MAXIMAL-HAUSDORFF SPACES

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§1. Introduction

§1. 1. Bicompact spaces form a very important class of topological spaces; the principal properties relating to them will be found in Alexandroff and Hopf, *Topologie*, Kap. II. Among the bicompact spaces, the Hausdorff bicompact spaces occupy, as is well known, a special position; their characteristic property has been stated in the following form by Dr. R. Vaidyanathaswamy in his *Treatise on Set Topology*, p. 104 (in the press):

A Hausdorff bicompact space is both minimal-bicompact and maximal-Hausdorff.

The Author further adds:

It is not known whether there exist topologies on an infinite set R which are minimal-bicompact without being Hausdorff or maximal-Hausdorff without being bicompact.

The purpose of the present paper is to supply a partial answer to this question by proving that a maximal-Hausdorff space need not necessarily be bicompact.

The first important result obtained is that a maximal-Hausdorff space is necessarily *H*-closed. That the only regular maximal-Hausdorff spaces are the Hausdorff bicompact spaces follows from a well-known theorem of Alexandroff and Hopf, namely that a *H*-closed regular space is bicompact. A method of strengthening a Hausdorff space at an irregular point is shown in Theorem IV, which leads to some negative results. In this connection it is remarkable that a space constructed by Urysohn for a different purpose happens to be maximal-Hausdorff without being bicompact. This is proved by the necessary and sufficient condition enunciated and proved for a Hausdorff space to be maximal-Hausdorff, namely that every neighbourhood of every point of the space should contain an open domain neighbourhood of the point whose complement is *H*-closed.

I am under a permanent debt of gratitude to Dr. R. Vaidyanathaswamy, Reader in Mathematics, University of Madras, for having indefatigably

helped me during the arduous discussions I have had with him on the subject and particularly in the enunciation and proof of the last mentioned theorem.

§1. 2. Strength of Topologies.

We are concerned only with topological spaces R of any sort of elements defined by neighbourhood systems satisfying the well-known postulates of Hausdorff, *viz.*,

H_1. *Each point of R has at least one neighbourhood and is contained in every one of its neighbourhoods.*

H_2. *If $U(p)$, $V(p)$ are any two neighbourhoods of p then there exists a neighbourhood of p which is contained in $U(p) \cdot V(p)$.*

H_3. *To each neighbourhood $U(p)$ of p there exists a neighbourhood $V(p)$ of p such that $U(p)$ contains a neighbourhood of each point of $V(p)$.*
It is well known that any neighbourhood system satisfying H_1, H_2, H_3 introduced in the set R defines uniquely a topological structure in R. In this structure

(i) for any set $X \subset R$ we have $\bar{X}$ is the set of points of R every neighbourhood of each of which has non-null intersection with X;

(ii) a set is open if it contains a neighbourhood of each of its points;

(iii) any point is an interior point of every one of its neighbourhoods.

Let there be two neighbourhood systems the U-system and the V-system introduced in R, both of course satisfying H_1, H_2, H_3. The two systems are said to be *equivalent* if and only if they define the same topological structure in R. The necessary and sufficient condition for this is that *for each point p of R, every $U(p)$ contains a $V(p)$ and every $V(p)$ contains a $U(p)$.*

The totality of open sets of a topological space R, when each of them is considered as a neighbourhood of each of its points, form a neighbourhood system of the space equivalent to the given neighbourhood system and will be conveniently used to replace the given system.

A topological space R is said to be (*a*) a T_1-space if every point $p \in R$ has a neighbourhood not containing any other assigned point q; (*b*) a T_2- or *Hausdorff space* if every pair of distinct points are *H-separable*, *i.e.*, have disjoint neighbourhoods; (*c*) a *Urysohn space* if every pair of distinct points are U-separable, *i.e.*, have neighbourhoods whose closures are disjoint. Two distinct points of R are said to be *U-inseparable* if they possess no neighbourhoods whose closures are disjoint. A point p of R is said to be *regular* if every neighbourhood of the point contains the closure of another neighbourhood of the point; the space itself is called regular if every point of the space is regular. An irregular point of R is called, after M. H. Stone, *semi-regular* if every neighbourhood of the point contains the interior of the closure of another neighbourhood of the point. It is obvious that every Urysohn space is a Hausdorff space and that every regular space is a Urysohn space; whereas, not all Urysohn spaces are regular spaces.

