Moduli for principal bundles over algebraic curves: II

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Abstract. We classify principal bundles on a compact Riemann surface. A moduli space for semistable principal bundles with a reductive structure group is constructed using Mumford's geometric invariant theory.

Keywords. Principal bundles; compact Riemann surface; geometric invariant theory; reductive algebraic groups.

4. Reduction to a quotient space problem

In this section we reduce the problem of constructing coarse moduli schemes for the functors $F^r_s$ to one of proving the existence of a good quotient of certain universal spaces under the action of a full linear group.

We recall the definition of a good quotient ([22], Definition 1.5, p. 516).

4.1. Definition

Let $\alpha: H \times T \to T$ be an action of the algebraic group $H$ on the scheme $T$. A morphism $p: T \to Y$ is called a good quotient of $T$ modulo $H$ if the conditions (i), (ii) and (iii) below are satisfied.

i) $p$ is surjective, affine and $H$-invariant.

ii) $p_*(O_T^H) = O_Y$, where $O_T^H$ is the sheaf of $H$-invariant functions on $T$.

iii) If $Z$ is a closed $H$-stable subset of $T$ then $p(Z)$ is closed in $Y$; further if $Z_1$, $Z_2$ are two closed $H$-stable subsets of $T$ such that $Z_1 \cap Z_2 = \emptyset$, then $p(Z_1) \cap p(Z_2) = \emptyset$.

If in addition the condition (iv) below is also satisfied we call $p: T \to Y$ a geometric quotient.

iv) $p(x_1) = p(x_2) \iff$ orbit of $x_1 = $ orbit of $x_2$ (or equivalently, in view of (iii), all orbits are closed).

4.2. Remark. A good quotient is a categorical quotient, i.e. given any $H$-invariant morphism $f: T \to Z$ there is a unique morphism $\overline{f}: Y \to Z$ such that $f = \overline{f} \circ p$ ([22], p. 516).

4.3. Notation. Let $\alpha: H \times T \to T$ be an action of the algebraic group $H$ on the scheme $T$. Then for morphisms $h: S \to H$ and $\iota: S \to T$ we denote by $h[\iota]$ the composite $S ^{h[\iota]} H \times T \to H \times T$. For any morphism $f: S_1 \to S_2$ we denote by $\overline{f}$ the product $f \times \text{id}_S: S_1 \times X \to S_2 \times X$.

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If $E$ is a bundle (or more generally a scheme) over a scheme $M$ and $m:S \to M$ is a morphism we denote by $E_m$ the pull back $m^*:E$. For $E' \to M \times X$ we write $E'_m$ instead of $E'_m$.

4.4. DEFINITION

Let $S$ be a set of isomorphism classes of $G$-bundles on $X$. Let $\tilde{F}^S$ be the sheaf associated to the functor $F^S:(\text{Sch}) \to (\text{Sets})$ which associates to a scheme $T$ the set of isomorphism classes of families of $G$-bundles in $S$ parametrized by $T$. On morphisms $F^S$ is defined to be pulling back. Let $M$ be a scheme and $H$ an algebraic group acting on $M$ by $\alpha:H \times M \to M$. Let $H \setminus M$ be the sheaf associated to the presheaf $H \setminus M(T) = \text{quotient set} \text{Hom}(T,H) \setminus \text{Hom}(T,M)$. We call $M$ a universal space with group $H$ for the set $S$ if there is an isomorphism of sheaves $\Phi: \tilde{F}^S \to H \setminus M$.

4.5. PROPOSITION

Let $S'$ be the set of isomorphism classes of semistable $G$-bundles of topological type $\tau$. Suppose there is a universal space $M$ with group $H$ for the set $S'$ and $\Phi: \tilde{F}^S \to H \setminus M$ is the isomorphism of sheaves. Then a good quotient of $M$ modulo $H$, if it exists, gives a coarse moduli scheme for the functor $\tilde{F}^S$ (see Definitions 3.2 and 3.9 in Part I) in a natural way.

Proof. Suppose $\pi:M \to Y$ is a good quotient of $M$ modulo $H$. Clearly $F^S = F^S$ (see Definition 3.1). Therefore we have a morphism $\Phi: \tilde{F}^S \to H \setminus M$. Let $h_M, h_Y$ be the functors represented by $M, Y$ respectively. The morphism $h_Y \to h_Y$ induced by $\pi:M \to Y$ gives rise to a morphism $\Psi: H \setminus M \to h_Y$ because of the $H$-invariance of $\pi$. We claim that the morphism $\eta = \Psi \circ \Phi: \tilde{F}^S \to h_Y$ goes down to a morphism $\bar{\eta}: \tilde{F}^S \to h_Y$ making $Y$ the coarse moduli scheme for $\tilde{F}^S$.

Suppose the family $\mathcal{F} \to S \times X$ in $S'$ has an admissible reduction of structure group to $P = M \cdot U$. Then by Proposition 3.5 we have a family $\mathcal{F} \to (C \times S) \times X$ in $S'$ such that $p_S(\mathcal{F})|_{C \times S} \times X$ is isomorphic to $\mathcal{F} \to S \times X$. Therefore $\eta_S(\mathcal{F}) = C \times S \to Y$ coincides with $\eta_S(\mathcal{F}) = C \times S \to Y$. It follows that $\eta$ goes down to a morphism $\bar{\eta}: \tilde{F}^S \to h_Y$.

That $\bar{\eta}: \tilde{F}^S(\text{Spec} C) \to \text{Hom}(\text{Spec} C, Y)$ is surjective follows from the fact that $\pi:M \to Y$ is surjective. To check injectivity we only have to show that if $E_1$ and $E_2$ are two semistable $G$-bundles of type $\tau$ on $X$ (considered as a family parametrized by $\text{Spec} C$) such that $\eta(E_1) = \eta(E_2)$ then $E_1$ and $E_2$ are equivalent. Let the point $m_i \in M$ represent $\Phi_c(E_i)$. Then by the property (iii) in the definition of a good quotient (Definition 4.1), $\overline{C}_1$ and $\overline{C}_2$, the closures of the $H$-orbits $C_1$ and $C_2$ of $m_1$ and $m_2$ respectively, intersect in $M$. Let $m \in C_1 \cap C_2$. We take the canonical reduced scheme structures on $C_i$ and $\overline{C}_i, i = 1, 2$. Let $[id_M] \in H \setminus M$ be the class of the identity morphism of $M$. The element $\Phi^{-1}([id_M]) \in \tilde{F}^S(M)$ then gives for some neighbourhood $U$ of $m \in M$ a faithfully flat morphism $f:U' \to U$ of schemes and a family of $G$-bundles $\mathcal{F} \to U' \times X$ in $S'$. Let $C_i = f^{-1}(C_i)$ and $\overline{C}_i = f^{-1}(\overline{C}_i)$. Since $f$ is faithfully flat, $C_i$ is dense open in $\overline{C}_i$. Since $\Phi$ is a morphism it follows easily that for the family $\mathcal{F}|_{\overline{C}_i} \to \overline{C}_i \times X, \mathcal{F}_x \approx E_x \forall x \in C_i$. Therefore by Proposition 3.24(i) $\mathcal{F}_m$ is equivalent to $E_1 \forall m \in C_i$, and in particular for $m' \in C_i$ such that $f(m') = m$. This proves that $E_1$ and $E_2$ are equivalent.
To verify the condition (ii) for coarse moduli scheme (Definition 3.2) suppose $Z$ is a scheme and $\chi: \tilde{\mathcal{F}}_{ss} \to h_{\gamma}$ a morphism. Then it is easy to see that corresponding to $\Phi_{M}^{-1}([id_{M}])$ the morphism $\chi$ gives a morphism $g:M \to Z$ which is $H$-invariant. Then $Y$ being the categorical quotient of $M$ modulo $H$(Remark 3.2), $g$ induces $\tilde{g}: Y \to Z$ such that $g = \tilde{g} \circ \pi$. If $h_{\gamma}: h_{\gamma} \to h_{\xi}$ is the corresponding morphism of functors it follows that $h_{\gamma}$ is the unique morphism which satisfies $\chi = h_{\gamma} \circ \tilde{g}$.

This proposition reduces the problem of constructing coarse moduli schemes for $\tilde{\mathcal{F}}_{ss}$ to one of constructing suitable universal spaces and then proving the existence of quotients. To construct universal spaces for $G$-bundles we will start with the universal spaces for vector bundles provided by the Quot schemes ([19], §6). These spaces have a stronger universal property (which we have formulated as a definition; see Definition 4.6 below) which is essential for our construction. By taking an embedding of $G$ in some $GL(n, \mathbb{C})$ we will consider a $G$-bundle as a vector bundle (or $GL(n, \mathbb{C})$-bundle) with a reduction of structure group to $G$ and thus will construct universal spaces for $G$-bundles as schemes over the universal spaces for $GL(n, \mathbb{C})$-bundles.

4.6. DEFINITION

Let $\mathcal{S}$ be a set of isomorphism classes of $G$-bundles on $X$. Let $\mathcal{E} \to T \times X$ be a family of $G$-bundles in $\mathcal{S}$. Suppose an algebraic group $H$ acts on $T$ by $\alpha: H \times T \to T$ and also on $\mathcal{E}$ as a group of $G$-bundle isomorphisms compatible with $\alpha$, we have the commutative diagram

$$
\begin{array}{ccc}
H \times \mathcal{E} & \xrightarrow{\bar{\alpha}} & \mathcal{E}^* (\mathcal{E}) \\
\downarrow & & \downarrow \\
H \times T \times X & & \\
\end{array}
$$

where $\bar{\alpha} = \alpha \times id_{\mathcal{E}}$ (cf. 4.3). We call $\mathcal{E} \to T \times X$ a universal family with group $H$ for the set $\mathcal{S}$ if the following conditions hold.

i) Given any family of $G$-bundles $\mathcal{F} \to S \times X$ in $\mathcal{S}$ and a point $s_{0} \in S$ there exists an open neighbourhood $U$ of $s_{0}$ in $S$ and a morphism $t: U \to T$ such that $\mathcal{F}|_{U \times X} \simeq \mathcal{E}$ (cf. §4.3 for notation).

ii) Given two morphisms $t_{1}, t_{2}: S \to T$ and an isomorphism $\varphi: \mathcal{E}_{t_{1}} \simeq \mathcal{E}_{t_{2}}$ there exists a unique morphism $h: S \to H$ such that $t_{2} = h[t_{1}]$ and $\varphi = (h \times t_{1})^* (\alpha)$ (noting that $(h \times t_{1})^* (H \times \mathcal{E}) = \mathcal{E}_{t_{1}}$ and, since $t_{2} = h[t_{1}], (h \times t_{1})^* (\mathcal{E}_{*} \mathcal{E}) = \mathcal{E}_{t_{2}}$).

4.7. Remark. The condition (ii) in particular implies that the isotropy group $H_{x}$ at $x \in T$ is precisely the automorphism group of $\mathcal{E}_{x}$.

4.7.1. Remark. If $\mathcal{E} \to T \times X$ is a universal family with group $H$ for $\mathcal{S}$ it is clear that $T$ is a universal space with group $H$ for $\mathcal{S}$.

4.8. Let $A, A'$ be two algebraic groups and $\rho: A' \to A$ a homomorphism. Let $\mathcal{E} \to T \times X$ be a family of $A$ bundles. Let $\Gamma(\rho, \mathcal{E}): (Sch/T) \to (Sets)$ be the functor defined by
$\Gamma(\rho, \mathcal{E})(\tau: S \to T) =$ the set of isomorphism classes of pairs $(\mathcal{E}', \varphi)$ where $\mathcal{E}' \to S \times X$ is an $A'$ bundle and $\varphi: \rho_* \mathcal{E}' \to \mathcal{E}$ is an isomorphism of $A$ bundles. The pair $(\mathcal{E}', \varphi_1)$ is isomorphic to the pair $(\mathcal{E}'_2, \varphi_2)$ if there is an $A'$ bundle isomorphism $\psi: \mathcal{E}'_1 \to \mathcal{E}'_2$ such that the diagram

\[ \begin{array}{ccc}
\rho_* \mathcal{E}'_1 & \xrightarrow{\rho_* \psi} & \rho_* \mathcal{E}'_2 \\
\varphi_1 \downarrow & & \downarrow \varphi_2 \\
\mathcal{E} & & \mathcal{E}
\end{array} \]

commutes. Note that if $\rho$ is injective such a $\psi$, if it exists, is unique for, since $A'$ acts faithfully on $A$ $\rho_* \psi = \varphi_2^{-1} \circ \varphi_1$ uniquely determines $\psi$. On morphisms $\Gamma(\rho, \mathcal{E})$ is defined as pulling back.

Let $\tau$ be a topological $A'$ bundle on $X$. Let $\Gamma^i(\rho, \mathcal{E})$ be the subfunctor of $\Gamma(\rho, \mathcal{E})$ defined by

$$\Gamma^i(\rho, \mathcal{E})(S) = \left\{ (\mathcal{E}', \varphi) \in \Gamma(\rho, \mathcal{E})(S) \mid \mathcal{E}'_S \text{ is topologically isomorphic to } \tau \forall s \in S \right\}$$

We then have the following lemma (cf. [17], Proposition 9, §3-5, p. 18).

4.8.1. Lemma. If $\rho: A' \to A$ is injective the functor $\Gamma(\rho, \mathcal{E})$ is representable by a $T$-scheme $T' \to T$ of locally finite type and a universal pair $(\mathcal{U}, u) \in \Gamma(\rho, \mathcal{E})(T')$. The functor $\Gamma^i(\rho, \mathcal{E})$ is representable by an algebraic subscheme $T'$ of $T'$ and the restriction of $(\mathcal{U}, u)$ to $T'$.

Proof. Since $\rho$ is injective we identify $A'$ with its image in $A$. Let $\Gamma = \Gamma(\rho, \mathcal{E})$.

Let $\Gamma: (\text{Sch}/T) \to (\text{Sets})$ be the functor such that $\Gamma((f: S \to T)) = \text{Hom}_{S \times X}(S \times X, \mathcal{E}/A) \to \text{Hom}_{T \times X}(S \times X, \mathcal{E}/A)$. We define a morphism of functors $\Phi: \Gamma \to \Gamma$ as follows.

Let $\sigma \in \Gamma(S) = \text{Hom}_{S \times X}(S \times X, \mathcal{E}/A)$. Define $\Phi_S(\sigma) = (\sigma \cdot \mathcal{E}/f, \varphi_2)$ where $\varphi_2: \rho_* \mathcal{E} \to \mathcal{E}_f$ is induced by $(S \times X)_{\rho_* \mathcal{E}/A} \times \mathcal{E}/f \to (S \times X, e) \to e.a$ where $s \in S, x \in X, e \in \mathcal{E}$, and $a \in A$.

We can also define an inverse morphism $\Psi: \Gamma \to \Gamma$. Let $(\mathcal{E}', \varphi) \in \Gamma((f: S \to T))$. Then the fiber bundle associated to $\rho_* \mathcal{E}'$ with fiber $A/A'$ is canonically isomorphic to the fiber bundle associated to $\mathcal{E}$ with fiber $A/A'$. Since $A'$ leaves the coset $(A')$ of $A/A'$ invariant we have a canonical section $\sigma$ of $\rho_* \mathcal{E}'/A'$. Using $\varphi$ this gives a section, again denoted by $\sigma$, of $\mathcal{E}'/A'$. Define $\Psi_S((\mathcal{E}', \varphi)) = \sigma$.

