \ldots, x_N)$.

It follows from ([10], Chapter 4, §4, equation $(***)_N$, p. 88; cf. also [19], p. 309) that $\mu^M((y_1, \ldots, y_N), \lambda) = n \cdot \Sigma_{i=1}^N \mathrm{rk}(W_i) - r \cdot N(\mathrm{rk} W)$ where W_i is the image of W in y_i. (In [10] the calculation is made for the Grassmannian of subspaces. It is easy to translate it to the Grassmannian of *quotient* spaces which we need here.) Since $W = H^0(X, V' \otimes L^m)$ generates V' (by our choice of m, cf. §3.11), rk $W_i = \mathrm{rk}(V')\ \forall i$. Therefore

$$\mu^M((y_1, \ldots, y_N), \lambda) = n \cdot N(\mathrm{rk} V') - r \cdot N(\mathrm{rk} W). \tag{1}$$

Applying Riemann–Roch we get

$$\mathrm{rk}\, W = \mathrm{rk}\, H^0(X, V' \otimes L^m) = (\mathrm{rk}\, V')(md_0 + 1 - g)$$

and

$$n = \mathrm{rk}\, H^0(X, V \otimes L^m) = r \cdot (md_0 + 1 - g).$$

Therefore $(\mathrm{rk}\, W)/n = (\mathrm{rk}\, V')/r$. Hence from (1) we have

$$\mu^M((y_1, \ldots, y_N), \lambda) = 0. \tag{2}$$

To calculate $\mu^\wedge((x_1, \ldots, x_N), \lambda)$ let $x = x_i$ and $y = y_i$ and let $g \in P(\lambda)$ such that $\lambda_g = g \lambda g^{-1} \subset P_y$ (cf. Proof of Lemma 5.5.2). Then $\mu^\wedge(x, \lambda) = \mu^\wedge(x, \lambda_g)$. It follows from ([10] Chapter 2, §2, pp. 55–56) that $P(\lambda) = P_W$, the stabilizer of W in $SL(n, \mathbb{C})$. Therefore $I_n = gW \oplus gW' = W \oplus gW'$ and λ_g acts on W by the character $\lambda_g(z) = z^{\mathrm{rk} W'}$ and on gW' by $\lambda_g(z) = z^{-\mathrm{rk} W}$, $z \in \mathbb{C}^*$. Since $\lambda_g \subset P_y$, λ_g leaves invariant $\ker(I_n \to y)$ and hence we can find a set of linearly independent elements $e_1, \ldots, e_q, e_{q+1}, \ldots, e_r$ such that $e_1, \ldots, e_q \in W$ and $e_{q+1}, \ldots, e_r \in gW'$ and $\bar{e}_1, \ldots, \bar{e}_q$ the images of $e_1, \ldots, e_q$ under $I_n \to y$ form a basis for the fiber $V'_{x'} \subset y$ of V' over $x' \in X$ (where y is the fiber of V over x') and $\bar{e}_1, \ldots, \bar{e}_q, \bar{e}_{q+1}, \ldots, \bar{e}_r$ form a basis for y. Then

$$\lambda_g^r(z) \cdot \bar{e}_i = \begin{cases} z^{r(\mathrm{rk} W)} \bar{e}_i & \text{for } 1 \leqslant i \leqslant q \\ z^{-r(\mathrm{rk} W')} \bar{e}_i & \text{for } q+1 \leqslant i \leqslant r. \end{cases}$$

For the action of λ_g on y we then have $\lambda_g^r = \lambda' \cdot \lambda''$ where $\lambda'(z)\bar{e}_i = z^t \bar{e}_i$; $t = q(\mathrm{rk}\, W) - (r-q)\mathrm{rk}\, W'$ and

$$\lambda''(z)\bar{e}_i = \begin{cases} z^{r(\mathrm{rk}W)-t}\bar{e}_i = z^{(r-q)\cdot n}\bar{e}_i, & 1 \leqslant i \leqslant q \\ z^{-r(\mathrm{rk}W')-t}\bar{e}_i = z^{-qn}\bar{e}_i, & q+1 \leqslant i \leqslant r. \end{cases}$$