Let there be two systems of neighbourhoods the U-system and the V-system introduced in a set R both satisfying H_1, H_2, H_3. (i) We say, after Dr. R. Vaidyanathaswamy, that the V-system is *stronger* than the U-system at a point $p \in R$, if every $V(p)$ contains a $U(p)$ and if there is at least one $U(p)$ which contains no $V(p)$. Suppose two different topological structures f, ϕ are defined in a set R. Let us denote by $(R;\ f)$ the topological space R whose topological structure is f. (ii) The topology ϕ is said to be *stronger* than f, if every open set of $(R;\ \phi)$ is also an open set of $(R;f)$ and if there is at least one open set of $(R;f)$ which is not open in $(R;\ \phi)$. (iii) A Hausdorff space $(R;\ f)$ is said to be *maximal-Hausdorff* if any topology on R which is stronger than f renders the space non-Hausdorff.

Let the U-system and the V-system define respectively the topological structures f and ϕ in a set R. We may suppose that $\{U\}$ and $\{V\}$ represent the absolute neighbourhood systems of the topologies f and ϕ respectively. Then we have

Theorem I.

If ϕ is stronger than f, then there is at least one point at which the V-system of neighbourhoods is stronger than the U-system.

Since ϕ is stronger than f, there is at least one set A which is open in f but not open in ϕ. Let p be a boundary point of A in ϕ-topology. Now any neighbourhood $V(p)$ of p in ϕ is open in ϕ, therefore also open in f and hence is a $U(p)$. But since A is open in f there is a $U(p)$, say $U_0(p)$, which is contained in A. Since p is a boundary point of A in ϕ there is no $V(p)$ which is contained in A. Hence $U_0(p)$ contains no $V(p)$. So $\{V\}$ is stronger than $\{U\}$ at the point p.

Theorem II.

If ϕ is stronger than f, then there exists a topology ψ which is equivalent to ϕ at just one point $p \in R$ and equivalent to f at all other points of R (it being assumed that f is a T_1-topology).

For, there is a set G which is open in f but not open in ϕ and hence it has a boundary point p in ϕ. Take the ϕ-neighbourhoods for the point p and the f-neighbourhoods for all other points of R. The system of neighbourhoods thus defined satisfy evidently H_1, H_2. To show that it satisfies also H_3, let us observe that a $\phi(p)$ contains a ϕ-neighbourhood of every one of its points. But a ϕ-neighbourhood of any point other than p is also a f-neighbourhood of that point. Hence any $\psi(p)$ contains a ψ-neighbourhood of every one of its points. Thus H_3 is satisfied by $\{\psi(p)\}$. If q is any point other than p, then the neighbourhoods of q which are disjoint with p by themselves form a system of neighbourhoods of the point equivalent to the given system, since the topology f is T_1. We see immediately that the ψ-system for points other than p also satisfies H_3. The topology defined by the new system of neighbourhoods satisfies the conditions of the theorem.

Corollary.

It follows from this theorem that, in order to show that the Hausdorff space $(R;f)$ is maximal-Hausdorff, it is sufficient to show that the neighbourhood system of the space cannot be strengthened at any point without destroying the Hausdorff character of the space. For, if there exists a Hausdorff topology ϕ stronger than f, then there would exist a topology ψ which is equivalent to f at all points except at a single point at which it is equivalent to ϕ. Then ψ will also be Hausdorff and would be obtained from f by strengthening at a single point.

§2. Regular Maximal-Hausdorff Spaces

§2·1. H-closed spaces.