It is easy to check that $\Phi \circ \Psi = \text{id}_{\tau}$ and $\Psi \circ \Phi = \text{id}_{\Gamma}$. Thus the functors $\Gamma$ and $\Gamma'$ are isomorphic. We shall show that the functor $\Gamma'$ is representable using the results of ([TDTE, IV]).

By Chevalley's semi-invariants theorem ([2], Theorem 5.1, p. 161) there is a representation of $A$ on a vector space $V$ with a line $l \subset V$ such that $A'$ is the stabilizer of $l$ in $A$. Let $\chi^{-1}$ be the character by which $A'$ acts on $l$. Then the line bundle $L$ on $A'/A'$ associated to the $A'$-bundle $A \to A/A'$ is the ample line bundle corresponding to the embedding of $A/A'$ in $\mathbb{P}(V)$. Then the line bundle $L'$ on $\mathcal{E}/A'$ associated to the $A'$-bundle $\mathcal{E} \to \mathcal{E}/A'$ by
the character $\chi$ is relatively ample for the morphism $\mathcal{E}/A' \to T \times X$, for it corresponds to the embedding $\mathcal{E}/A' \subset \mathcal{P}((\mathcal{E}(V))$ induced by $A/A' \subset \mathcal{P}(V)$. Therefore, we see that $\mathcal{E}/A'$ is quasi-projective over $T \times X$ and hence $\mathcal{E}/A' \to T$ is also quasi-projective. Therefore it follows from ([TDTE, IV] §4, C, pp. 19–20) that $\Gamma'$ is representable by a scheme $T'' (= \Pi_{T \times X_T}(\mathcal{E}/A')/T' \times X$, in the notation of ([TDTE, II], C, n° 2, pp. 12, 13)) of locally finite type. In fact $T''$ is an open subscheme of $\text{Hilb}_{\mathcal{E}/A'/T}$ whose closed points correspond to subschemes of $\mathcal{E}/A'$ which map isomorphically onto $t \times X$, for some $t \in T$, under the projection $\mathcal{E}/A' \to T \times X$ (loc. cit.). Therefore if a section $\sigma: t \times X \to \mathcal{E}/A'$ in $\Gamma(t \subset T)$ is such that the $A'$-bundle $\sigma^*(\mathcal{E})$ on $t \times X \cong X$ is of topological type $\tau$ then the Hilbert polynomial of the subscheme $\sigma(X) \cong X$ of $\mathcal{E}/A'$ corresponding to the section $\sigma$ (with respect to the ample line bundle $\mathcal{L}$ on $\mathcal{E}/A'$) is determined by $\tau$ since the restriction of $\mathcal{L}$ to $\sigma(X) \cong X$ is topologically isomorphic to $\mathcal{O}_{\mathcal{E}/A'}(\tau)$ so that its degree depends only on $\tau$. Since subschemes with a fixed Hilbert polynomial are represented by an algebraic subscheme of $\text{Hilb}$ ([TDTE, IV] pp. 17, 20) it follows that $\Gamma'$ is represented by an algebraic subscheme $T'$ of $T''$.

4.8.2. Remark. If $A'$ and $A$ are reductive groups then $A/A'$ is affine and we can take a representation of $A$ in $V$ such that the character $\chi$ is trivial, so that $A/A'$ is embedded in $V$ itself. In this case, therefore, it follows that $T''$ is itself already algebraic.

4.8.3. Remark. If $P$ is a parabolic subgroup of $G$ and $\mathcal{E} \to Y$ is a $G$-bundle on a scheme $Y$ then $\mathcal{E}/P \to Y$ is a projective morphism. Since $G/P$ is projective, this follows as in the proof of the lemma above (taking $A = G$ and $A' = P$).

4.9. If $\rho$ is not an injection the functor $\Gamma(\rho, \mathcal{E})$ may not be a sheaf. Let $\tilde{\Gamma}(\rho, \mathcal{E})$ be the sheaf associated to the functor $\Gamma(\rho, \mathcal{E})$. The following lemma shows that we can construct a universal space for $A'$-bundles starting from a universal family $\mathcal{E}$ of $A$-bundles when $\tilde{\Gamma}(\rho, \mathcal{E})$ is representable and if $\Gamma(\rho, \mathcal{E})$ itself is representable, then we can actually construct a universal family for $A'$-bundles. So taking an embedding $G \subset GL(n, \mathbb{C})$ and starting with a universal family for vector bundles we can construct a universal family for $G$-bundles. But then to prove the existence of coarse moduli scheme for $\mathcal{E}$ we have to prove the existence of a good quotient of the parameter scheme. For this it is convenient to take the adjoint representation. The existence of a good quotient reduces to proving that a certain morphism is proper and if we take the adjoint representation this follows from the (local) rigidity of the Lie algebra structure of a semisimple Lie algebra (see §5 below). But the adjoint representation is not faithful and hence we construct universal families in two steps, first from vector bundles to $G/Z$-bundles and then from $G/Z$-bundles to $G$-bundles. This involves the representability of the functor $\Gamma(\rho, \mathcal{E})$ where (essentially) $\rho$ is the projection $G \to G/Z$. But this functor is not a sheaf (e.g. $\mathbb{C}^* \to 1 = \mathbb{C}^*/\mathbb{C}^*$, cf. ([TDTE, V, §1])) and we are forced to take the associated sheaf $\tilde{\Gamma}(\rho, \mathcal{E})$ which we can prove to be representable by identifying it with a suitable Picard functor (Lemma 4.15.1) if $\tilde{\Gamma}(\rho, \mathcal{E})$ alone is representable we can construct only a universal space for $G$-bundles even starting from a universal family for $G/Z$-bundles. But by Proposition 4.5 this is enough to prove the existence of a coarse moduli scheme for $\mathcal{E}$.

4.10. Lemma. Suppose the family $\mathcal{E} \to T \times X$ is a universal family with group $H$ for a set $\mathcal{F}$ of $A$-bundles. Also suppose that the sheaf $\Gamma(\rho, \mathcal{E})$ is representable by a scheme $M$. 

(i) The group \( H \) can be made to act on \( M \) in a natural way and \( M \) with this action of \( H \) then becomes a universal space with group \( H \) for the set \( \mathcal{S} \) of \( A' \)-bundles which give \( A \)-bundles in \( \mathcal{S} \) on extending the structure group by \( \rho : A' \to A \).

(ii) Moreover if \( \rho : A' \to A \) is an injection so that \( \Gamma(\rho, \mathcal{S}) \) itself is representable and we have a universal pair \((\mathcal{U}, \underline{a}) \in \Gamma(\rho, \mathcal{S})(M)\) (Lemma 4.8.1.1) the group \( H \) can be made to act in a natural way, on \( \mathcal{U} \) as a group of \( A' \)-bundle isomorphisms compatible with its action on \( M \) (ii) above. With this action of \( H, \mathcal{U} \) then becomes a universal family with group \( H \) for \( \mathcal{S}' \).

**Proof.** (i) To give the action of \( H \) on \( M \) we describe the action of \( \text{Hom}(S, H) \) on \( \text{Hom}(S, M) \) for any scheme \( S \). Let \( h \in \text{Hom}(S, H) \) and \( m \in \text{Hom}(S, M) \). Let \( \pi : M \to T \) be the structural morphism and \( t = \pi \circ m \). Since \( M \) represents the sheaf \( \tilde{\Gamma}(\rho, \mathcal{S}) \) corresponding to the morphism \( m \) we have an open covering \( \{ U_i \} \) of \( S \), faithfully flat morphisms \( f_i : U_i \to U_i, A' \)-bundles \( \mathcal{E}_i \to U_i \times X \) and \( A \)-bundles isomorphisms \( \varphi_i : \rho_{\ast} \mathcal{E}_i \to (t \circ f_i) \ast \mathcal{E} \). We define \( h[m] \) to be the morphism from \( S \) to \( M \) corresponding to the element \( (\mathcal{E}_i, \varphi_i) \) in \( \tilde{\Gamma}(\rho, \mathcal{S})(S) \) where \( \bar{\alpha}_{h \times t} = (h \times t) \ast (\bar{\alpha}) \) and \( \bar{\alpha} : H \times \mathcal{E} \to \mathcal{E} \ast \mathcal{E} \) gives the action of \( H \) on \( \mathcal{E} \) (Definition 4.6). Then it is easy to see that we have indeed an action of \( H \) on \( M \) and that \( h[T] = \pi \circ h[m] \). To prove that \( M \) is a universal space let \( \mathcal{E}' \to S \times X \in \mathcal{F}'(S) \) i.e. a family of \( A' \)-bundles in \( \mathcal{S}' \). By extending the structure group by \( \rho \) we get a family of \( A \)-bundles \( \rho_{\ast} \mathcal{E}' \) in \( \mathcal{S} \). Since \( T \) is universal this gives an open covering \( \{ U_i \} \) of \( S \) and morphisms \( f_i : U_i \to T \) such that \( \mathcal{E}_i \to \rho_{\ast} \mathcal{E}' \mid_{U_i \times X} \). This then gives morphisms \( f_i : U_i \to M \). Using condition (ii) of Definition 4.6 satisfied by \( \mathcal{E}' \to T \times X \) these \( f_i \) are seen to define an element of \( H \setminus \text{M}(S) \). It is easy to check that by associating this element of \( H \setminus \text{M}(S) \) to \( \mathcal{E}' \in \mathcal{F}'(S) \) we have an isomorphism of sheaves \( \tilde{\Gamma} \longrightarrow H \setminus \text{M} \) (see proof of (ii) below, locally, in the faithfully flat topology, the arguments run on the same lines).

(ii) In this case we can define \( h[m] \) as the morphisms from \( S \) to \( M \) corresponding to the pair \((\mathcal{U}_m, \bar{\alpha}_{h \times t} \circ u_m) \). Therefore by definition the pair \((\mathcal{U}_m, u_{\mathcal{M}[m]}) \) is isomorphic to \((\mathcal{U}_m, u_{\mathcal{M}[m]}) \), and hence there is an isomorphism (which is unique since \( \rho \) is injective, cf. §4.8) \( \tilde{\beta}_{h \times t} : \mathcal{U}_m \to \mathcal{U}_{\mathcal{M}[m]} \) making the diagram

\[
\begin{array}{ccc}
\rho_{\ast} \mathcal{U}_m & \longrightarrow & \rho_{\ast} \mathcal{U}_{\mathcal{M}[m]} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
\mathcal{E}_1 & \longrightarrow & \mathcal{E}_{\mathcal{M}[1]}
\end{array}
\]

commutative. Taking \( S = H \times M \) and \( h, m \) to be the projections \( p_H : H \times M \to H, p_M : H \times M \to M \) respectively, in the above we get the action \( p_H[p_M] = \tilde{\beta} : H \times M \to M \) of \( H \) on \( M \) and \( \tilde{\beta}_{p_H \times p_M} = \tilde{\beta} : H \times \mathcal{U} \to \tilde{\beta} \ast \mathcal{U} \) which gives the action of \( H \) on \( \mathcal{U} \).

The condition (i) of Definition 4.6 for \( \mathcal{U} \) follows immediately from the universal properties of \( M \) and \( \mathcal{E} \). To check condition (ii) of Definition 4.6, let \( m_1, m_2 : S \to M \) and \( \phi : \mathcal{U}_m \to \mathcal{U}_{m_1} \) an isomorphism be given. Let \( t_1 = \pi \circ m_1 \) and \( t_2 = \pi \circ m_2^2 \). Define \( \phi' \) by the commutative diagram:
By the universal property of $\mathcal{E}$ there is a unique morphism $h:S \to H$ such that $t_2 = h[t_1]$ and $\varphi' = \tilde{\mathcal{E}}_{h \times t_1}$. Now $m_2$ is defined by the pair $(\mathcal{U}_{m_2}, u_{m_2})$ and $h[m_1]$ by the pair $(\mathcal{U}_{m_1}, \tilde{\mathcal{E}}_{h \times t_1} \circ u_{m_1})$. But from the diagram (\ast) and the observation above, it follows that $\varphi: \mathcal{U}_{m_1} \to \mathcal{U}_{m_2}$ gives an isomorphism of the pairs $(\mathcal{U}_{m_1}, \tilde{\mathcal{E}}_{h \times t_1} \circ u_{m_1})$ and $(\mathcal{U}_{m_2}, u_{m_2})$. Therefore $m_2 = h[m_1]$, as required to be shown.

4.11. We shall now construct universal families for semistable vector bundles. This is essentially contained in ([19], §6; [20], §3). We have made it a little more explicit to suit our purposes.

Let $\mathcal{S}^{r,d}$ be the set of isomorphism classes of semistable vector bundles of rank $r$ and degree $d$. Let $L$ be an ample line bundle on $X$ of degree $d_0$. Since $\mathcal{S}^{r,d}$ is bounded i.e. there is a family of vector bundles on $X$ parametrized by an algebraic scheme in which every element of $\mathcal{S}^{r,d}$ occurs ([19], Proposition 3.2, p. 307), we can find an integer $m_0$ such that for any $m \geq m_0$ and every $V \in \mathcal{S}^{r,d}$, $r' \leq r$, $H^i(X, V \otimes L^m) = 0$ for $i > 0$ and $H^0(X, V \otimes L^m)$ generates $V \otimes L^m$. Then $H^0(X, V \otimes L^m)$ has same rank, say $n$, for all $V \in \mathcal{S}^{r,d}$.

4.11.1. Let $P \subset GL(n, \mathbb{C})$ be the parabolic subgroup defined as the stabilizer of the subspace $\mathbb{C}^{n-r} \subset \mathbb{C}^n$. The decomposition $\mathbb{C}^n = \mathbb{C}^{n-r} \oplus \mathbb{C}^r$ gives a homomorphism $P \to GL(n, \mathbb{C})$. Given a $P$-bundle on a scheme $Y$ the representations $P \subset GL(n, \mathbb{C})$ and $P \to GL(r, \mathbb{C})$ give rise to vector bundles $V_1$ and $V_2$ (of ranks $n$ and $r$ respectively) on $Y$ and there is a surjective homomorphism $V_1 \to V_2$ induced by the $P$-equivariant projection $\mathbb{C}^n \to \mathbb{C}^r$. Conversely given two vector bundles $V_1$ and $V_2$ of rank $n$ and $r$ respectively and a surjective homomorphism $V_1 \to V_2 \to 0$ we can construct a $P$-bundle as follows. For any open set $U \subset Y$ and a faithfully flat morphism $f: U \to U$ associate the set of all vector bundle isomorphisms $\varphi_1, \varphi_2$ making the diagram

\[ U \times \mathbb{C}^n \longrightarrow U \times \mathbb{C}^r \longrightarrow 0 \]

\[ \varphi_1 \downarrow \quad \varphi_2 \downarrow \]

\[ f^*V_1 \longrightarrow f^*V_2 \longrightarrow 0 \]

commutative. Thus we get a sheaf $\mathcal{F}$ for the faithfully flat topology and the trivial group scheme $Y \times P$ over $Y$ given by $P$ acts on $\mathcal{F}$ by $(\varphi_1, \varphi_2)p = (\varphi_1 \circ p, \varphi_2 \circ p)$, $p \in P$. Then $\mathcal{F}$ is a principal homogeneous space under $Y \times P$ and by descent ([SGA, 3] exp. 24V) it is representable by a $P$-bundle over $Y$. Thus we can consider a $P$-bundle as a surjective homomorphism $V_1 \to V_2 \to 0$ of vector bundles. Then an isomorphism of the $P$-bundles $V_1 \to V_2 \to 0$ and $V'_1 \to V'_2 \to 0$ is given by a commutative diagram
Since $GL(n, \mathbb{C}) \to GL(n, \mathbb{C})/P$ is locally trivial ([3], Theorem 4.13, p. 90) any $P$-bundle is in fact locally trivial ([17], Theorem 2, §4.3, p. 1–24).