As in Lemma 5.5.2, eq. (6)

$$\mu^{\Lambda}(x, \lambda') = \mu^{M}(y, \lambda_g) = \mu^{M}(y, \lambda).$$

To calculate $\mu^{\Lambda}(x, \lambda'')$ we shall use ([10] Proposition 2.3; cf. also [22], §2). Let $\tilde{x}$ be a point in $y^* \otimes y^* \otimes y$ which lies above $x \in Q_y(\bar{Y}) \subset \mathbb{P}(y^* \otimes y^* \otimes y)$. Let

$$\tilde{x} = \sum_{i,j,k=1}^{r} x_{ijk}\, \bar{e}_i^* \otimes \bar{e}_j^* \otimes \bar{e}_k,$$

where $\bar{e}_i^*$ form the dual basis to $\bar{e}_i$. If we think of $\tilde{x}$ as a Lie algebra structure on y then the x_{ijk} are the 'structure constants' and we have

$$[\bar{e}_i, \bar{e}_j] = \sum_{k=1}^{r} x_{ijk}\bar{e}_k.$$

From the fact that $\bar{e}_1, \ldots, \bar{e}_q$ span an abelian ideal in y we get that $x_{ijk} = 0$ whenever i, j, k satisfy any one of the following three conditions: (1) both i and j are $\leqslant q$ and k arbitrary (abelian) (2) $i \leqslant q, j$ arbitrary and $k \geqslant q+1$ (ideal) (3) i arbitrary, $j \leqslant q$ and $k \geqslant q+1$ (symmetric to (2)). Therefore x_{ijk} may not be zero only in the following cases:

Case i. $i \leqslant q, j \geqslant q+1$ and $k \leqslant q$.
Case ii. $i \geqslant q+1, j \leqslant q$ and $k \leqslant q$.
Case iiia. $i \geqslant q+1, j \geqslant q+1$ and $k \leqslant q$.
Case iiib. $i \geqslant q+1, j \geqslant q+1$ and $k \geqslant q+1$.

Let

$$\lambda''(z)(\bar{e}_i^* \otimes \bar{e}_j^* \otimes \bar{e}_k) = z^{a_{ijk}} \cdot \bar{e}_i^* \otimes \bar{e}_j^* \otimes \bar{e}_k, a_{ijk} \in \mathbb{Z}.$$

Then it is easy to see that $a_{ijk} = q$ in cases (i), (ii), and (iiib) and $a_{ijk} = q + r$ in case (iiia). Therefore in every case when $x_{ijk} \neq 0, \lambda''(z)$ acts by a strictly positive power, viz. q or $q+r$, of z. It follows by ([10], Proposition 2.3) that $\mu^{\Lambda}(x, \lambda'') < 0$. Now using eq. (8) of Lemma 5.5.2 and (2) above, we get $\mu((x_1, \ldots, x_N), \lambda) < 0$ contradicting the semistability of $(x_1, \ldots, x_N)$. Therefore we conclude that the Killing form $\bar{e}$ must be nondegenerate on all fibers. Then the Lie algebra structure of all fibers is semisimple. But the Lie algebra structure of a semisimple Lie algebra is (locally) rigid, i.e., interpreting $\bar{Y}$ as Lie algebra structures on y, if $x \in \bar{Y}$ gives a semisimple Lie algebra structure on y then $x \in Y$ ([16], §§3,4, Corollary 4.3, pp. 413–415). This shows that the image of s lies in $Q_0(Y)$. Therefore the image of the closed point of Spec A under Spec $A \to S_1$ lies in R_1 as was to be shown.

5.7. *Lemma. Let* $\pi: R_3 \to Q(\bar{Y})_{ss}^N$ *be the composite* $R_3 \xrightarrow{\pi_3} J_\tau \times R_2 \to R_2 \xrightarrow{\pi_2} R_1 \to Q(\bar{Y})_{ss}^N$. *Then for* $r_3 \in R_3$, $\pi(r_3)$ *is a stable point of* $Q(\bar{Y})_{ss}^N$ *if and only if the G-bundle* $E \to X$ *corresponding to* r_3 *is stable.*

Proof. Let $\pi_3(r_3)=(j,r_2)$. Then the Ad G-bundle E' corresponding to $r_2\in R_2$ is obtained from the G-bundle E by the extension of structure group $G\to \operatorname{Ad} G = G/Z$. Therefore E' is stable if and only if E is stable ([14], Proposition 7.1, p. 146).

Then point $\pi(r_3)$ in $Q(\bar{Y})^N_{ss}$ is stable if and only if the orbit of $\pi(r_3)$ under $SL(n,\mathbb{C})$ is closed and the isotropy of $\pi(r_3)$ in $SL(n,\mathbb{C})$ is finite ([10], Amplification 1.11; [12], § 2, Theorem 2(a), p. 193).

The morphism $R_2\to Q(\bar{Y})^N_{ss}$ is an $SL(n,\mathbb{C})$-equivariant finite morphism. Moreover it is easy to see that if two Ad G-bundles E'_1 and E'_2 give rise to isomorphic Aut $\mathscr{G}'$-bundles under the extension of structure group Ad $G \hookrightarrow$ Aut $\mathscr{G}'$ then E'_1 is stable if and only if E'_2 is stable. It then follows that it is enough to show that the Ad G-bundle E' is stable if and only if the $SL(n,\mathbb{C})$ orbit of r_2 in R_2 is closed and the isotropy of r_2 in $SL(n,\mathbb{C})$ is finite.