We say that a topological space R can be *topologically imbedded* in another topological space S if R is homeomorphic to a subset of S. We say that a Hausdorff space R is *H-closed* if it is closed in every Hausdorff space in which it can be topologically imbedded. It is obvious that a finite set of R is always H-closed; the set (1, 2, 3,....) of the real number space is not H-closed. Alexandroff and Hopf (*loc. cit.*, p. 90) have proved

§2·2. *A necessary and sufficient condition for a Hausdorff space R to be H-closed is that from every open covering* $\{G\}$ *of R, it should be possible to select a finite number of elements* $G_1, G_2, \ldots G_n$ *say, such that*

$$\sum_{1}^{n} \bar{G}_i = R.$$

§2·3. Theorem III

A Maximal-Hausdorff space is necessarily H-closed.

If the Hausdorff space R is not H-closed then we shall show that its topology can be strengthened at any point p, without derogation to the Hausdorff character of the space. Now by §2.2 there exists an open covering $\{G\}$ of R such that for every choice of the finitely-many elements

$G_1, G_2, \ldots. G_n$ of $\{G\}$ the open set $F = (R - \sum_1^n \overline{G_i})$ is non-null. Now the family of open sets F are such that the intersection of any two F's is a F. Moreover, if we choose for a neigbourhood $U(p)$ of any point p, an element of $\{G\}$ which contains p then $U(p)$ is disjoint with $\{R - \overline{U}(p)\}$ which by definition is a F. Hence there is at least one F which is disjoint with a $U(p)$. Since the intersection of any two F's is non-null it follows that if a $U(p)$ is disjoint with a F then that $U(p)$ can contain no F. Let us now define a system of neighbourhoods $\{V(p)\}$ for p where the $V(p)$'s are sets of the form $\{U(p) + \text{a set } F\}$. Now the $V(p)$'s are open sets of the space R which contain p. Since the intersection of any two $U(p)$'s is a $U(p)$ and the intersection of any two F's is a F, it follows that the intersection of any two $V(p)$'s is a $V(p)$. Further there is at least one $U(p)$ which contains no $V(p)$. Hence the system $\{V(p)\}$ is stronger than the system $\{U(p)\}$. Let now q be any point of R other than p. Then there exists a $U(p)$ disjoint with a $U(q)$ and there exists a F disjoint with a $U(q)$. Hence there exists a $V(p)$ disjoint with a $U(q)$. Thus the V-topology is a Hausdorff topology which is stronger than the U-topology. Hence the U-topology is not maximal-Hausdorff. Q.E.D.

§**2·4.** The condition of Theorem III is not sufficient. The following example of Alexandroff and Hopf (*loc. cit.*, p. 31), constructed for a different purpose, exhibits a H-closed space which is not maximal-Hausdorff.

Example 1.—Let R be the unit length $0 \leqslant p \leqslant 1$ of the Arithmetical continuum, D the set of all points $1/n$, $n = 1, 2, 3, \ldots.$ If $0 < p < 1$, then let $J(p)$ be an arbitrary open interval of R which contains p; if $p = 0$ or 1, let $J(p)$ be a half-interval $0 \leqslant t < a$ or $a < t \leqslant 1$ respectively, where a lies between 0 and 1 and is otherwise arbitrary. When $p \neq 0$, let $J(p)$ be a neighbourhood of p. Let the sets of the form $\{J(0) - D\}$ be the neighbourhoods of the point 0.

The space R thus constructed is a Hausdorff space in which 0 is an irregular point. It is H-closed. For, consider any open covering $\{G\}$ of R. Evidently the G's may be open intervals of the form $J(p)$ if they do not contain 0 and of the form $\{J(0) - D\}$ if they contain 0. We see that $\overline{G(0)}$ is always an open interval of the form $J(0)$. The subspace $\{R - \overline{G(0)}\}$ can always be covered by the closures of a finite number of G's. Hence R itself can be covered by the closures of a finite number of G's.

Evidently the given neighbourhood system is weaker at the point 0 than, and equivalent at all other points to, the neighbourhood system of the space R under its usual topology where the neighbourhoods of the point 0 are the half-open intervals $J(0)$. The usual topology of R is a Hausdorff topology and hence the given topology of R is not maximal-Hausdorff. Q.E.D.