4.11.2. The trivial bundle $I_n = X \times \mathbb{C}^n \to X$ thought of as a family of vector bundles on $X$ parametrized by a point, with the group $GL(n, \mathbb{C})$ acting in the natural way on $I_n$ gives a universal family with group $GL(n, \mathbb{C})$ for the singleton set $\{I_n\}$. (Condition (ii) of Definition 4.6 is obviously satisfied and condition (i) follows from ([11], Lecture 7, p. 51, (ii) and (iii)).) So applying Lemma 4.8.1 with $A' = P \subset GL(n, \mathbb{C}) = A$ and fixing the topological type of the $P$-bundle such that we get by the extension of structure group $P \to GL(r, \mathbb{C})$ vector bundles of degree $d$ (since the topological type of a vector bundle on $X$ is determined by its degree and rank and any extension splits topologically this condition fixes the topological type of the $P$-bundle; cf. 4.11.1), we get a universal family of $P$-bundles $\mathcal{U} \to M \times X$ with group $GL(n, \mathbb{C})$ for the set of $P$-bundles of the fixed topological type which give $I_n$ on extending the structure group by $P \to GL(n, \mathbb{C})$, parametrized by an algebraic scheme $M$. Let $\mathcal{O}$ be the $GL(r, \mathbb{C})$-bundle obtained from $\mathcal{U}$ by the extension of structure group $P \to GL(r, \mathbb{C})$.

4.11.3. By our constructions in §4.8 it follows that $M$ is an open subscheme of the locally finite type scheme $\text{Hom}(X, G_{n,r})$ which represents the functor $\Gamma$, $\Gamma(S) = \text{Hom}(S \times X, G_{n,r})$ where $G_{n,r} = GL(n, \mathbb{C})/P$ is the Grassmannian of $r$-dimensional quotients of $\mathbb{C}^n$. Then $\mathcal{U}$ is the pull back of the $P$-bundle $GL(n, \mathbb{C}) \to GL(n, \mathbb{C})/P$ by the universal section in $\text{Hom}(M \times X, G_{n,r})$ and $\mathcal{O}$ is the pull back of the $GL(r, \mathbb{C})$-bundle $Q \to G_{n,r}$ obtained from $GL(n, \mathbb{C}) \to G_{n,r}$ by the extension of structure group $P \to GL(r, \mathbb{C})$. The $P$-bundle $\mathcal{U}$ corresponds to the surjection $I_n \to \mathcal{O} \to 0$ which is the pull back of the surjection $G_{n,r} \times \mathbb{C}^n \to Q \to 0$ induced by $\mathbb{C}^n \to \mathbb{C}^r$. (Note that $GL(n, \mathbb{C}) \to G_{n,r}$ becomes trivial when we extend the structure group by $P \subset GL(n, \mathbb{C})$.)

4.11.4. Let

$$R = \left\{ q \in M \mid \begin{array}{c} \mathcal{O}_q \text{ is semistable and the canonical map} \\
I_n \to H^0(X, \mathcal{O}_q) \text{ is an isomorphism} \end{array} \right\}.$$ 

It follows from the semi-continuity theorem and the fact that the points corresponding to semistable bundles form an open subset of the parameter scheme in any family of vector bundles ([19], Corollary 7.2, p. 332) that $R$ is an open subset of $M$. We take $R$ with the open subscheme structure induced from $M$. Clearly $R$ is stable under the action of $GL(n, \mathbb{C})$. We denote the restriction of $\mathcal{O}$ to $R \times X$ also by $\mathcal{O}$. Let $\mathcal{O}(-m) = \mathcal{O} \otimes p_X^*(L^{-m})$ where $p_X: R \times X \to X$ is the projection. Let $GL(n, \mathbb{C})$ act on $\mathcal{O}(-m)$ by its action on $\mathcal{O}$ and the trivial action on $p_X^*(L^{-m})$.

*Editor's Note: For the definition of $Q$ see Introduction, in part I. It is the principal $GL(r)$ bundle associated to the universal quotient bundle on $G_{n,r}$. 
4.11.5. PROPOSITION

The family $\mathcal{O}(-m) \to R \times X$ with this action of $GL(n, \mathbb{C})$ gives a universal family with group $GL(n, \mathbb{C})$ for the set $\mathcal{S}^{r,d}$ of semistable vector bundles of rank $r$ and degree $d$.

Proof. Let $\mathcal{F} \to S \times X$ be a family in $\mathcal{S}^{r,d}$ and $s_0 \in S$. The direct image $p_{s_0}(\mathcal{F}(m))$ is locally free ([11], Lecture 7, p. 51). Choose a trivialization $p_{s_0}(\mathcal{F}(m)|_U) \approx U \times \mathbb{C}^n$ in a neighbourhood $U$ of $s_0$. Then we have a surjection $p_{s_0}^*(U \times \mathbb{C}^n) = I_n \to \mathcal{F}(m)|_{U \times X} \to 0$. This gives a reduction of structure group of the trivial $GL(n, \mathbb{C})$ bundle $I_n$ to $P(\mathcal{F}(m)|_{U \times X})$. Then by the universal property of the $P$-bundle $\mathcal{U}$ on $R \times X$ we get a morphism $f: U' \to R$, $s_0 \in U' \subset U$, inducing the $P$-bundle $I_n \to \mathcal{F}(m)|_{U' \times X} \to 0$. Extending the structure group by $P \to GL(r, \mathbb{C})$ and tensoring by $L^{-m}$ we get $f(\mathcal{O}(-m)) \approx \mathcal{F}|_{U' \times X}$.

To check condition (ii) of Definition 4.6, let $r_1, r_2: S \to R$ and $\varphi: \mathcal{O}_{r_1}(-m) \to \mathcal{O}_{r_2}(-m)$ be given. Let $\varphi': \mathcal{O}_{r_1} \to \mathcal{O}_{r_2}$ be the isomorphism induced by $\varphi$.

Then $\varphi'$ induces an isomorphism $\varphi'': p_{s_0}(\mathcal{O}_{r_1}) \approx p_{s_0}(\mathcal{O}_{r_2})$. But by commutativity under base change (since $H^1(X, \mathcal{O}_q) = 0$ for $q \in R([11], Lecture 7, p. 51)$), $p_{s_0}(\mathcal{O}_{r_1})$ is canonically isomorphic to $r_i^*(p_{r_s}(\mathcal{O}))$, $i = 1, 2$. Since $p_{r_s}(\mathcal{O})$ is canonically isomorphic to the trivial bundle we have the commutative diagram

$$
\begin{array}{ccc}
r^*_1(I_0) & \to & \mathcal{O}_{r_1} \to 0 \\
\downarrow \varphi'' & & \downarrow \varphi' \\
r^*_2(I_0) & \to & \mathcal{O}_{r_2} \to 0
\end{array}
$$

Now use the universal property for the $P$-bundle $\mathcal{U}$ (cf. 4.11.1).

4.11.6. Remark. If $\mathcal{U}$ is any universal family of vector bundles with group H for a set $\mathcal{S}$ of vector bundles then $\mathcal{U} \otimes L_1$ is a universal family with group $H$ ($H$ acting trivially on $L_1$) for the set $\mathcal{S} \otimes L_1 = \{ V \otimes L_1 | V \in \mathcal{S} \}$, $L_1$ being any line bundle. Therefore it follows from Proposition 4.11.5 that $\mathcal{O}(q)$ is a universal family for $\mathcal{S}^{r,d} + rd_0(m + q)$, $q \in \mathbb{Z}$.

4.11.7. Remark. It is easy to see that the scheme $R$ above is the same as scheme $R^{ss}$ which Seshadri constructs in ([20], §3; [19], §6).

4.12. PROPOSITION

The set of isomorphism classes of semistable G-bundles of a fixed topological type $\tau$ is bounded, i.e. there exists a family $\mathcal{E} \to M \times X$ of $G$-bundles such that given any semistable $G$-bundle $E$ of type $\tau$ there is an $m \in M$ such that $E \approx \mathcal{E}_m$.

Proof. Let $\rho: G \to GL(V)$ be a faithful representation. Let $V = V_1 \oplus \cdots \oplus V_n$ be a decomposition into irreducible subspaces. Let $r_i$ be the rank of $V_i$ and $d_i$ be the degree of the vector bundle $\rho_{i*}(\tau)$ where $\rho_i: G \to GL(V_i)$ gives the action of $G$ on $V_i$. Let $u_i: M_i \times X$ be
the universal family for the set $\mathcal{G}_{r,d}$ and $U = U_1 \times \cdots \times U_r \to (M_1 \times \cdots \times M_r) \times X$. Let $\varepsilon : M \times X$ represent the functor $\Gamma(\rho', U)$ (§§ 4.8, 4.8.1), where $\rho' = \rho_1 \times \cdots \times \rho_r : G \to GL(V_1) \times \cdots \times GL(V_r)$; use Proposition 3.17 to see that this $\rho$ satisfies what we require.

4.13. Let $\text{Aut } \mathcal{G}$ be the group of Lie algebra automorphisms of $\mathcal{G}$. Let $\dim \mathcal{G} = r$. The group $\text{Aut } \mathcal{G}$ may not be connected and $\text{Ad } G = G/Z$ is the connected component of identity of $\text{Aut } \mathcal{G}$. Let $\rho : \mathbb{C}^* \times \text{Aut } \mathcal{G} \to GL(\mathcal{G})$ be the natural inclusion. Note that, in $GL(\mathcal{G}), \mathbb{C}^* \times \text{Aut } \mathcal{G}$ is trivial. Applying Lemma 4.8.1 for this $\rho$ and the universal family $\mathcal{O} \to R \times X$ for $\mathcal{G}_{r,\text{md}_0}$ to get a universal family $\varepsilon : \to R' \times X$ with group $GL(n, \mathbb{C})$ for $\mathbb{C}^* \times \text{Aut } \mathcal{G}$-bundles which under the extension of structure group $\rho$ give semistable vector bundles of degree $\text{md}_0$. Since any $\mathcal{G}$-bundle gives a vector bundle of degree zero on extension of structure group by $\text{Aut } \mathcal{G} \to GL(\mathcal{G})$ (Aut $\mathcal{G}$ is contained in the orthogonal group corresponding to the Killing form) it follows that for the extension of structure group $\mathbb{C}^* \times \text{Aut } \mathcal{G} \to \mathbb{C}^*, \varepsilon$ gives a family of line bundles $\varepsilon'(\mathbb{C}^*)$ of degree $\text{md}_0$ on $X$. Let $J_{\text{md}_0}$ be the Jacobian of $X$ of line bundles of degree $\text{md}_0$. Then by the universal property of $J_{\text{md}_0}$ we have a morphism $R' \to J_{\text{md}_0}$ corresponding to $\varepsilon'(\mathbb{C}^*) \to R' \times X$. Let $R_1$ be the fiber over $L^* \in J_{\text{md}_0}$ under this morphism. Restricting $\varepsilon'$ to $R_1 \times X$ and extending the structure group by $\mathbb{C}^* \times \text{Aut } \mathcal{G} \to \text{Aut } \mathcal{G}$ we get an $\mathcal{G}$-bundle $\varepsilon_1 \to R_1 \times X$. The action of $GL(n, \mathbb{C})$ on $\varepsilon' \to R' \times X$ gives an action of $GL(n, \mathbb{C})$ on $\varepsilon_1 \to R_1 \times X$.

4.13.1. PROPOSITION

The $\mathcal{G}$-bundle $\varepsilon_1 \to R_1 \times X$, with the natural action of $GL(n, \mathbb{C})$, constructed above is a universal family with group $GL(n, \mathbb{C})$ for the set of isomorphism classes of $\mathcal{G}$-bundles which give semistable vector bundles of degree zero on extension of structure group by the inclusion $\text{Aut } \mathcal{G} \to GL(\mathcal{G})$. Moreover the parametrizing scheme $R_1$ is non-singular.

Proof. The universal property of $\varepsilon_1$ is clear from the above discussion. The non singularity of $R_1$ is proved in the following two lemmas (4.13.3 and 4.13.4).

4.13.2. Remark. The elements of the tensor space $\mathcal{H} = \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^* = \text{Hom} (\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G})$ give algebra structures on $\mathcal{G}$. Those elements of $\mathcal{H}$ which give algebra structures which satisfy the Jacobi identity and skew symmetry form a closed subvariety of $\mathcal{H}$ and give Lie algebra structures on $\mathcal{G}$. The points of the variety $Y = GL(\mathcal{G})/\mathbb{C}^* \times \text{Aut } \mathcal{G} \subset \mathbb{P}(\mathcal{H})$ then give Lie algebra structures on $\mathcal{G}$, determined up to a scalar and isomorphic to the original Lie algebra structure of $\mathcal{G}$. If $0 \neq h \in \mathcal{H}$ is such that $\{h\} \in X$ and the Lie algebra structure of $\mathcal{G}$ given by $h$ is semisimple then $\{h\} \in Y$ ([16], Corollary 4.3, p. 514). This fact will be crucial for us in proving the existence of quotient space in the next section (cf. proof of Lemma 5.6) and is the reason why we have chosen the adjoint representation for constructing universal families.

Let $r_1 \in R_1 \subset R'$ and $r \in R$ its image. Then the $\mathbb{C}^* \times \text{Aut } \mathcal{G}$-bundle $\varepsilon_r$ gives the vector bundle $\mathcal{O}$, under the extension of structure group $\mathbb{C}^* \times \text{Aut } \mathcal{G} \to GL(\mathcal{G})$ and the $\mathcal{G}$-bundle $\varepsilon_r'$ gives the vector bundle $\mathcal{O},(-m)$ under $\text{Aut } \mathcal{G} \to GL(\mathcal{G})$. If we take the embedding $Y \subset \mathbb{P}(\mathcal{H}), r_1$ gives a section $r_1 : X \to Q_r(Y) \subset \mathbb{P}(\mathcal{O}_r \otimes \mathcal{O}_r \otimes \mathcal{O}_r)$ such that $r_1^*(\Lambda) = L^{-m}$ where $\Lambda$ is the tautological line bundle on $\mathbb{P}(\mathcal{O}_r \otimes \mathcal{O}_r \otimes \mathcal{O}_r)$.
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4.13.3. Lemma. The schemes $R$ and $R'_1$ are nonsingular and $\dim R = n^2 + r^2(g - 1)$ and $\dim R'_1 = n^2 + (r + 1)(g - 1)$.