Suppose E' is a stable Ad G-bundle. Then gr $E' = E'$ (Proposition 3.12) and it follows from Proposition 3.24(i) that the $SL(n,\mathbb{C})$ orbit of r_2 is closed (cf. proof of Proposition 4.5). Since the group Aut E' of Ad G-bundle automorphisms of E' is finite ([14], Proposition 3.2, p. 136) it follows from Remark 4.7 that the isotropy of r_2 in $SL(n,\mathbb{C})$ is finite.

Conversely suppose that the $SL(n,\mathbb{C})$ orbit of r_2 is closed and the isotropy of r_2 finite. From Proposition 3.24(ii) it follows that the closure of the orbit of r_2 always contains a point r'_2 which is isomorphic to gr E'(cf. proof of Proposition 3.5). Therefore if the orbit of r_2 is closed then $E'\approx$ gr E'. By Proposition 3.15 this implies that E' is a unitary bundle E'_ρ corresponding to a unitary representation $\rho:\pi_1(X)\to\bar{K}$, where $\bar{K}$ is a maximal compact subgroup of Ad G. If ρ is not irreducible then there is a subgroup $S\subset\bar{K}$, with dim $S>0$, which commutes with ρ ([14], Definition 1.2, cf. also § 2, p. 131). Then the group S gives rise to a group of automorphisms of $E'_\rho = E'$ of dimension >0. This contradicts the finiteness of the isotropy at r_2 (Remark 4.7). Therefore ρ is irreducible. Then $E'=E'_\rho$ is stable by ([14], Proposition 2.2, p. 133).

5.8. PROPOSITION

Let $\mathscr{E}\to S\times X$ be an arbitrary family of G-bundles on X parametrized by a scheme S. Then the set S_{ss} (resp. S_s) of points $s\in S$ such that $\mathscr{E}_s$ is semistable (resp. stable) is an open subset of S.

Proof. Since a G-bundle E is semistable (resp. stable) if and only if the Ad G-bundle E' obtained from E by the extension of structure group $G\to$ Ad G is so ([14], Proposition 7.1, p. 147), we can assume that $G=$ Ad G. Let $\mathscr{E}(\mathscr{G})$ be the vector bundle associated to $\mathscr{E}$ by the adjoint representation $G\hookrightarrow GL(\mathscr{G})$. Since the question is local on S we can assume that for $m\gg 0$, $\mathscr{E}(\mathscr{G})\otimes p^*_X(L^m)$ is a quotient of I_n for a suitable n, $I_n\to\mathscr{E}(\mathscr{G})\otimes p^*_X(L^m)\to 0$. This then gives a morphism $f:S\times X\to G_{n,r}$ the Grassmannian of r dimensional quotients of I_n and the G-bundle $\mathscr{E}\to S\times X$ gives a morphism $\tilde{f}:S\times X\to Q(Y)\subset Q(\bar{Y})$ in the obvious way. By choosing an N-tuple $(x_1,\ldots,x_N)$ of points of X we get

$$\begin{array}{ccc} & & Q(Y)^N\subset Q(\bar{Y})^N \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \\ S & \xrightarrow{\quad f\quad} & G_{n,r} \end{array}$$

We can choose the N-tuple $(x_1,\ldots,x_N)$, $N \gg 0$, such that $f(s)$ is a semistable (resp. stable) point of $G^N_{n,r}$ if and only if $\mathscr{E}_s(\mathscr{G})$ is a semistable (resp. stable) vector bundle ([19], Theorem 7.1(3), also Corollary 7.2). It follows then from Corollary 3.18 and Lemma 5.5.2 that the set S_{ss} is $\bar{f}^{-1}(Q(\bar{Y})^N_{ss})$. Since $Q(\bar{Y})^N_{ss}$ is open in $Q(\bar{Y})^N$ this proves that S_{ss} is open is S.

Again making S smaller if necessary we can assume that the family $\mathscr{E}|S_{ss}$ is induced by a morphism $S_{ss} \to R_2$. It follows from Lemma 5.7 that the points in R_2 corresponding to stable bundles is open since it is the inverse image of the open subset $Q(\bar{Y})^N_s$ of $Q(\bar{Y})^N_{ss}$ under the morphism $R_2 \to Q(\bar{Y})^N_{ss}$. Therefore S_s is open in S.

5.9. *Theorem. The functor* $\bar{\mathbf{F}}^\tau_{ss}$ (*Definition* 3.9) *has a coarse moduli scheme* M^τ. *The scheme* M^τ *is irreducible, projective, normal and Cohen-Macaulay. The dimension of* M^τ *is* $\dim Z + (g-1)\cdot\dim G$. *The set* M^τ_s *of points in* M^τ *corresponding to stable G-bundles is an open* (*and hence dense*) *subset of* M^τ.