§2·5. **Corollary**

The only regular maximal-Hausdorff spaces are the Hausdorff bicompact spaces.

This follows from Theorem III and the following theorem due to Alexandroff and Hopf (*loc. cit.*, p. 91) *viz.*,

A H-closed regular space is bicompact.

§ 3. Irregular Maximal-Hausdorff Spaces

§3·1. The following method of strengthening a Hausdorff space at an irregular point is often useful.

Theorem IV

The topology of a Hausdorff irregular space can be strengthened at any irregular point p by choosing the regular neighbourhoods of p as its new neighbourhoods; the strengthened topology is necessarily T_1 and would be Hausdorff if p is U-separable from every other point of the space.

The intersection of two regular neighbourhoods of p is evidently a regular neighbourhood of p. Since we are not altering the neighbourhoods at other points, it follows that the regular neighbourhoods of p satisfy H_1, H_2, H_3 and can be taken as new neighbourhood system at p.

Let q be any other point and $U(p)$, $U(q)$ disjoint neighbourhoods of p, q (which exist as the space is Hausdorff); as $(R-q) \supset \overline{U(p)}$, it follows that $(R-q)$ is a regular neighbourhood of p disjoint with q. Thus the strengthened topology is T_1. If every point q other than p is U-separable from q, there are neighbourhoods $U(p)$, $U(q)$ of p and q such that $\overline{U(p)}.\overline{U(q)} = 0$. Hence Ext. $U(q)$, which contains $\overline{U \cdot p)}$ is a regular neighbourhood of p disjoint with the neighbourhood $U(q)$ of q. Thus p, q are H-separable in the strengthened topology.

§3·2. **Corollary I**

A Hausdorff space with only one irregular point cannot be maximal-Hausdorff.

If p be the only irregular point of a Hausdorff space R, then p is U-separable from every other point of R. For, if r be any point of R other than p, then there exist neighbourhoods $U(p)$, $U(r)$ such that $0 = U(p).U(r) = \overline{U(p)}.U(r)$. But r is a regular point of R. Hence there exists a neighbourhood $U_1(r)$ of r whose closure is contained in $U(r)$. It follows that $\overline{U(p)} \cdot \overline{U_1(r)} = 0$, *i.e.*, p and r are U-separable. The given space can be strengthened as explained in Theorem IV and the strengthened topology is Hausdorff by the same Theorem.

Example 1 given above, illustrates the present corollary. As further illustration we may consider the following space constructed by Dr. R. Vaidyanathaswamy (*loc. cit.*, p. 99) for a different purpose; the space is regular except at the point O, which is semi-regular.

Example 2.—Take the Cartesian plane with the rectangular axes $X'OX$, $Y'OY$. Consider the semi-circles $POP'S$ lying to the right of Y-axis. Define the neighbourhoods of O to be the union of O with the open semi-circles POP's. The neighbourhoods of the other points are the usual ones. The space thus constructed is a Hausdorff space for which, as can be easily seen, O is a semi-regular point. Replacing the given neighbourhoods of O by its usual ones, which form a stronger system than the given system, the space still remains Hausdorff. Hence the given space is not maximal-Hausdorff.

§3·3. **Corollary II**

If, in a Hausdorff irregular space, there exists at least one irregular point which is U-separable from every other point of the space, then the space cannot be maximal-Hausdorff.

For, if p be an irregular point of the Hausdorff space R, which is U-separable from every other point of the space, on replacing the given neighbourhood system at p by the system of regular neighbourhoods of p, we see, by Theorem IV, that p is H-separable from every other point of R and so R is Hausdorff in the strengthened topology. Hence the given space is not maximal-Hausdorff. Q.E.D.

§3·4. **Corollary III**

If every pair of distinct irregular points of a Hausdorff space are U-separable, then the space cannot be maximal-Hausdorff.

This is only a case with much stronger hypothesis than Corollary II. As an illustration of the present Corollary, let us consider the following example.