Proof. We use the notation of §4.11.3. Let $Y = GL(\mathcal{G})/C^* \times \text{Aut} \mathcal{G}$ and $Q(Y)$ be the fiber bundle with fiber $Y$ associated to $Q$ considered as a $GL(\mathcal{G}) = GL(r, C)$-bundle. The scheme $R$ is an open subscheme of $\text{Hom}(X, G_{nr})$ and $R_1$ is an open subscheme of $\text{Hom}(X, Q(Y))$. The morphism $R_1 \to R$ is induced by $\varphi: Q(Y) \to G_{nr}$. By associating a morphism $f: X \to Q(Y)$ to its graph $\Gamma_f$ in $X \times Q(Y)$, $\text{Hom}(X, Q(Y))$ becomes an open subscheme of $\text{Hilb}(X \times Q(Y))$ ([TDTE, IV], §4, pp. 19, 20). The graph $\Gamma_f \approx X$ of $f \in R_1$ is a nonsingular complete subvariety of the non-singular variety $X \times Q(Y)$. Therefore the obstruction to the smoothness of the Hilbert scheme at $\Gamma_f$ is an element of $H^1(\Gamma_f, N_{\Gamma_f})$, where $N_{\Gamma_f}$ is the normal bundle of $\Gamma_f$ in $X \times Q(Y)$ ([TDTE, IV], §5; cf. also [9]). Identifying $\Gamma_f$ with $X$ by the projection $X \times Q(Y) \to X, N_{\Gamma_f} \approx f^*(T_q)$ where $T_q$ is the tangent bundle of $Q(Y)$. We shall show that $H^1(X, f^*(T_q)) = 0$ from which it will follow that $R'_1$ is nonsingular.

We have the following diagram of vector bundles on $X$, which is commutative and exact (in the obvious sense)

\[
\begin{array}{cccccc}
0 & \to & K & \to & (\varphi f)^*(\text{Ad } Q) & \to & f^*(T'_q) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & (\varphi f)^*(M) & \to & f^*(T_q) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & (\varphi f)^*(T_q) & \to & (\varphi f)^*(T_q) & \to & 0 & & \\
& & 0 & \to & 0 & & & &
\end{array}
\]

(1)

where $T_q$ (respectively $T'_q$) is the tangent bundle of $Q(Y)$ (resp. $G_{nr}$) and $T_q'$ is the tangent bundle along fibers of $\varphi: Q(Y) \to G_{nr}$, the canonical map.

The first column is the pull back by $\varphi f$ of the Atiyah exact sequence of $Q \to G_{nr}$. The horizontal arrows from the first to the second column are induced by the differential of the projection $Q \to Q(Y) = Q/C^* \times \text{Aut} \mathcal{G}$.

Let $\varphi f \in R$ correspond to the quotient

\[
0 \to H_f \to I_n \to F_f \to 0,
\]

(2)
i.e. $F^*(Q) = F_f$. Then $(\varphi f_\ast)(T_g) = H^*_g \otimes F_f$ ([TDTE] IV. § 5).

It is easy to check the following identifications:

$$
0 \rightarrow (\varphi f_\ast)(\text{Ad } Q) \rightarrow (\varphi f_\ast)(M) \rightarrow (\varphi f_\ast)(T_g) \rightarrow 0
$$

$\approx$

$$
0 \rightarrow F^*_g \otimes F_f \rightarrow H^*_g \otimes F_f \rightarrow H^*_g \otimes F_f \rightarrow 0
$$

(3)

where the second row is obtained from (2) by dualizing and tensoring by $F_f$.

Also, $K$ is the adjoint bundle of the $C^* \times \text{Aut } G$ bundle obtained from the reduction of structure group of $F_f$ to $C^* \times \text{Aut } G$ corresponding to $f$.

From the cohomology exact sequence of the bottom row of (3) we get $H^1(X, (\varphi f_\ast)M) = H^1(X, I_g \otimes F_f)$ is $n$-copies of $H^1(X, F_f)$. But $F_f \in \mathcal{G}_{\text{rmd}}$ and hence $H^1(X, F_f) = 0$. Therefore $H^1(X, (\varphi f_\ast)M) = 0$. Also $H^1(X, H^*_g \otimes F_f) = 0$ and therefore $H^1(X, (\varphi f_\ast)(T_g)) = 0$. This proves that $R$ is nonsingular (for the same reason that $H^1(X, f^*_q(T_g)) = 0$ proves $R_1$ is nonsingular).

From diagram (1), taking cohomology, we have

$$H^1((\varphi f_\ast)M) \rightarrow H^1(f^*_qT_g) \rightarrow 0.$$  

(All cohomologies over $X$). This proves that $H^1(X, f^*_qT_g) = 0$ as claimed.

By ([TDTE] IV. § 5) the Zariski tangent space to $R$ (resp. $R_1$) at $\varphi f$ (resp. $f$) can be canonically identified with $H^0(X, H^*_g \otimes F_f)$ (resp. $H^0(X, f^*_qT_g)$). From the bottom row of (3) we get that deg $H^0(X, H^*_g \otimes F_f) = n \cdot \text{deg } F_f = n \cdot \text{rmd}_g$. Applying Riemann–Roch: $\dim H^0(X, H^*_g \otimes F_f) = n \cdot \text{rmd}_g + r(n - r)(1 - g)$. Since $n = \dim H^0(X, F_f) = \text{rmd}_g + r(1 - g)$ we have $\dim R = \dim H^0(X, H^*_g \otimes F_f) = n^2 + r^2(g - 1)$.

From the exact sequence

$$0 \rightarrow H^0(f^*_qT_g) \rightarrow H^0(f^*_qT_g) \rightarrow H^0(f^*_qT_g) \rightarrow H^1(f^*_qT_g) \rightarrow 0$$

(all cohomologies over $X$) we get

$$\dim H^0(f^*_qT_g) = \dim H^0(f^*_qT_g) + \dim H^0(f^*_qT_g) - \dim H^1(f^*_qT_g).$$

We apply Riemann–Roch to $f^*_qT_g$, noting that deg $f^*_qT_g = 0$ (since in the first row of (1) both $K$ and $(\varphi f_\ast)\text{Ad } Q$ being adjoint bundles have degree zero) and $r \cdot f^*_qT_g = r^2 - r - 1 (= \dim Y)$, to get $\dim R_1 = \dim H^0(X, f^*_qT_g) = n^2 + r^2(g - 1) + (r^2 - r - 1)(1 - g) = n^2 + (r + 1)(g - 1) - g$.

4.13.4. Lemma. The scheme $R_1$ is nonsingular and $\dim R_1 = n^2 + (r + 1)(g - 1) - g$.

Proof: Since $R_1$ and $J^\text{md}$ are non-singular and $R_1$ is the fiber over $L^\infty eJ^\text{md}$, it is enough to check that the differential of $\psi: R_1 \rightarrow J^\text{md}$ is surjective at any point $r \in R_1$. Let $t: \text{Spec } C[e] \rightarrow J^\text{md}$ be a tangent to $J^\text{md}$ at $L^\infty$. We have to show that we can lift this morphism to $R_1$ such that the unique closed point of $\text{Spec } C[e]$ goes to $r$. For this we use the universal properties of $R$ and $R_1$. Let $r \in R_1$, corresponding to the $C^* \times \text{Aut } G$-bundle $M \times X$ where $M$ is a line bundle on $X$ and $E \rightarrow X$ is an $\text{Aut } G$-bundle. Let $\mathcal{P} \rightarrow J^\text{md} \times X$ be the Poincaré bundle. Consider the $C^* \times \text{Aut } G$-bundle $t^*\mathcal{P} \times_X E \rightarrow \text{Spec } C[e] \times X$ where $E = \text{Spec } C[e] \times E$. Then $H^1(X, (t^*\mathcal{P} \otimes E)_o) = H^1(X, L^\infty \otimes E(G')) = 0$, where $o$ denotes the closed point of $\text{Spec } C[e]$. Therefore the direct image
**4.14. PROPOSITION**

Let $\mathcal{E}_2 \to R_2 \times X$ be the Ad $G$-bundle representing the functor $\Gamma(\rho, \mathcal{E}_1)$ where $\rho : \text{Ad } G \to \text{Aut } \mathcal{G}$ is the inclusion and $\mathcal{E}_1 \to R_1 \times X$ is the universal Aut $\mathcal{G}$-bundle constructed in the preceding theorem. Then $\mathcal{E}_2$ is a universal family with group $GL(n, \mathbb{C})$ (for the action of $GL(n, \mathbb{C})$ provided by Lemma 4.10) for the set of all isomorphism classes of semistable Ad $G$-bundles.

The natural morphism $\pi_2 : R_2 \to R_1$ is étale and finite and hence $R_2$ is nonsingular.

**Proof.** Since Ad $G$ is semisimple there are only finitely many isomorphism classes of topological Ad $G$-bundles (Lemma 3.8.1, $R_2$ is a scheme of finite type. The universal property for $\mathcal{E}_2$ follows from Lemma 4.10. We note that $\mathcal{E}_2$ is universal for all semistable Ad $G$-bundles because by Corollary 3.18 for any Ad $G$-bundle $E, E(\mathcal{G})$ is semistable if and only if $E$ is semistable. (Note that $E(\mathcal{G})$ is semistable if and only if $E(\mathcal{G})$ is semistable.)

Since Ad $G$ is the connected component of identity of Aut $\mathcal{G}$ the morphism $GL(\mathcal{G})/\text{Ad } G \to GL(\mathcal{G})/\text{Aut } \mathcal{G}$ is an étale covering. Therefore it follows that $\mathcal{E}_1/\text{Ad } G \to R_1 \times X$ is an étale covering. Taking $Y = R_1 \times X, \bar{Y} = \mathcal{E}_1/\text{Ad } G$ and $T = R_1$ in the following lemma we see that $\pi_2 : R_2 \to R_1$ is étale and finite.

**4.14.1. Lemma.** Let $Y \to T$ and $\bar{Y} \to T$ be schemes over a scheme $T$ and $\bar{Y} \to Y$ a $T$-morphism which is an étale (surjective) covering. Suppose $Y$ is flat and projective over $T$ and $\bar{Y}$ is quasi-projective over $T$ so that the functor $\pi : (\text{locally noetherian schemes}/T) \to (\text{Sets})$ defined by

$$
\pi(f : S \to T) = \text{Hom}_{\mathcal{X}/T}(S \times_Y Y, S \times_T \bar{Y}) = \text{Hom}_T(S \times_Y Y, \bar{Y})
$$

is representable by a locally finite type $T$-scheme $\pi : \Pi \to T$ ([TDTE, IV], §4). Assume also that $Y \to T$ is faithfully flat. Then $\pi$ is étale and finite.

**Proof.** To prove $\pi : \Pi \to T$ is étale, let $A$ be a scheme, $A_0$ a subscheme of $A$ defined by a nilpotent ideal and

![Diagram](image)

a commutative diagram. We only have to show that the morphism corresponding to the broken arrow exists uniquely ([EGA IV], Definitions 17.1.1, 17.3.1). Consider the commutative diagram
where the morphism $A_0 \times Y \to \bar{Y}$ is induced by $A_0 \to \Pi$. Since $A_0 \times Y$ in $A \times Y$ is also defined by a nilpotent ideal and $\bar{Y} \to Y$ is étale the broken arrow in the diagram can be realized uniquely and hence by the universal property of $\Pi$ the broken arrow in the preceding diagram can also be realized.

It follows from ([SGA I], Exposé I, §5, Corollary 5.3) that $\pi: \Pi \to T$ is quasi finite (i.e. has finite fibers). Since a proper and quasi finite morphism is finite ([EGA III], Corollary 4.4.11) it now suffices to show that $\pi$ is proper. We use the valuation criterion for this.

Let $A$ be a discrete valuation ring over $C$ with residue field $C$ and quotient field $K$. Suppose we are given morphisms

$$\begin{array}{ccc}
\text{Spec } K & \longrightarrow & \Pi \\
\downarrow & & \downarrow \\
\text{Spec } A & \longrightarrow & T
\end{array}$$

Again by the universal property of $\Pi$ this gives

$$\begin{array}{ccc}
\text{Spec } K \times Y & \longrightarrow & \bar{Y} \\
\downarrow & & \downarrow \\
\text{Spec } A \times Y & \longrightarrow & Y
\end{array}$$

By taking the base change of the étale covering $\bar{Y} \to Y$ by $\text{Spec } A \times Y \to Y$ we get an étale covering over $\text{Spec } A \times Y$ which, by the above diagram, has a section over the open subset $\text{Spec } K \times Y$ which is dense ($Y \to T$ being faithfully flat) and hence is trivial. This proves that the broken arrow in this diagram, and hence in the preceding diagram, can be realized.

4.15. Let $G'$ be the commutator subgroup $[G, G]$ of $G$. Let $\tau$ be a topological $G$-bundle on $X$. The group $G/G'$ is a torus isomorphic to $C^\times$. Therefore fixing an isomorphism $G/G' \simeq C^\times$, a $G/G'$-bundle can be considered as a $q$-tuple of line bundles. Let $d_1, \ldots, d_q$ be the degrees of the topological line bundles $L_1, \ldots, L_q$ corresponding to the $G/G'$-bundle obtained from $\tau$ by the extension of structure group $G \to G/G'$. Let $J_\tau = J^{d_1} \times \cdots \times J^{d_q}$, where $J^{d_i}$ is the Jacobian of $X$ of line bundles of degree $d_i$. Let $U_i \to J^{d_i} \times X$ be the Poincaré bundle and $U_i \to J_\tau \times X$ be $(id_{J_\tau} \times \Delta_X)^*(U_1 \times \cdots \times U_q)$ where $\Delta_X: X \to X \times \cdots \times X$ is the diagonal embedding and $U_1 \times \cdots \times U_q \to J_\tau \times X \times \cdots \times X$ is
the external product of $U_i \to J \times X$. Then $U_i$ is a $G/G'$-bundle and making $G/G'$ act trivially on $J$, and in the natural way on $U_i$ (since $G/G'$ is abelian this action gives $G/G'$-bundle automorphisms) it is easy to see, using the universal property of the Jacobian, that $U_i$ is a universal family with group $G/G'$ for the set of isomorphism classes of $G/G'$-bundles of the topological type determined by the degrees $d_1, \ldots, d_d$. The $G/G' \times G/Z$-bundle $\delta'_2 = U_i \times \delta_2 \to J_i \times R_2 \times X$ is then a universal family with group $G/G' \times GL(n, \mathbb{C})$ for semistable $G/G' \times G/Z$-bundles of suitable topological type, in the obvious way.

4.15.1. Lemma. The sheaf $\overline{\Gamma}(\rho, \delta'_2)$ corresponding to the natural projection $\rho: G \to G/G' \times G/Z$ and the family $\delta'_2 \to R_2 \times X$, $R_2 = J \times R_2$, is representable by a scheme $R_3$, étale and finite over $R_2$.

Proof. We will prove the lemma by identifying $\overline{\Gamma}(\rho, \delta'_2)$ with a product of Picard functors and using the representability theorems for Picard functors. The idea is simply to generalize the fact that given a projective bundle $P \to X$ to give a vector bundle $V \to X$ such that $\mathbb{P}(V) = P$ is equivalent to giving a tautological line bundle on $P$. For this purpose we construct an algebraic group $H$ which is related to $Ad G$ similar to the way in which $GL(n, \mathbb{C})$ is related to $PGL(n, \mathbb{C}) = Ad SL(n, \mathbb{C})$.