Proof. By Propositions 4.5 and 4.15.2 it follows that to prove the existence of a coarse moduli scheme for $\mathbf{F}^\tau_{ss}$ it is enough to prove that a good quotient of R_3 modulo $GL(n,\mathbb{C})$ exists. Since $R_3 \to J_\tau \times R_2$, $R_2 \to R_1$ and $R_1 \to Q(\bar{Y})^N_{ss}$ are all finite $GL(n,\mathbb{C})$-equivariant morphisms it follows from Lemma 5.1 that a good quotient of R_3 modulo $GL(n,\mathbb{C})$ exists if a good quotient of $Q(\bar{Y})^N_{ss}$ modulo $GL(n,\mathbb{C})$ (or $SL(n,\mathbb{C})$, since the scalars act trivially on $Q(\bar{Y})^N_{ss}$) exists. We know that a good quotient of $Q(\bar{Y})^N_{ss}$ exists and is projective ([10], Theorem 1.10; [22], Theorem 1.1(B)). Therefore $\bar{\mathbf{F}}^\tau_{ss}$ has a coarse moduli scheme M^τ. Moreover since $R_3 \to J_\tau \times R_2$ etc. are finite morphisms it follows from Lemma 5.1 that M^τ is projective (noting that a scheme finite over a projective scheme is projective).

Since M^τ is a categorical quotient of the non-singular (and hence normal) scheme R_3 it follows that M^τ is normal ([10], Chapter 0, §2, p. 5). Since $q: R_3 \to M^\tau$ is a good quotient R_3 can be covered by G-invariant affine open subsets Spec A_i such that the corresponding quotients Spec A_i^G form an affine open covering for M^τ.

Since R_3 is nonsingular the $\mathbb{C}$-algebras A_i are regular and hence by ([8], Main theorem) it follows that A_i^G are Cohen-Macaulay. Therefore M^τ is Cohen-Macaulay.

It follows easily from § 5.4.1 and Lemmas 5.7 and 5.1 that M^τ_s is open in M^τ and $q: R^s_3 = q^{-1}(M^\tau_s) \to M^\tau_s$ is a geometric quotient.

Since $R^s_3 \to M^\tau_s$ is a geometric quotient modulo $SL(n,\mathbb{C})$ we have $\dim M^\tau = \dim R_3 - \dim SL(n,\mathbb{C})$. Since $R_3 \to J_\tau \times R_2$ and $R_2 \to R_1$ are étale and finite $\dim R_3 = \dim J_\tau + \dim R_1$. Therefore using Lemma 4.13.4, $\dim M^\tau = (g\cdot\dim Z) + (n^2 + (r+1)(g-1) - g) - (n^2 - 1)$. Noting that $r = \dim G - \dim Z$, we get $\dim M^\tau = \dim Z + (g-1)\dim G$.

We need the following lemma to complete the proof of the theorem.

5.9.1. *Lemma. Let S be a complex analytic space and* $\mathscr{E} \to S \times X$ *be a complex analytic family of semistable G-bundles of topological type* τ *parametrized by S. Then there is an analytic morphism* $f_{\mathscr{E}}: S \to M^\tau$ *such that for any* $s \in S$, $f_{\mathscr{E}}(s)$ *is the equivalence class of* $\mathscr{E}_s$.

Proof. This follows easily from the fact that the functors **Pic** and '$\Gamma(\rho,\mathscr{E})$' etc. used in our construction of universal families are representable in the analytic category also and are represented by the same universal spaces as in the algebraic category (cf. [5]).

We will now continue with the proof of Theorem 5.9. Since we have shown that M^τ is normal, to show that it is irreducible it is enough to show that it is connected. Let

$m_1, m_2 \in M^\tau$ and let E_1, E_2 be semistable G-bundles belonging to the equivalence classes m_1, m_2 respectively. Then by ([14], Proposition 4.2 p. 142) there is a complex analytic family of semistable G-bundles $\mathscr{E} \to S \times X$ parametrized by an open *connected* subspace S of the complex plane $\mathbb{C}$ such that for some $s_1, s_2 \in S$ we have $\mathscr{E}_{s_i} = E_i, i = 1, 2$. (In [14], Proposition 4.2 this is stated only for stable bundles but the same proof goes through for semistable bundles also, noting that the proof of ([14], Proposition 4.1, p. 138) with a little modification gives that for a complex analytic family $\mathscr{F} \to S \times X$ the set of $s \in S$ such that $\mathscr{F}_s$ is *semi*stable is an analytic open subset of S.) Now applying Lemma 5.9.1 we get that M^τ is connected and hence irreducible.

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