Example 3.—The set R of real numbers is topologised by ascribing to each number x as its neighbourhoods, the sets $(I_x - S)$ if x is irrational and the sets $\{x + (I_x - S)\}$ if x is rational, where I_x is any open interval containing x and S is the set of rational numbers.

The space thus defined is a Hausdorff space in which every point is irregular. Any pair of distinct points, however, are U-separable. The space, therefore, by Corollary III, is not maximal-Hausdorff and this may also be verified by the fact that the given neighbourhood system is weaker at every point than the neighbourhood system of the usual topology of the real number space which is a Hausdorff topology.

§3.5. An open set g of a topological space is an *open domain* of the space if and only if it satisfies any one of the following equivalent conditions: (i) g is the exterior of some open set; (ii) g is the exterior of its own exterior; (iii) g is the interior of its closure; (iv) g is the interior of some closed set. For fuller information *vide* Dr. R. Vaidyanathaswamy, *loc. cit.*, p. 91. A family of open sets $\{G\}$ of a topological space is called, after Dr. R. Vaidyanathaswamy, a *proximate covering* of the space if it covers the whole of the space with the possible exception of a *non-dense* set. It follows that the union of all sets of a proximate open covering is a *dense open* set.

3·6. Theorem V

A maximal-Hausdorff space is necessarily semiregular.

Any two distinct points p, q of the Hausdorff space have neighbourhoods $U(p)$, $U(q)$ such that $0 = U(p).U(q) = \overline{U(p)}.U(q) = U(p).\overline{U(q)}$. Hence $Int\ \overline{U(p)}$ is disjoint with $U(q)$ and similarly $Int\ \overline{U(q)}$ is disjoint with $U(p)$. Now if $U_1(p)$, $U_2(p)$ be any two neighbourhoods of p, then $\overline{U_1(p)}.\overline{U_2(p)} \supset \overline{U_1(p).U_2(p)}$ so that $Int\ \overline{U_1(p)}.Int\ \overline{U_2(p)} \supset Int\ \overline{U_1(p).U_2(p)}$. It follows that the system of open domains $\{Int\ \overline{U(p)}\}$ satisfy H_1, H_2, H_3 and if these be taken to form a new system of neighbourhoods for p then the system $Int\ \{\overline{U(p)}\}$ will be stronger than the system $\{U(p)\}$ if the point p is not semiregular. Hence if a Hausdorff space R contains at least one irregular point which is not semiregular then the topology of the space can be strengthened by replacing its neighbourhoods by the system of open domains containing the point as its new neighbourhoods without destroying the Hausdorff character of the space and so the given space would not be maximal-Hausdorff. If in the above argument the point p is a semiregular point so that its neighbourhoods are already open domains then $\{Int\ \overline{U(p)}\}$ will be equivalent to $\{U(p)\}$. We see therefore that, in order that a Hausdorff irregular space may be maximal-Hausdorff, the space should necessarily be semiregular. Q.E.D.

§3.7. We are now in a position to prove the necessary and sufficient condition for a Hausdorff space to be maximal-Hausdorff.

Theorem VI

The necessary and sufficient condition for a Hausdorff semiregular space R to be maximal-Hausdorff is that every neighbourhood of every point of the space should contain an open domain neighbourhood of the point whose complement is H-closed.

The condition is necessary.