Let $T$ be a maximal torus of $G' = [G, G]$. Then $Z \cdot T$ is a maximal torus for $G$ and $Z \cap T$ is the finite group $Z' = Z[G']$. Let $\chi_1, \ldots, \chi_d$ be a set of characters of $Z \cdot T$ such that the homomorphism $t \mapsto (\chi_1(t), \ldots, \chi_d(t))$ from $Z \cdot T$ to $\mathbb{C}^d$ is injective on $Z'$. Let $F$ be the finite subgroup $\{(\chi_1(t), \ldots, \chi_d(t), t^{-1}) \mid t \in Z'\}$ of $\mathbb{C}^d \times G$. Define $H$ to be the quotient group $(\mathbb{C}^d \times G)/F$. Let $A$ be the quotient group $(\mathbb{C}^d \times Z')/F$. Then we have the diagram which is commutative and exact.

\[
\begin{array}{cccccc}
1 & 1 \\
\downarrow & & \downarrow & & \\
1 & \to & Z' & \to & G & \to & G/G \times G/Z & \to & 1 \\
\downarrow & & \downarrow & & \rho' & = & n & \downarrow & \\
1 & \to & A & \to & H & \to & G/G \times G/Z & \to & 1 \\
\alpha & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
A/Z' & \to & H/G & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
1 & 1 & & & & & & & \\
\end{array}
\]  

(1)

Note that both $A$ and $A/Z'$ are isomorphic to $\mathbb{C}^d$ since $Z'$ is a finite subgroup of $A$ and we can choose isomorphisms such that we have the commutative diagram

\[
\begin{array}{cccccc}
1 & \to & Z' & \to & A & \to & A/Z' & \to & 1 \\
\downarrow & \approx & \downarrow & \approx & \downarrow & \approx & \downarrow & \approx & \\
1 & \to & C_n & \to & \mathbb{C}^d & \to & \mathbb{C}^d & \to & 1 \\
\times n & & & & & & & & \\
\end{array}
\]  

(2)

where $n = (n_1, \ldots, n_d)$, $n_i \in \mathbb{N}$,

\[C_n = \{(z_1, \ldots, z_l) \in \mathbb{C}^d \mid |z_i^n| = 1, \forall i\}\]

and $\times n$ is the homomorphism $(z_1, \ldots, z_l) \mapsto (z_1^n, \ldots, z_l^n)$. 

Let $B$ be a Borel subgroup of $G$ containing $Z\cdot T$. The characters $\chi_1, \ldots, \chi_t$ extend to $B$ and give a homomorphism $B \to A \approx (\mathbb{C}^* \times Z')/F$ by sending $b \in B$ to $(\chi_1(b), \ldots, \chi_t(b), 1) \cdot F$. Clearly $H$ is a connected reductive algebraic group and $B_H = (\mathbb{C}^* x B)/F$ is a Borel subgroup of $H$. Again the characters $\chi_1, \ldots, \chi_t$ extend to $B_H$ by defining $\chi_i((z_1, \ldots, z_t, b) \cdot F) = z_1 \cdot \chi_i(b)$ and we have a homomorphism $B_H \to A$ defined by $(z_1, \ldots, z_t, b) \cdot F \mapsto (z_1 \chi_1(b), \ldots, z_t \chi_t(b), 1) \cdot F$. We then have the commutative diagram

$$
\begin{array}{ccc}
B_H & \to & A \\
\downarrow & & \downarrow \\
H/G & \approx & A/Z'
\end{array}
$$

(3)

The image $\bar{B} = B/Z$ of $\bar{G} = G/Z$ is a Borel subgroup of $\bar{G}$ and $G/B \to \bar{G}/\bar{B}$ is an isomorphism. The projection $\mathbb{C}^* x G \to G$ induces an isomorphism $H/B_H \to G/B$ which is the inverse of the isomorphism $G/B \to H/B_H$ induced by the inclusion $G \hookrightarrow H$. We then have the commutative diagram

$$
\begin{array}{ccc}
H & \leftarrow & G \\
\downarrow & & \downarrow \\
H/B_H & \approx & G/B \approx \bar{G}/\bar{B}
\end{array}
$$

(4)

Let $M \to G/B$ be the $A$-bundle obtained from the $B$-bundle $G \to G/B$ by the extension of structure group $B \to A$. Note that $M$ is also the $A$-bundle obtained from the $B_H$-bundle $H \to H/B_H$ by the extension of structure group $B_H \to A$ as follows from the diagrams (3) and (4). The group $H$ operates on $G/G'$ (by left multiplication through $\rho'$) and on $G/B \approx H/B_H$. Since $M \to G/B \approx H/B_H$ is a bundle associated to $H \to H/B_H$, $H$ operates on $M$ also, compatibly with its action on $H/B_H$. Further $H$ is precisely the automorphism group of the 'structure' consisting of $G/G' \times G/B$ and $M$, i.e., to put it more precisely, given an isomorphism $\phi: G/G' \times G/B \to G/G' \times G/B$ induced by an element of $G$ and an $A$-bundle isomorphism $\varphi_M: M \to M$ over $\varphi_2$, where $\varphi_2: G/B \to G/B$ is given by $\varphi$, then there exists an unique $h \in H$ whose action gives $\phi$ and $\varphi_M$. The existence of such an $h$ is clear and for uniqueness note that since $\cap_{g \in G} gBg^{-1} = Z$, only $Z'$ acts trivially on $G/G' \times G/B$ and the action of $Z'$ on $M$ is faithful since it is given by the characters $\chi_1, \ldots, \chi_t$.

Let $\mathbf{P}_M$ be the sheaf associated to the functor $\mathbf{P}_M: (\text{Sch}/R'_2) \to (\text{Sets})$ which associates to $f: S \to R'_2$ the set of isomorphism classes of $A$-bundles on $\mathbf{f}^*(\mathcal{E}_2/B)$, where $\mathbf{f} = f \times id_X: S \times X \to R'_2 \times X$, such that for every point $(s, x) \in S \times X$, and for any trivialization $\varphi_{(s, x)}: \mathcal{E}_{(s, x)} \cong \mathcal{E}_{2,(s, x)} / B \to G/B$ there exists an isomorphism $\tilde{\varphi}_{(s, x)}$ of $A$-bundles over $\varphi_{(s, x)}$: ...
We define $P_M$ on morphisms to be pull back. We shall show that $\tilde{\Gamma}(\rho', \delta'_2)$ is isomorphic to $\tilde{P}_M$.

Let $\Lambda \in P_M(f:S \rightarrow R'_2)$. Consider the functor $F:(\text{Sch}/S \times X) \rightarrow (\text{Sets})$ which associates to $f':S' \rightarrow S \times X$ the set of pairs $(\phi, \phi_M)$, where $\phi$ is an isomorphism of the trivial $(G/G' \times G'/Z)$-bundle on $S'$ with $f'^*(\tilde{f}^*(\delta_2'))$ and $\phi_M$ is an isomorphism of $A$-bundles such that

$$
\phi
$$

commutes. On morphisms $F$ is defined to be pull back. Let $H = S \times X \times H$ be the constant group scheme over $S \times X$ (considered as a functor) defined by $H$. Since $H$ is the automorphism group of the structure consisting of $G/G' \times G/B$ and $M \rightarrow G/B$, as explained above, it is easy to see that $F$ is a principal homogeneous space under $H$.

Since $H$ is an affine algebraic group scheme over $S \times X$ it follows by descent ([SGA 3], exposé XXIV) that $F$ is representable. Thus we get an $H$-bundle $\delta_A$ on $S \times X$ and, from the construction, there is a natural isomorphism $\phi': \rho'_* \delta_A \rightarrow \tilde{f}^*(\delta'_2)$. We define the morphism $\Phi: \tilde{P}_M \rightarrow \tilde{\Gamma}(\rho', \delta'_2)$ by setting $\Phi_S(\Lambda) = (\delta_A, \phi)$. We now construct an inverse $\Psi: \tilde{\Gamma}(\rho', \delta'_2) \rightarrow \tilde{P}_M$ for $\Phi$. Let $(\delta, \phi) \in \tilde{\Gamma}(\rho', \delta'_2)$ (f: $\tilde{S} \rightarrow R'_2$). Thus $\delta : S \times X$ is an $H$-bundle and $\phi : \rho'_* \delta \rightarrow \tilde{f}^*(\delta'_2)$ is an isomorphism.

Now $\phi$ induces an isomorphism $\delta/B_H \cong \tilde{f}^*(\delta'_2(G/B))$ (see diagrams (3) and (4)). Note that $\delta'_2(G/B) = \delta'/B$. Define $\Lambda : \tilde{f}(\delta'_2/B)$ to be the $A$-bundle obtained from the $B_H$-bundle $\delta \rightarrow \delta/B_H$ by the extension of structure group $B_H \rightarrow A$ (see diagram (3)). Define $\Psi_S((\delta, \phi)) = \Lambda$. Then it is straightforward to check that $\Psi$ is a morphism inverse to $\Phi$.

Let $\tilde{P}$ be the sheaf associated to the functor $P:(\text{Sch}/R'_2) \rightarrow (\text{Sets})$ which associates to $f:S \rightarrow R'_2$ the set of isomorphism classes of $A$-bundles on $f'^*(\delta'_2(G/B))$. Since $A \cong C^* \times \cdots \times C^*$ (diagram (2)) an $A$-bundle is just an $l$-tuple of line bundles and hence $P \cong \text{Pic} \times \cdots \times \text{Pic}$ the $l$-fold product of the relative Picard functor of $\delta'_2(G/B)/R'_2$ ([TDTE, V], §I). Since the morphism $\delta'_2(G/B) = \delta'/B \rightarrow R'_2$ is projective (Remark 4.8.3), flat and smooth with irreducible fibers, $\text{Pic}$ is representable by a locally finite type scheme $P$ over $R'_2$ ([TDTE V], Theorem 3.1; [TDTE VI], Theorem 4.1, Corollary 4.2) and hence so is $\tilde{P}$. Clearly $P_M$ is a subfunctor of $P$.

Let $\tilde{F}_1$ (resp. $\tilde{F}_2$) be the sheaf associated to the functor $F_1$ (resp. $F_2$):$(\text{Sch}/R'_2) \rightarrow (\text{Sets})$ which associates to $f:S \rightarrow R'_2$ the set of isomorphism classes of $H G$-bundles on
S × X (resp. on \( \tilde{f}^*(\mathcal{E}'_2(G/B)) \)). The morphism \( \mathcal{E}'_2(G/B) \to R'_2 \times X \) induces a morphism \( F_1 \to F_2 \). Since \( \mathcal{E}'_2(G/B) \to R'_2 \times X \) is projective and flat with irreducible fibers by ([TDTE VI], Proposition 2.1 and [EGA III], Corollary 7.8.8) it follows that \( F_1 \to F_2 \) is a monomorphism.

The exact sequence \( 1 \to G \to H \to H/G \to 1 \) gives morphisms \( \Gamma(\rho, \mathcal{E}_2') \to \Gamma(\rho', \mathcal{E}_2') \to F_1 \) and an exact sequence (part of which is)

\[
H^0(S \times X, H/G) \to H^1(S \times X, G) \to H^1(S \times X, H) \to H^1(S \times X, H/G)
\]

([17], §3.6, Propositions 11 and 12; see also [SGA I], exposé XI, §4). Since \( H = Z[H] \cdot G \) it follows that \( H^0(S \times X, H/G) \) operates trivially on \( H^1(S \times X, G) \). Therefore \( H^1(S \times X, G) \to H^1(S \times X, H) \) is an injection. This means that \( \Gamma(\rho, \mathcal{E}_2') \to \Gamma(\rho', \mathcal{E}_2') \) is a monomorphism.

We use the isomorphism \( H/G \to A/Z' \) of diagram (1) and the isomorphism \( A/Z' \cong C^{*l} \) of diagram (2) to get an isomorphism of \( F_2 \) with \( P \). We have the commutative diagram

\[
\begin{array}{cccc}
F_1 & \xleftarrow{\mathcal{E}_2} & \Gamma(\rho', \mathcal{E}_2') & \to \to \Gamma(\rho, \mathcal{E}_2') & F_2 \\
\downarrow & & \downarrow & & \downarrow \\
P_M & \xleftarrow{P} & P & \to \to P
\end{array}
\]

where \( P \to P \) is induced by \( \times n \) of diagram (2).

It follows from ([TDTE VI], Theorem 2.5) that the morphism \( P \to P \) corresponding to \( P \to P \) is étale. Corresponding to the trivial \( H/G \)-bundle on \( R'_2 \times X \) we get a morphism \( R'_2 \to P \). Let \( R'_3 \) be the fiber product

\[
\begin{array}{cccc}
R'_3 & \xrightarrow{\mathcal{E}_2} & P \\
\downarrow & & \downarrow \\
R'_2 & \to & P
\end{array}
\]

Then it follows from ([TDTE VI], Corollary 4.2 and [EGA II], Corollary 5.4.3) that \( R'_3 \) is proper over \( R'_2 \). Also \( R'_3 \) is étale over \( R'_2 \).

Using the fact that an \( H \)-bundle \( \mathcal{E} \) comes from a \( G \)-bundle if and only if \( \mathcal{E} (H/G) \) is trivial, it is straightforward to check that \( R'_3 \) represents the sheaf \( \tilde{\mathcal{T}}(\rho, \mathcal{E}_2') \). Let \( R_3 \) be the subscheme of \( R'_3 \) corresponding to \( G \)-bundles of type \( \tau \). Then clearly \( R_3 \to R'_2 \) is again étale and finite (since topological type is a discrete invariant).

4.15.2. PROPOSITION

The scheme \( R_3 \) with the action of \( GL(n, \mathbb{C}) \subset G/G' \times GL(n, \mathbb{C}) \) (given by Lemma 4.10) is a universal space with group \( GL(n, \mathbb{C}) \) for semistable \( G \)-bundles of topological type \( \tau \).

The \( GL(n, \mathbb{C}) \)-equivariant morphism \( R_3 \to R'_2 \) is étale and finite and hence \( R_3 \) is nonsingular.
5. Existence of a quotient space

5.1. Lemma. Suppose a reductive algebraic group $H$ acts on the schemes $Y$ and $Z$. If $f: Y \to Z$ is an affine $H$-equivariant morphism and $Z$ has a good quotient $q: Z \to \tilde{Z}$ modulo $H$ then $Y$ has a good quotient $q: Y \to \tilde{Y}$ modulo $H$ and the induced morphism $\tilde{f}: \tilde{Y} \to \tilde{Z}$ is affine.

If moreover $f$ is finite then $\tilde{f}$ is also finite. When $f$ is finite and $p: Z \to \tilde{Z}$ is a geometric quotient then $q: Y \to \tilde{Y}$ is also a geometric quotient.

Proof. Let $\{U_i\}$ be a covering of $\tilde{Z}$ by affine open sets. Let $U_i = (p \circ f)^{-1}(U_i)$ Then $\{U_i\}$ is a covering of $Y$ by affine $H$-invariant open sets. Since $U_i$ is affine there exists a good quotient $q_i: U_i \to \tilde{U}_i$ of $U_i$ modulo $H$, with $\tilde{U}_i$ affine ([22], Theorem 1.1(A)). We shall now give a patching up data for $\{U_i\}$. Let $f_i: \tilde{U}_i \to U_i$ be the morphism induced by $f$. Let $f_i: \tilde{U}_i \to \tilde{U}_j$ be the morphism induced by $f$. Let $U_{ij} = f_i^{-1}(U_i \cap U_j)$. Then $q_i^{-1}(U_{ij}) = (p \circ f)^{-1}(U_i \cap U_j) = U_i \cap U_j$. Since, being a good quotient is local with respect to the base ([20], §3, Property 1, p. 356) $q_i: U_i \cap U_j \to U_{ij}$ is a good quotient of $U_i \cap U_j$ modulo $H$. Interchanging $i$ and $j$, $q_j: U_i \cap U_j \to U_{ij}$ is also a good quotient of $U_i \cap U_j$ modulo $H$. A good quotient, being a categorical quotient, is unique. Therefore we have natural isomorphisms $h_{ij}: \tilde{U}_{ij} \to \tilde{U}_{ji}$. The $h_{ij}$'s satisfy the cocycle condition and we can patch up the $\tilde{U}_i$ by $h_{ij}$ along $\tilde{U}_{ij}$ to get a prescheme $\tilde{Y}$. The $q_i$ patch up to give a morphism $q: Y \to \tilde{Y}$ and the $f_i$ a morphism $\tilde{f}: \tilde{Y} \to \tilde{Z}$. Clearly $\tilde{f}$ is affine and $\tilde{Z}$ being a (separated) scheme $\tilde{Y}$ is also a (separated) scheme ([EGA II], Proposition 1.2.4). Again since being a good quotient is local with respect to the base $q: Y \to \tilde{Y}$ is a good quotient.

To show that $\tilde{f}$ is finite if $f$ is finite we can assume that $Y$ and $Z$ are affine. So let $Y = \text{Spec } A$ and $Z = \text{Spec } B$ and $f$ be given by the homomorphism $f: B \to A$ making $A$ into a finite $B$-module. Then $\tilde{Z} = \text{Spec } B^H$ and $\tilde{Y} = \text{Spec } A^H$ ([22], Theorem 1.1(A)) and $\tilde{f}$ is given by $B^H \to A^H$ the restriction of $f$ (where $A^H$, $B^H$ are the rings of invariants under $H$). We have to show that $A^H$ is a finite $B^H$-module. Since $A$ is a finite $B$-module there exists $a_1, \ldots, a_n \in A^H$ such that any $a \in A^H$ can be written as $a = \Sigma_{i=1}^n f(b_i) a_i$ with $b_i \in B$. Applying Reynolds' operator $P$ on both sides we get $a = \Sigma f(P(b_i)) a_i$ by Reynolds' identity and functoriality (cf. [10], Chapter 1, Theorem 1.19). Since $P(b_i) \in B^H$ this proves that $A^H$ is a finite $B^H$-module. The last assertion of the lemma is easily verified.

5.2. It follows immediately from the preceding lemma and the results of section 4 to that prove the existence of a good quotient for $R_3$ modulo $GL(n, \mathbb{C})$ it is enough to prove the existence of a good quotient of $R_1$ modulo $GL(n, \mathbb{C})$ (or equivalently, modulo $SL(n, \mathbb{C})$ since the scalars act trivially on $R_1$, see Remark 4.13.2). We shall prove this by using Mumford's theory of stable and semistable points for actions of reductive groups ([10]).

5.3. Let $ad \in g^* \otimes g^* \otimes g = \text{Hom}(g^* \otimes g^*, g') = \mathcal{H}$ be the tensor which gives the Lie algebra structure of $g'$, i.e., for $x, y \in g'$, $ad(x, y) = [x, y]$. For the natural action of $GL(g')$ on the tensor space $\mathcal{H}$ the stabilizer of the line (ad) generated by $ad$ is $C^* \times \text{Aut } g'$, where $\text{Aut } g'$ is the group of Lie algebra automorphisms of $g'$. Therefore
\[
\begin{array}{c}
R_1 \times X \\
\downarrow \\
R \times X \\
\downarrow \\
\vdots
\end{array} \quad \xrightarrow{f_1} \quad Q(Y) \\
\downarrow \\
\downarrow \\
\downarrow \\
f
\quad \xrightarrow{f} \quad G_{n,r}
\]

Now the morphisms \( Y \subset \bar{Y} \subset P(H) \) give rise to the morphisms \( Q(Y) \subset Q(\bar{Y}) \subset Q(P(H)) \). Since \( Y \subset \bar{Y} \) is an open immersion \( Q(Y) \subset Q(\bar{Y}) \) is an open immersion and since \( \bar{Y} \subset P(H) \) is a closed immersion \( Q(\bar{Y}) \subset P(Q(H)) \) is also a closed immersion (§2.4). Let \( R = Q(H) = Q^* \otimes Q^* \otimes Q \). Let \( \Lambda \) be the relatively ample line bundle \( O_{P(R)}(1) \) on \( P(R) \). Let \( \Lambda \) denote also its restriction to \( Q(\bar{Y}) \). Consider the functor
\[
\Gamma : (\text{Sch}/R) \to (\text{Sets})
\]
defined by
\[
\Gamma(S \to R) = \{ \sigma \in \text{Hom}_{G_{n,r}}(S \times X, Q(\bar{Y})) | (\sigma^*(\Lambda))_{|S \times X} \cong L^{-m}, s \in S \}.
\]

This is representable by an algebraic scheme \( S_1 \) over \( R \). Let \( g_1 : S_1 \times X \to Q(\bar{Y}) \) be the universal section. It is easy to see that the functor represented by \( R_1 \) is a subfunctor of \( \Gamma \) (Remark 4.13.2). Since \( Q(Y) \) is open in \( Q(\bar{Y}) \) it follows that \( R_1 \subset S_1 \) is an open immersion. We have the commutative diagram
\[
\begin{array}{c}
R_1 \times X \\
\downarrow \\
S_1 \times X \\
\downarrow \\
R \times X \\
\downarrow \\
\vdots
\end{array} \quad \xrightarrow{f_1} \quad Q(Y) \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\vdots \\
\quad \xrightarrow{f} \quad G_{n,r}
\]

5.3.1. Lemma. The morphism \( S_1 \to R \) is proper.

Proof. We make use of the valuative criterion for properness. Let \( A \) be a discrete valuation ring over \( \mathbb{C} \) with residue field \( \mathbb{C} \) and quotient field \( K \). Suppose we have morphisms

\[
\begin{array}{ccc}
\text{Spec } K & \xrightarrow{\varphi_K} & S_1 \\
\downarrow & & \downarrow \\
\text{Spec } A & \xrightarrow{\varphi} & R
\end{array}
\]

By the universal properties of \( R \) and \( R_1 \) (recall that \( R \) and \( R_1 \) represent the corresponding functors of sections over the category of locally noetherian schemes ([TDTE IV])). We have

\[
\begin{array}{c}
\text{Spec } K \\
\downarrow \\
\text{Spec } A
\end{array} \quad \xrightarrow{\varphi_K} \quad S_1 \\
\downarrow \\
\downarrow \\
\text{Spec } A \quad \xrightarrow{\varphi} \quad R
\]

\[
(1)
\]
On Spec $K \times X$, we have the exact sequence
\[0 \to \tilde{\phi}_K^*(\Lambda) \to \tilde{\phi}_K^*(\mathbb{R}) \to \tilde{\phi}_K^*(\mathbb{R}/\Lambda) = V_K \to 0.\]

We can extend the quotient vector bundle $V_K$ to Spec $A \times X$ as a quotient coherent sheaf $V$ of $\tilde{\phi}^*(\mathbb{R})$ flat over Spec $A$ (cf. [EGA IV], Proposition 2.8.1). Let $V_0$ be the restriction of $V$ to Spec $\mathbb{C} \times X \cong X$ corresponding to the closed point of Spec $A$. We then have the surjection $\tilde{\phi}_K^*(\mathbb{R}) \to V_0 \to 0$ of coherent sheaves on $X$. Note that $\tilde{\phi}_K^*(\mathbb{R})$ is a semistable vector bundle (by Proposition 3.17 since $\tilde{\phi}_K^*(\mathbb{Q})$ is semistable of degree $-r^3md_0$ and therefore $\mu(\tilde{\phi}_K^*(\mathbb{R})) = -md_0$). From the definition of $S_1$ and the flatness of $V$ over Spec $A$ it follows that $\deg V_0 = -r^3md_0 - md_0$. Since $X$ is a curve we can write $V_0 = V_0' \oplus T$ where $V_0'$ is locally free and $T$ is torsion. Since $\tilde{\phi}_K^*(\mathbb{R})$ is semistable, $V_0'$ being a quotient $\mu(V_0) \geq -md_0$. Therefore $\deg V_0' \geq -(r^3 + 1)md_0$. On the other hand $T$ being a torsion sheaf $\deg T > 0$ and since $\deg V_0 = -(r^3 + 1)md_0$ we have $\deg V_0' \leq -(r^3 + 1)md_0$. This shows that $\deg V_0 = -(r^3 + 1)md_0$. Therefore $T = 0$ and $V_0 = V_0'$. Hence $V_0'$ is locally free. Then $V$ is a vector bundle on Spec $A \times X$ (cf. [19], Lemma 6.17). By the universal property of $\mathbb{P}(\mathbb{R})$, $V$ determines a morphism Spec $A \times X \to \mathbb{P}(\mathbb{R})$ (cf. [EGA II], Theorem 4.2.4; note that in our notation $\mathbb{P}$ stands for the dual of what $\mathbb{P}$ stands for in this reference). But since $Q(Y)$ is closed in $\mathbb{P}(\mathbb{R})$ this morphism goes into $Q(Y)$ proving that the broken arrow in diagram (2) can be realized. This immediately implies that the broken arrow in diagram (1) can be realized.

5.4. We now recall briefly some definitions and results from Mumford's 'geometric invariant theory' ([10]; see also [22]). Let a reductive group $H$ act on a projective algebraic scheme $Y$. Let $\Lambda \to Y$ be an ample line bundle on $Y$ and $H$ act on $\Lambda$ also as a group of line bundle isomorphisms compatible with its action on $Y$ i.e. $\Lambda$ has a $H$-linearization ([10], Chapter I, §3). Then a point $y \in T$ is called semistable if for some $m > 0$ there is a $H$-invariant section $seH^0(Y, \Lambda^m)$ such that $s(y) \neq 0$. If moreover every orbit of $H$ in $Y_s = \{x \in Y | s(x) \neq 0\}$ is closed and of the same dimension as $H$, $y$ is called a (properly) stable point ([10], Chapter I, §4).

5.4.1. The set $Y_{ss}$ (resp. $Y_s$) of semistable (resp. stable) points is open in $Y$ and there exists a good quotient $p: Y_{ss} \to \overline{Y}_{ss}$ modulo $H$ such that $\overline{Y}_{ss}$ is projective.

There is an open subscheme $\tilde{Y}_s \subset \overline{Y}_{ss}$ such that $p:p^{-1}(\tilde{Y}_s) \to \tilde{Y}_s$ is a 'geometric quotient' ([10], Chapter I, Theorem 1.10 and the remark on p. 40; [22], Theorem 1.1(B)).

5.4.2. Let $\lambda: \mathbb{C} \to G$ be a 1-PS (i.e. 1-parameter subgroup) and $y \in Y$. Then $\lim_{t \to 0} \lambda(t)y = y_0$ exists (since $Y$ is projective) and $y_0$ is fixed by $\lambda$. Let $\varphi \Mapsto \rho'$ be the character by which $\lambda$ operates on $\Lambda_y$. Then we define $\rho'(y, \lambda) = -r$. (In this definition $\gamma$ can be an arbitrary line bundle not necessarily ample.) Then a point $y \in Y$ is semistable
(resp. stable) if and only if \( \mu^\Lambda(y, \lambda) \geq 0 \) (resp. > 0) for every 1-PS \( \lambda \) of \( G \) ([10], Chapter 2, §1, Theorem 2.1).

5.4.3. Let \( H \) act on the projective algebraic scheme \( Y' \) and \( f: Y' \to Y \) an \( H \)-equivariant morphism. Then \( \mu^\Lambda(y', \lambda) = \mu^\Lambda(f(y'), \lambda), y' \in Y' \) ([10], p.49).

5.4.4. Given a 1-PS \( \lambda \) of \( G \) there is a parabolic subgroup \( P(\lambda) \supseteq \lambda \) such that \( \mu^\Lambda(y, \lambda) = \mu^\Lambda(y, g^\lambda g^{-1}) \) for every \( g \in P(\lambda) \). This \( P(\lambda) \) depends only on \( \lambda \) and not on the scheme \( Y \) or \( \Lambda \) ([10], Chapter 2, §2, [22], Lemma 3.1, Proposition 3.1).

5.5. Now by the usual 'diagonal argument' we can choose an \( N \)-tuple \((x_1, \ldots, x_N)\) of points in \( X \) for \( N \) sufficiently large such that the morphism \( R_1 \to Q(\overline{Y})^N = Q(\overline{Y}) \times \cdots \times Q(\overline{Y}) \), \( N \) factors, given by evaluating \( f_1 \) at the points \( x_i \), i.e. \( R_1 \ni r \to (f_1(r, x_1), \ldots, f_1(r, x_N)) \), is an injection ([19], p.326). It is proved in ([20], §3, Lemma 2) that we can choose the \( N \)-tuple \((x_1, \ldots, x_N)\) such that under the morphism \( R \to G^N_{n,r} = Z \) (defined by \( R \ni r \to (f_1(r, x_1), \ldots, f_1(r, x_N)) \)) \( R \) goes into the open subscheme of semistable points \( Z_{ss} \) of \( Z \) for the action of \( SL(n, C) \) (and the natural polarization of the Grassmannian) and moreover such that the morphism \( R \to Z_{ss} \) is proper.

We have the commutative diagram

\[
\begin{array}{ccc}
R_1 & \longrightarrow & Q(\overline{Y})^N \\
\downarrow & & \downarrow \\
S_1 & \longrightarrow & \mathcal{P}(\mathbb{R})^N \\
\downarrow & & \\
R & \longrightarrow & G^N_{n,r} \\
\end{array}
\]

We shall give a suitable ample line bundle on \( Q(\overline{Y})^N \) and prove that \( R_1 \) goes into the semistable points of \( Q(\overline{Y})^N \) for the natural action of \( SL(n, C) \) on \( Q(\overline{Y})^N \).

Let \( M \) be the very ample line bundle on the Grassmannian \( G^N_{n,r} \) corresponding to the natural embedding of \( G^N_{n,r} \) in \( \mathcal{P}(\Lambda^{n-r} C^n) \). There is a natural \( SL(n, C) \)-linearization of \( M \) given by the action of \( SL(n, C) \) on \( \Lambda^{n-r} C^n \). As defined in §5.3, let \( \Lambda \) be the tautological line bundle on \( \mathcal{P}(\mathbb{R}) \) corresponding to the vector bundle \( \mathbb{R} \). Then \( \Lambda \) is relatively ample for the morphism \( p: Q(\overline{Y}) \to G^N_{n,r} \). Therefore the line bundle \( p^*(M)^a \otimes \Lambda^b \) is ample on \( Q(\overline{Y}) \) for \( a \gg b \) ([EGA II] §4.6, [22], §5).

The action of \( SL(n, C) \) on \( G^N_{n,r} \) has a natural lift to an action on the bundle \( Q \) and hence on the associated bundle \( \mathbb{R} = Q^* \otimes Q^* \otimes Q \). This gives an \( SL(n, C) \)-linearization of \( \Lambda \). By ([22], Proposition 5.1) we can choose the positive integers \( a, b \) with \( b/a \) sufficiently small such that \( p(\overline{Y})^{ss} = Z_{ss} \) where \( Q(\overline{Y})^{ss} \) is the set of semistable points of \( Q(\overline{Y})^N \) for the action of \( SL(n, C) \) and the ample line bundle on \( Q(\overline{Y})^N \) which is the product of the line bundles \( p^*(M)^a \otimes \Lambda^b \) on each factor \( Q(\overline{Y}) \).

5.5.1. Lemma. The point \((ad)\in \mathcal{P}(\mathbb{G}^* \otimes \mathbb{G}^* \otimes \mathbb{G}^*)\) (cf. §5.3) is semistable for the natural action of \( SL(\mathbb{G}) \) and the line bundle \( O(1) \) on \( \mathcal{P}(\mathbb{G}^* \otimes \mathbb{G}^* \otimes \mathbb{G}^*) \).