For, suppose the condition is not satisfied. Then there exists a point p such that it has a neighhourhood $U_0(p)$* which contains no open domain-neighbourhood whose complement is H-closed. We shall prove that the topology can be strengthened at the point p without destroying the Hausdorff character of the space. In the first place, $(R-p)$ cannot be H-closed, for, if it were H-closed p would be an isolated point and would therefore be an open domain-neighbourhood contained in every neighbourhood of p contrary to hypothesis. We will now find an open covering of $(R-p)$ which contains no finite proximate covering of $(R-p)$. Now by hypothesis, $[S=R-U_0(p)]$ is not H-closed. Hence there is a covering of S by relative open sets $\{g_\alpha\}$ which contains no finite proximate covering of S. Let G_α be an open set of R such that $G_\alpha . S = g_\alpha$. Now for any point $q \epsilon S$ there is a $U(q) \subset g_\alpha$ and disjoint with a neighbourhood of p. This is possible since R is Hausdorff. For any point $r \epsilon U_0(p)$ choose an open set $U(r) \subset U_0(p)$ which is disjoint with a neighbourhood of p, which is again possible since R is Hausdorff. Consider the covering of $(R-p)$ constituted by the family of open sets consisting of the $U(q)$'s and the $U(r)$'s. No finite number of $\overline{U(q)}$'s can cover S for, otherwise the corresponding g_α's would constitute a finite proximate covering of S. But $S-\sum_1^m \overline{U(q)}$ is a relative open domain of S and therefore will have an interior. This interior can never be covered by any finite number of $\overline{U(r)}$'s since every $\overline{U(r)} \subset \overline{U(p)}$. Hence $F=(R-p)-\sum_1^m \overline{U(q)} - \sum_1^n \overline{U(r)}$ is never null; since each $U(q)$ and $U(r)$ is disjoint with a $U(p)$, a finite number of $U(q)$'s and $U(r)$'s are always disjoint with a $U(p)$, it follows that every F contains an open set containing p. Evidently the intersection of two F's is a F. Further $U_0(p)$ contains no F, for otherwise the closures of a finite number of $U(q)$'s and $U(r)$'s would cover S. Now the open sets $(p+F)$ satisfy the conditions for being taken for a new system of neighbourhoods for p. As each $p+F$ contains a $U(p)$ we see that $p+F$ is

* We may assume, without loss of generality, that $U_0(p)$ is an open domain neighbourhood of p.

stronger than or equivalent to the original system of neighbourhoods of p. But it is not equivalent to the original system since $U_0(p)$ contains no $p+F$. Hence the new system of neighbourhoods of p is stronger than the original system and so the V-topology is stronger than the given topology of R. Further the point p is H-separable from every point of the space under its new neighbourhood system for from the formation of the F's it is clear that there is a F disjoint with a $U(q)$ for any point $q \epsilon S$ and also a F disjoint with a $U(r)$ for any point $r \epsilon U_0(p)$. Hence the strengthened topology is Hausdorff and the given topology of R is not maximal-Hausdorff. Q.E.D.

The condition is sufficient. Suppose there exists a system $\{V(p)\}$ for a point p which is stronger than $\{U(p)\}$ so that there is a $U(p)$, say $U_0(p)$, which contains no $V(p)$. Then p would be a boundary point of $U_0(p)$ in the V-topology *i.e.*, a contact point of $[R - U_0(p)]$. But $[R - U_0(p)]$ is H-closed. Hence $[R - U_0(p)] + p$ cannot be a Hausdorff space; so that in the V-topology, p fails to satisfy the H-separability axiom with points in $\{R - U_0(p)\}$. Q.E.D.

The theorem is evidently verified in the case of Hausdorff bicompact spaces, where, as is well known, every closed subset is H-closed.

We now proceed to illustrate the theorem by consideration of a Hausdorff space constructed by Urysohn for a different purpose (*vide* Urysohn, Über die Mächtigkeit zusammenhängender Mengen; *Mathematische Annalen*, Band **94**, p. 268).

Example 4.—Let the space R be defined as follows:

$$R = a + b + \{a_{ij}\} + \{b_{ij}\} + c_i \ldots . (i, j = 1, 2, \ldots .)$$

A point of R shall be said to be of 0th, 1st or 2nd order according as the letter which represents the point contains no element, only one element or two elements respectively in the suffix.

The neighbourhoods are defined as follows:

$$U(a_{ij}) = a_{ij};\quad U(b_{ij}) = b_{ij};$$

(*i.e., the points of the second order are isolated points.*)

$$\left.\begin{aligned} U^n(c_i) &= c_i + \sum_{j=n}^{\infty} (a_{ij} + b_{ij}) \\ U^n(a) &= a + \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} a_{ij} \\ U^n(b) &= b + \sum_{j=1}^{\infty} \sum_{i=n}^{\infty} b_{ij} \end{aligned}\right\} n = 1, 2, 3 \ldots$$

The neighbourhoods satisfy H_1, H_2, H_3.

H_1: evident.

H_2: follows from the fact that for every point x

$$U^n(x) . U^{n+h}(x) = U^{n+h}(x)$$

H_3: follows from the fact that only the points of the second order—which are in fact identical with their neighbourhoods—can be contained in a neighbourhood of any other point.

H-separability: is evidently satisfied for any two points of the same order; that it is satisfied for all other points also is seen from the following equalities:

1. (*a point of the second order and a point of the first order*):

$$U(a_{pq}) . U^{q+1}(c_i) = 0, \quad U(b_{pq}) . U^{q+1}(c_i) = 0;$$

2. (*second order and* 0*th order*):

$$U(a_{pq}) . U^{p+1}(a) = 0, \quad U(b_{pq}) . U^{p+1}(b) = 0;$$

$$U(a_{pq}) . U^1(b) = 0, \quad U(b_{pq}) . U^1(a) = 0;$$

3. (*first order and* 0*th order*):

$$U^1(c_p) . U^{p+1}(a) = 0, \quad U^1(c_p) . U^{p+1}(b) = 0.$$

However for arbitrary n and m,

$$\begin{aligned} \overline{U^n(a)} . \overline{U^m(b)} &\supset \overline{U^{n+m}(a)} . \overline{U^{n+m}(b)} \\ &\supset \overline{\left(\sum_{j=1}^{\infty} \sum_{i=n+m}^{\infty} a_{ij}\right)} . \overline{\left(\sum_{j=1}^{\infty} \sum_{i=n+m}^{\infty} b_{ij}\right)} \\ &\supset \overline{\left(\sum_{j=1}^{\infty} a_{(n+m)j}\right)} . \overline{\left(\sum_{j=1}^{\infty} b_{(n+m)j}\right)} \\ &\supset c_{n+m}. \end{aligned}$$

Thus, in the space R, the points a and b are U-inseparable and consequently they are both irregular points of the space. As a matter of fact, a and b are both semiregular points of the space. Incidentally we have here a simple example of a semiregular space which is not regular. (Compare in this connection Example 3 above, and M. H. Stone, "Application of the theory of Boolean rings to General Topology," *Trans. Amer. Math. Society*, 1937, **41**, p. 452, where, the author who was the first to recognise the importance of semiregular spaces, has constructed a rather complicated example of a semiregular space which is not regular.)

In the space constructed above, a and b are U-inseparable and the other points are all regular. The space is H-closed. To prove this let us take an arbitrary open covering $\{G\}$ of R. Let us denote by $G(a)$, $G(b)$ a pair of elements of $\{G\}$, which contain a and b respectively. $\overline{G(a)}$ and $\overline{G(b)}$ will each contain all but a finite number, say n and m respectively, of the c_i's. Suppose $n > m$. Then the c_i's which are not covered simultaneously by $\overline{G(a)}$ and $\overline{G(b)}$ will be n in number. These c_i's will be covered by the closures of a finite number of the G's, *say* $\overline{G_1}, \overline{G_2}, \ldots \overline{G_k}$. It is easy to see that $R - [\overline{G(a)} + \overline{G(b)} + \overline{G_1} + \overline{G_2} + \ldots + \overline{G_k})]$ is a finite set of a_{ij}'s and b_{ij}'s. Hence, etc.

By a similar reasoning, we can show that the complement of every neighbourhood of a and b are H-closed. The complement of every neighbourhood of every a_{ij}, b_{ij}, c_i which are all regular points are also H-closed. Hence the condition of the theorem is satisfied in the space R and R is therefore maximal-Hausdorff. Being a Hausdorff irregular space, R cannot evidently be bicompact.

We have thus proved that a maximal-Hausdorff space need not necessarily be bicompact.