Proof. Let \( \varphi: (\mathbb{G}^* \otimes \mathbb{G}^* \otimes \mathbb{G}^*) = \text{Hom}(\mathbb{G}, \text{End } \mathbb{G}) \to \mathbb{G}^* \otimes \mathbb{G}^* = (\mathbb{G} \otimes \mathbb{G})^* \) be defined by \( \varphi(f)(x \otimes y) = \text{trace}(f(x) \circ f(y)), f \in \text{Hom}(\mathbb{G}, \text{End } \mathbb{G}) \) and \( x, y \in \mathbb{G} \). Then \( \varphi \) is an \( SL(\mathbb{G}) \)-equivariant morphism. Choose an arbitrary linear space isomorphism of \( \mathbb{G}^* \)
with $\mathcal{F}$. Then we get an isomorphism $\mathcal{F}' \otimes \mathcal{F}'' \cong \text{End}(\mathcal{F})$. Let $\det: \mathcal{F}' \otimes \mathcal{F}'' \cong \text{End}(\mathcal{F}') \to \mathbb{C}$ be the determinant map. Then $\det \circ \phi$ is an $\text{SL}(\mathcal{F})$-invariant polynomial on $\mathcal{F}' \otimes \mathcal{F}'' \otimes \mathcal{F}'$ and $(\det \circ \phi)(ad)$ is the determinant of the Killing form of $\mathcal{F}'$ and hence is non-zero, $\mathcal{F}'$ being semisimple.

5.5.2. Lemma. The point $(x_1, \ldots, x_N) \in \mathbb{Q}(Y)^N \subset \mathbb{Q}(\bar{Y})^N$ is semistable if and only if $(p(x_1), \ldots, p(x_N))$ is semistable in $G_{n,r}'$. 

Proof. Let $(p(x_1), \ldots, p(x_N)) = (y_1, \ldots, y_N)$. If $(x_1, \ldots, x_N)$ is semistable then by our choice of $p^*(M)^r \otimes \Lambda^b, (y_1, \ldots, y_N)$ is semistable ([22] Proposition 5.1).

Suppose $(y_1, \ldots, y_N)$ is semistable. Let $\lambda$ be a 1-PS of $\text{SL}(n, \mathbb{C})$. Let $P(\lambda)$ be the canonical parabolic subgroup associated to $\lambda$ (§ 5.4.4). Let $x \in \mathbb{Q}(\bar{Y})$ and $y = p(x) \in G_{n,r}$. Then $y$ is an $r$-dimensional quotient of $\mathbb{C}^n$. Let $P_r$ be the maximal parabolic subgroup of $\text{SL}(n, \mathbb{C})$ which is the stabilizer of $y \in G_{n,r}$. By Bruhat's lemma $P_y \cap P(\lambda)$ contains a maximal torus $T$ of $\text{SL}(n, \mathbb{C})$. Since $\lambda \subset P(\lambda)$ there is a $g \in P(\lambda)$ such that $g\lambda g^{-1} \subset T$. Then

$$\mu(x, \lambda) = \mu(x, g\lambda g^{-1}),$$

where $\mu$ stands for $\mu^p(M)^r \otimes \Lambda^b$ (§ 5.4.4). Since $T \subset P_y$ we can choose a basis $e_1, \ldots, e_r, e_{r+1}, \ldots, e_n$ for $\mathbb{C}^n$ such that the images $\bar{e}_i$ of $e_i$ in $y, i = 1, \ldots, r$ form a basis for $y$ and such that $T$ becomes the group of diagonal matrices with respect to $e_1, \ldots, e_n$. Now

$$\mu(x, \lambda) = a_\mu^M(y, \lambda) + b\lambda^\mu(y, \lambda) = a_\mu^M(y, g\lambda g^{-1}) + b\lambda^\mu(x, g\lambda g^{-1}).$$

(2)

We shall first calculate $\lambda^\mu(x, \lambda) = \mu^\lambda(x, \lambda'_g)$ where $\lambda'_g = g\lambda g^{-1}$. From the definition of $\lambda'_g(2.1$). Let $\lambda'_g(z) = z^s e_i, z \in \mathbb{C}^*, s \in \mathbb{Z}, i = 1, \ldots, r$. Define the 1-PS $\lambda'$ of the center of $\text{GL}(y)$ by $\lambda'(z) = z^s e_i, z \in \mathbb{C}^*, s = s_1 + \cdots + s_r$.

Since $\lambda'$ leaves $y$ invariant it follows easily from definitions (cf. § 5.4.3) that for calculating $\mu^\lambda(x, \lambda'$ we can restrict our attention to $y$ or equivalently the subspace generated by $e_1, \ldots, e_r$. For the action on the quotient space $y$ we have $\lambda'_g = \lambda' \lambda''$ where $\lambda''(z) = z^s e_i, z \in \mathbb{C}^*, i = 1, \ldots, r$. Note that $\lambda'' \subset \text{SL}(y)$. Since $\lambda'$ is in the center of $\text{GL}(y)$ it is easy to see that

$$\mu^\lambda(x, \lambda'_g) = \mu^\lambda(x, \lambda') + \mu^\lambda(x, \lambda'')$$

(4)

and that

$$\mu^\lambda(x, \lambda') = s.$$  

(5)

Moreover we get from ([10], Chap. 4, §4, eq.(4), p. 67) that

$$\mu^M(y, \lambda'_g) = s.$$  

(6)

Therefore

$$\mu^\lambda(x, \lambda) = \frac{1}{r} \{\mu^M(y, \lambda'_g) + \mu^\lambda(x, \lambda'')\}$$

$$= \frac{1}{r} \{\mu^M(y, \lambda) + \mu^\lambda(x, \lambda'')\}$$
with $\lambda''$ a 1-PS of $SL(y)$, by (1), (4), (5) and (6). This gives
\[
\mu(x, \lambda) = a\mu^M(y, \lambda) + b\mu^N(x, \lambda)
\]
\[= \left(a + \frac{b}{r}\right)\mu^M(y, \lambda) + \frac{b}{r}\mu^N(x, \lambda''). \tag{7}\]
Writing $x_i$ for $x$ in (7) and summing through $i$ we get
\[
\mu(x_1, \ldots, x_N, \lambda) = \left(a + \frac{b}{r}\right)\mu^M((y_1, \ldots, y_N), \lambda)
\]
\[+ \frac{b}{r} \sum_{i=1}^{N} \mu^N(x_i, \lambda_i''). \tag{8}\]
Since $(y_1, \ldots, y_N)$ was assumed to be semistable $\mu^M((y_1, \ldots, y_N), \lambda) \geq 0$. Since $x_i \in Q(Y)$ it follows from Lemma 5.5.1 that $\mu^N(x_i, \lambda_i'') \geq 0 \forall i$ (cf. 5.4.3). Therefore $\mu((x_1, \ldots, x_N), \lambda) \geq 0$ which proves $(x_1, \ldots, x_N)$ is semistable (§ 5.4.2).

5.5.3. Lemma. Under the morphism $R_1 \to Q(Y)_N \subset Q(\overline{Y})_N$ maps into the open subscheme $Q(\overline{Y})_{ss}$ and hence $R_1 \to Q(\overline{Y})_N$ factors as $R_1 \to Q(\overline{Y})_{ss} \to Q(\overline{Y})_N$.

Proof. Since under the morphism $R \to G_{nr} = Z$, $R$ maps into $Z_{ss}$ this follows immediately from the preceding lemma.

5.6. Lemma. The injective morphism $R_1 \to Q(\overline{Y})_{ss}$ is proper.

Proof. We use the valuation criterion. Let $A$ be a discrete valuation ring over $\mathbb{C}$ with residue field $\mathbb{C}$ and quotient field $K$. Suppose we are given
\[
\begin{array}{c}
\text{Spec } K \\
\downarrow
\end{array}
\begin{array}{c}
R_1 \\
\downarrow
\end{array}
\begin{array}{c}
\text{Spec } A \\
\downarrow
\end{array}
\begin{array}{c}
Q(\overline{Y})_{ss}
\end{array}
\]  

We have to complete the broken arrow.

From (1) and the diagram in § 5.5, using the facts that $p(Q(\overline{Y})_N) \subset Z_{ss}$ and the morphism $R \to Z_{ss}$ and $S_1 \to R$ are proper ([20], § 3, Lemma 2 and Lemma 5.3.1) it follows that we can get a lift Spec $A \to S_1$ giving the commutative diagram
\[
\begin{array}{c}
\text{Spec } K \\
\downarrow
\end{array}
\begin{array}{c}
R_1 \\
\downarrow
\end{array}
\begin{array}{c}
\text{Spec } A \\
\downarrow
\end{array}
\begin{array}{c}
Q(\overline{Y})_{ss}
\end{array}
\]

We will be through if we show that the closed point of Spec $A$ maps into $R_1$ under this morphism Spec $A \to S_1$.

Let $V = c \cdot (-m)$ be the vector bundle (semistable of degree zero) corresponding to the image of the closed point of Spec $A$ in $R$ under the composite Spec $A \to S_1 \to R$. The image
of the closed point of Spec \( A \) in \( S_1 \) then gives a section of \( Q_0(\overline{Y}) \subset P(V^* \otimes V^* \otimes V) \rightarrow X \) which actually comes from a section \( s \in H^0(X, V^* \otimes V^* \otimes V) \) with \( s(x) \neq 0 \) for \( x \in X \). This section \( s \), since its image is in \( Q_0(\overline{Y}) \), gives a Lie algebra structure on the fibers of \( V \) (cf. Remark 4.13.2). Let \( s \in H^0(X, V^* \otimes V^*) \) correspond to the Killing form of the Lie algebra structure given by \( s \). We shall show that \( s \) is nondegenerate on all fibers.

Suppose \( s \) is not nondegenerate on all fibers. Then the homomorphism \( V \rightarrow V^* \) induced by \( s \) has a nontrivial kernel sheaf. Since both \( V \) and \( V^* \) are semistable vector bundles of degree zero the kernel is actually a subbundle \( V_1 \), semistable and of degree zero ([19], Proposition 3.1). Then \( V_1 \) is a solvable ideal in \( V \), i.e. the fibers of \( V_1 \) are solvable ideals in the fibers of \( V \) ([18], Chapter VI, proof of Theorem 2.1). Again since \( V_1 \otimes V_1 \) and \( V_1 \) are semistable vector bundles of degree zero the image \( [V_1, V_1] \) of the morphism \( V_1 \otimes V_1 \rightarrow V_1, x \otimes y \mapsto [x, y] \), given by the Lie bracket operation, is a subbundle \( V_2 \) of \( V_1 \) of degree zero and semistable. Similarly \( V_3 = [V_2, V_2] \) etc. are all semistable vector bundles of degree zero. Since \( V_1 \) is solvable we arrive after a certain stage at a non-zero subbundle \( V' \), of degree zero and semistable, which is an abelian ideal in \( V \).

The inclusion \( V' \otimes L^m \subset V \otimes L^m \) induces \( W = H^0(X, V' \otimes L^m) \subset H^0(X, V \otimes L^m) = I_n \). Let \( W' \) be a supplement for \( W \) in \( I_n \) so that \( I_n = W \oplus W' \). Let \( \lambda \) be the 1-PS of \( SL(n, \mathbb{C}) \) which acts on \( W \) by the character \( \lambda(z) = z^{r \cdot \text{rk} W} \), and on \( W' \) by the character \( \lambda(z) = z^{-r \cdot \text{rk} W} \), \( z \in \mathbb{C}^* \). Let \( (x_1, \ldots, x_n) \) be the image of the closed point of \( \text{Spec} A \) in \( Q(\overline{Y}) \) and \( (y_1, \ldots, y_n) = (p(x_1), \ldots, p(x_n)) \). We shall now compute \( \mu((x_1, \ldots, y_n), \lambda) \) and show that it is \( < 0 \) contradicting semistability of \( (x_1, \ldots, x_n) \).

It follows from ([10], Chapter 4, § 4, equation (**)), p. 88; cf. also [19], p. 309) that \( \mu^M((y_1, \ldots, y_n), \lambda) = n \cdot \text{rk} W' - r \cdot \text{rk} W \) where \( W' \) is the image of \( W \) in \( y_i \). (In [10] the calculation is made for the Grassmannian of subspaces. It is easy to translate it to the Grassmannian of quotient spaces which we need here.) Since \( W = H^0(X, V' \otimes L^m) \) generates \( V' \) (by our choice of \( m \), cf. § 3.11), \( \text{rk} W' = \text{rk}(V') \forall i <\). Therefore

\[
\mu^M((y_1, \ldots, y_n), \lambda) = n \cdot \text{rk} W' - r \cdot \text{rk} W.
\]

Applying Riemann–Roch we get

\[
\text{rk} W = \text{rk} H^0(X, V' \otimes L^m) = (\text{rk} V')(md_0 + 1 - g)
\]

and

\[
n = \text{rk} H^0(X, V \otimes L^m) = r \cdot (md_0 + 1 - g).
\]

Therefore \( (\text{rk} W)/n = (\text{rk} V')/r \). Hence from (1) we have

\[
\mu^M((y_1, \ldots, y_n), \lambda) = 0.
\]

To calculate \( \mu^A((x_1, \ldots, x_n), \lambda) \) let \( x = x_i \) and \( y = y_i \) and let \( g \in P(\lambda) \) such that \( \lambda_g = g \lambda g^{-1} \subset P_r \) (cf. Proof of Lemma 5.5.2). Then \( \mu^A((x_1, \lambda)) = \mu^A((x_2, \lambda)) \). It follows from ([10], Chapter 2, § 2, pp. 55–56) that \( P(\lambda) = P_W \), the stabilizer of \( W \) in \( SL(n, \mathbb{C}) \). Therefore \( I_n = g W \oplus g W' = W \oplus g W' \) and \( \lambda_g \) acts on \( W \) by the character \( \lambda_g(z) = z^{r \cdot \text{rk} W} \) and on \( g W' \) by \( \lambda_g(z) = z^{-r \cdot \text{rk} W} \), \( z \in \mathbb{C}^* \). Since \( \lambda_g < P_y \), \( \lambda_g \) leaves invariant ker\((I_n \rightarrow y) \) and hence we can find a set of linearly independent elements \( e_1, \ldots, e_{q+1}, \ldots, e_r \), such that \( e_1, \ldots, e_{q} \in W \) and \( e_{q+1}, \ldots, e_r \in g W' \) and \( e_{q+1}, \ldots, e_r \), the images of \( e_1, \ldots, e_{q} \) under \( I_n \rightarrow y \) form a basis for the fiber \( V'_x \subset y \) of \( V' \) over \( x \in X \) (where \( y \) is the fiber of \( V \) over \( x \)) and \( e_{q+1}, \ldots, e_{q+1}, \ldots, e_r \), form a basis for \( y \). Then

\[
\lambda_g(z) \cdot \tilde{e}_i = \begin{cases} z^{r \cdot \text{rk} W} \cdot \tilde{e}_i & \text{for } 1 \leq i \leq q \\ z^{-r \cdot \text{rk} W} \cdot \tilde{e}_i & \text{for } q + 1 \leq i \leq r. \end{cases}
\]
For the action of \( \lambda_g \) on \( y \) we then have \( \lambda'_q = \lambda' \cdot \lambda'' \) where \( \lambda'(z)\bar{e}_i = z^r \bar{e}_i; \ t = q(rkW) - (r - q)rkW \) and
\[
\lambda''(z)\bar{e}_i = \begin{cases}
    z^{r(rkW) - t} \bar{e}_i = z^{(r-q)^m} \bar{e}_i, & 1 \leq i \leq q \\
    z^{-r(rkW) - t} \bar{e}_i = z^{-q^n} \bar{e}_i, & q + 1 \leq i \leq r.
\end{cases}
\]
As in Lemma 5.5.2, eq. (6)
\[
\mu^\lambda(x, \lambda') = \mu^\lambda(y, \lambda_g) = \mu^\lambda(y, \lambda).
\]
To calculate \( \mu^\lambda(x, \lambda'') \) we shall use ([10] Proposition 2.3; cf. also [22], §2). Let \( \bar{x} \) be a point in \( y^* \otimes y^* \otimes y \) which lies above \( x \in O(x, \bar{Y}) \subset \mathcal{P}(y^* \otimes y^* \otimes y) \). Let
\[
\bar{\bar{x}} = \sum_{i,j,k=1}^r x_{ijk} \bar{e}_i^* \otimes \bar{e}_j^* \otimes \bar{e}_k^*,
\]
where \( \bar{e}_i^* \) form the dual basis to \( \bar{e}_i \). If we think of \( \bar{x} \) as a Lie algebra structure on \( y \) then the \( x_{ijk} \) are the ‘structure constants’ and we have
\[
[\bar{e}_i, \bar{e}_j] = \sum_{k=1}^r x_{ijk} \bar{e}_k.
\]
From the fact that \( \bar{e}_1, \ldots, \bar{e}_q \) span an abelian ideal in \( y \) we get that \( x_{ijk} = 0 \) whenever \( i, j, k \) satisfy any one of the following three conditions: (1) both \( i \) and \( j \) are \( \leq q \) and \( k \) arbitrary (abelian) (2) \( i \leq q, j \) arbitrary and \( k \geq q + 1 \) (ideal) (3) \( i \) arbitrary, \( j \leq q \) and \( k \geq q + 1 \) (symmetric to (2)). Therefore \( x_{ijk} \) may not be zero only in the following cases:

Case i. \( i \leq q, j \geq q + 1 \) and \( k \leq q \).
Case ii. \( i \geq q + 1, j \leq q \) and \( k \leq q \).
Case iii. \( i \geq q + 1, j \geq q + 1 \) and \( k \leq q \).
Case iiii. \( i \geq q + 1, j \geq q + 1 \) and \( k \geq q + 1 \).

Let
\[
\lambda''(z)(\bar{e}_1^* \otimes \bar{e}_j^* \otimes e_k^*) = z^{a_{ijk}} \bar{e}_1^* \otimes \bar{e}_j^* \otimes e_k^*, a_{ijk} \in \mathbb{Z}.
\]
Then it is easy to see that \( a_{ijk} = q \) in cases (i), (ii), and (iii); and \( a_{ijk} = q + r \) in case (iii). Therefore in every case when \( x_{ijk} \neq 0 \), \( \lambda''(z) \) acts by a strictly positive power, viz. \( q \) or \( q + r \), of \( z \). It follows by ([10], Proposition 2.3) that \( \mu^\lambda(x, \lambda'') < 0 \). Now using eq. (8) of Lemma 5.5.2 and (2) above, we get \( \mu((x_1, \ldots, x_N), \lambda) < 0 \) contradicting the semistability of \((x_1, \ldots, x_N)\). Therefore we conclude that the Killing form \( \bar{\bar{x}} \) must be nondegenerate on all fibers. Then the Lie algebra structure of all fibers is semisimple. But the Lie algebra structure of a semisimple Lie algebra is (locally) rigid, i.e., interpreting \( \bar{Y} \) as Lie algebra structures on \( y \), \( x \in \bar{Y} \) gives a semisimple Lie algebra structure on \( y \) if \( x \in \bar{Y} \) gives a semisimple Lie algebra structure on \( y \). This shows that the image of \( s \) lies in \( Q_0(Y) \). Therefore the image of the closed point of \( \text{Spec} \ A \) under \( \text{Spec} \ A \rightarrow S_1 \) lies in \( R_1 \) as was to be shown.

5.7. Lemma. Let \( \pi: R_3 \rightarrow Q(\bar{Y})^N_\text{ss} \) be the composite \( R_3 \twoheadrightarrow J_x \times R_2 \rightarrow R_2 \rightarrow R_1 \rightarrow Q(\bar{Y})^N_\text{ss} \). Then for \( r_3 \in R_3 \), \( \pi(r_3) \) is a stable point of \( Q(\bar{Y})^N_\text{ss} \) if and only if the G-bundle \( E \rightarrow X \) corresponding to \( r_3 \) is stable.
Proof. Let $\pi_3(r_3) = (j, r_2)$. Then the Ad $G$-bundle $E'$ corresponding to $r_2 \in R_2$ is obtained from the $G$-bundle $E$ by the extension of structure group $G \to \text{Ad} \ G = G/Z$. Therefore $E'$ is stable if and only if $E$ is stable ([14], Proposition 7.1, p. 146).

Then point $\pi(r_3)$ in $Q(\tilde{Y}_n)$ is stable if and only if the orbit of $\pi(r_3)$ under $SL(n, C)$ is closed and the isotropy of $\pi(r_3)$ in $SL(n, C)$ is finite ([10], Amplification 1.11; [12], § 2, Theorem 2(a), p. 193).

The morphism $R_2 \to Q(\tilde{Y}_n)$ is an $SL(n, C)$-equivariant finite morphism. Moreover it is easy to see that if two Ad $G$-bundles $E'$ and $E'_2$ give rise to isomorphic $\text{Aut} \ \mathcal{G}$-bundles under the extension of structure group $Ad \ G \subset \text{Aut} \ \mathcal{G}$, then $E'_1$ is stable if and only if $E'_2$ is stable. It then follows that it is enough to show that the Ad $G$-bundle $E'$ is stable if and only if the $SL(n, C)$ orbit of $r_2$ in $R_2$ is closed and the isotropy of $r_2$ in $SL(n, C)$ is finite.

Suppose $E'$ is a stable Ad $G$-bundle. Then $gr \ E' = E'$ (Proposition 3.12) and it follows from Proposition 3.24(i) that the $SL(n, C)$ orbit of $r_2$ is closed (cf. proof of Proposition 4.5). Since the group $\text{Aut} \ E'$ of Ad $G$-bundle automorphisms of $E'$ is finite ([14], Proposition 3.2, p. 136) it follows from Remark 4.7 that the isotropy of $r_2$ in $SL(n, C)$ is finite.

Conversely suppose that the $SL(n, C)$ orbit of $r_2$ is closed and the isotropy of $r_2$ finite. From Proposition 3.24(ii) it follows that the closure of the orbit of $r_2$ always contains a point $r'_2$ which is isomorphic to gr $E'$ (cf. proof of Proposition 3.5). Therefore if the orbit of $r_2$ is closed then $E' \approx gr \ E'$. By Proposition 3.15 this implies that $E'$ is a unitary bundle $E'_1$ corresponding to a unitary representation $\rho: \pi_1(X) \to K$, where $K$ is a maximal compact subgroup of $Ad \ G$. If $\rho$ is not irreducible then there is a subgroup $S \subset K$, with dim $S > 0$, which commutes with $\rho$ ([14], Definition 1.2, cf. also § 2, p. 131). Then the group $S$ gives rise to a group of automorphisms of $E'_1$ of dimension $> 0$. This contradicts the finiteness of the isotropy at $r_2$ (Remark 4.7). Therefore $\rho$ is irreducible. Then $E' = E'_1$ is stable by ([14], Proposition 2.2, p. 133).

5.8. PROPOSITION

Let $\mathcal{E} \to S \times X$ be an arbitrary family of $G$-bundles on $X$ parametrized by a scheme $S$. Then the set $S_{s_0}$ (resp. $S_s$) of points $s \in S$ such that $\mathcal{E}_s$ is semistable (resp. stable) is an open subset of $S$.

Proof. Since a $G$-bundle $E$ is semistable (resp. stable) if and only if the Ad $G$-bundle $E'$ obtained from $E$ by the extension of structure group $G \to \text{Ad} \ G$ is so ([14], Proposition 7.1, p. 147), we can assume that $G = \text{Ad} \ G$. Let $\mathcal{E}(\mathcal{G})$ be the vector bundle associated to $\mathcal{E}$ by the adjoint representation $G \subset GL(\mathcal{G})$. Since the question is local on $S$ we can assume that for $m > 0$, $\mathcal{E}(\mathcal{G}) \otimes \mathcal{P}_n^m(L^n)$ is a quotient of $I_n$ for a suitable $n, I_n \to \mathcal{E}(\mathcal{G}) \otimes \mathcal{P}_n^m(L^n) \to 0$. This then gives a morphism $f: S \times X \to G_{n,r}$ the Grassmannian of $r$ dimensional quotients of $I_n$ and the $G$-bundle $\mathcal{E} \to S \times X$ gives a morphism $\tilde{f}: S \times X \to Q(\tilde{Y}) \subset Q(\tilde{Y})$ in the obvious way. By choosing an $N$-tuple $(x_1, \ldots, x_N)$ of points of $X$ we get

$$Q(\tilde{Y})^N \subset Q(\tilde{Y})^N$$
We can choose the \( N \)-tuple \((x_1, \ldots, x_N)\), \( N \gg 0 \), such that \( f(s) \) is a semistable (resp. stable) point of \( G_{s,r}^N \) if and only if \( s \) is a semistable (resp. stable) vector bundle ([19], Theorem 7.1(3), also Corollary 7.2). It follows then from Corollary 3.18 and Lemma 5.5.2 that the set \( S_{s,s} \) is \( \bar{f}^{-1} (Q(\tilde{Y})_N^N) \). Since \( Q(\tilde{Y})_N^N \) is open in \( Q(\tilde{Y})_N^N \) this proves that \( S_{s,s} \) is open.

Again making \( S \) smaller if necessary we can assume that the family \( \mathcal{E} \mid S_{s,s} \) is induced by a morphism \( S_{s,s} \to R_2 \). It follows from Lemma 5.7 that the points in \( R_2 \) corresponding to stable bundles is open since it is the inverse image of the open subset \( Q(\tilde{Y})_N^N \) of \( Q(\tilde{Y})_N^N \) under the morphism \( R_2 \to Q(\tilde{Y})_N^N \). Therefore \( S_{s,s} \) is open in \( S \).

5.9. Theorem. The functor \( F_{ss}^N \) (Definition 3.9) has a coarse moduli scheme \( M' \). The scheme \( M' \) is irreducible, projective, normal and Cohen-Macaulay. The dimension of \( M' \) is \( \dim Z + (g - 1) \cdot \dim G \). The set \( M'_s \) of points in \( M' \) corresponding to stable \( G \)-bundles is an open (and hence dense) subset of \( M' \).

Proof. By Propositions 4.5 and 4.15.2 it follows that to prove the existence of a coarse moduli scheme for \( F_{ss} \) it is enough to prove that a good quotient of \( R_3 \) modulo \( GL(n, \mathbb{C}) \) exists. Since \( R_3 \to J_1 \times R_2, R_3 \to R_1 \) and \( R_1 \to Q(\tilde{Y})_N^N \) are all finite \( GL(n, \mathbb{C}) \)-equivariant morphisms it follows from Lemma 5.1 that a good quotient of \( R_3 \) modulo \( GL(n, \mathbb{C}) \) exists if a good quotient of \( Q(\tilde{Y})_N^N \) modulo \( GL(n, \mathbb{C}) \) (or \( SL(n, \mathbb{C}) \), since the scalars act trivially on \( Q(\tilde{Y})_N^N \)) exists. We know that a good quotient of \( Q(\tilde{Y})_N^N \) exists and is projective ([10], Theorem 1.10; [22], Theorem 1.1(B)). Therefore \( F_{ss}^N \) has a coarse moduli scheme \( M' \). Moreover since \( R_3 \to J_1 \times R_2 \) etc. are finite morphisms it follows from Lemma 5.1 that \( M' \) is projective (noting that a scheme finite over a projective scheme is projective).

Since \( M' \) is a categorical quotient of the non-singular (and hence normal) scheme \( R_3 \) it follows that \( M' \) is normal ([10], Chapter 0, § 2, p. 5). Since \( \phi : R_3 \to M' \) is a good quotient \( R_3 \) can be covered by \( G \)-invariant affine open subsets \( \text{Spec} A_i \) such that the corresponding quotients \( \text{Spec} A_i^G \) form an affine open covering for \( M' \).

Since \( R_3 \) is nonsingular the \( \mathcal{C} \)-algebras \( A_i \) are regular and hence by ([8], Main theorem) it follows that \( A_i^G \) are Cohen-Macaulay. Therefore \( M' \) is Cohen-Macaulay.

It follows easily from § 5.4.1 and Lemmas 5.7 and 5.1 that \( M' \) is open in \( M' \) and \( q : R^*_3 = q^{-1}(M'_s) \to M^*_s \) is a geometric quotient.

Since \( R^*_3 \to M^*_s \) is a geometric quotient modulo \( SL(n, \mathbb{C}) \) we have \( \dim M^*_s = \dim R_3 - \dim SL(n, \mathbb{C}) \). Since \( R_3 \to J_1 \times R_2 \) and \( R_2 \to R_1 \) are étale and finite \( \dim R_3 = \dim J_1 + \dim R_1 \). Therefore using Lemma 4.13.4, \( \dim M^*_s = (g \cdot \dim Z) + (n_2 + (r + 1)(g - 1) - g) - (n^2 - 1) \). Noting that \( r = \dim G - \dim Z \), we get \( \dim M^*_s = \dim Z + (g - 1) \cdot \dim G \).

We need the following lemma to complete the proof of the theorem.

5.9.1. Lemma. Let \( S \) be a complex analytic space and \( \mathcal{E} \to S \times X \) be a complex analytic family of semistable \( G \)-bundles of topological type \( \tau \) parametrized by \( S \). Then there is an analytic morphism \( f_\tau : S \to M' \) such that for any \( s \in S, f_\tau(s) \) is the equivalence class of \( \mathcal{E}_s \).

Proof. This follows easily from the fact that the functors \( \text{Pic} \) and \( \Gamma(\rho, \mathcal{E}) \) etc. used in our construction of universal families are representable in the analytic category also and are represented by the same universal spaces as in the algebraic category (cf. [5]).

We will now continue with the proof of Theorem 5.9. Since we have shown that \( M' \) is normal, to show that it is irreducible it is enough to show that it is connected. Let
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$m_1, m_2 \in M$ and let $E_1, E_2$ be semistable $G$-bundles belonging to the equivalence classes $m_1, m_2$ respectively. Then by ([14], Proposition 4.2 p. 142) there is a complex analytic family of semistable $G$-bundles $\mathcal{E} \to S \times X$ parametrized by an open connected subspace $S$ of the complex plane $\mathbb{C}$ such that for some $s_1, s_2 \in S$ we have $\mathcal{E}_s = E_i, i = 1, 2.$ (In [14], Proposition 4.2 this is stated only for stable bundles but the same proof goes through for semistable bundles also, noting that the proof of ([14], Proposition 4.1, p. 138) with a little modification gives that for a complex analytic family $\mathcal{T} \to S \times X$ the set of $s \in S$ such that $\mathcal{T}_s$ is semistable is an analytic open subset of $S.$) Now applying Lemma 5.9.1 we get that $M$ is connected and hence irreducible.

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References

[1] Artin M, Grothendieck Topologies Notes on a seminar by M Artin (Spring 1962), Harvard University