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Vector bundles with a fixed determinant on an irreducible nodal curve

USHA N BHOSLE

Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India
E-mail: usha@math.tifr.res.in

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Abstract. Let M be the moduli space of generalized parabolic bundles (GPBs) of rank r and degree d on a smooth curve X. Let $M_{\bar{L}}$ be the closure of its subset consisting of GPBs with fixed determinant $\bar{L}$. We define a moduli functor for which $M_{\bar{L}}$ is the coarse moduli scheme. Using the correspondence between GPBs on X and torsion-free sheaves on a nodal curve Y of which X is a desingularization, we show that $M_{\bar{L}}$ can be regarded as the compactified moduli scheme of vector bundles on Y with fixed determinant. We get a natural scheme structure on the closure of the subset consisting of torsion-free sheaves with a fixed determinant in the moduli space of torsion-free sheaves on Y. The relation to Seshadri–Nagaraj conjecture is studied.

Keywords. Nodal curves; torsion-free sheaves; fixed determinant.

1. Introduction

Generalized parabolic vector bundles (GPBs) on a smooth curve X are vector bundles on X together with parabolic structures on finitely many disjoint divisors $D_j,\ j = 1, \ldots, m$ [1, 2]. There is an open subscheme M'' of the moduli space M of GPBs on which one can define a determinant morphism into the moduli space of generalized parabolic line bundles $\bar{L}$, the map does not extend to M. Let $M''_{\bar{L}}$ be its locally closed subset consisting of GPBs with a fixed determinant $\bar{L}$. In this note, we define a moduli functor and construct a coarse moduli scheme $M_{\bar{L}}$ for it. The moduli scheme contains $M''_{\bar{L}}$ as an open dense subscheme.

Let Y be an irreducible projective nodal curve with nodes y_j, $j = 1, \ldots, m$ and p: $X \to Y$ its desingularization with D_j the inverse image of y_j. Denote by U the moduli variety of torsion-free sheaves of rank r, degree d on Y. Let U' be the open subvariety of U corresponding to vector bundles on Y. There is a surjective morphism f from M onto U [1, 2]. The restriction of the morphism f to $M' = f^{-1}U'$ is an isomorphism onto the open subvariety U' of U. A GPB $\bar{L}$ gives a torsion-free sheaf $\mathcal{L}$ on Y. If $\mathcal{L}$ is locally free, let $U'_{\mathcal{L}}$ be the closed subset of U' consisting of vector bundles with fixed determinant $\mathcal{L}$. Using f, $U'_{\mathcal{L}}$ may be identified with $M'_{\bar{L}}$ and $M_{\bar{L}}$ can be regarded as compactified moduli variety of vector bundles on Y with determinant $\mathcal{L}$.

We show that $f(M_{\bar{L}}) = \overline{U'_{\mathcal{L}}}$, the closure of $U'_{\mathcal{L}}$ in U, thus giving $\overline{U'_{\mathcal{L}}}$ the scheme structure of an image subscheme of U. Let I_j denote the ideal sheaf at the node y_j. For a torsion-free sheaf F of rank r on Y, let $N = \Lambda^r F/(\text{torsion})$ where (torsion) denotes the torsion subsheaf. Then we show that for any $\bar{L}$, the image $f(M_{\bar{L}})$ can be described (as a set) by

$$f(M_{\bar{L}}) = \{F \in U\colon I_j^r \mathcal{L} \subset N \subset \mathcal{L}, \quad \forall j\}.$$

This gives a proof of a conjecture by Seshadri and Nagaraj (Conjecture (a), p. 136 of [3]). Proving Seshadri–Nagaraj conjecture was not the aim of this note. The conjecture was proved by Sun [6] by degeneration methods. However he does not get a scheme structure on $\overline{U'_{\mathcal{L}}}$ or a moduli functor (except in some low rank cases). Our aim is to give a moduli functor and an explicit construction of a projective moduli space for it which contains an open subvariety isomorphic to $U'_{\mathcal{L}}$ if $\mathcal{L}$ is a line bundle. We also deal with the case when $\mathcal{L}$ is torsion-free but not locally free. The construction is much simpler than that of Schmitt [4] and hence the moduli space is easier to study. For example, properties like reduced, irreducible, Cohen–Macaulay follow immediately for our moduli space. Normality is true in rank 2 and is expected to be true in general. These properties have been used in computation of Picard groups in the rank two case.

2. The moduli scheme of GPBs with fixed determinant

2.1.

Let X be a nonsingular projective curve over an algebraically closed base field k. Let $D_j,\ j = 1, \dots, m$ be disjoint divisors on X with $D_j = x_j + x'_j$, where x_j, x'_j are distinct closed points. We recall here some basics on generalized parabolic bundles (GPBs), details may be found in [1, 2].

DEFINITION 2.1

A generalized parabolic bundle (GPB, in short) of rank r and degree d on X is a vector bundle E of rank r and degree d on X together with r-dimensional vector subspaces $F_j(E)$ of $E_{x_j} \oplus E_{x'_j}$. For a subbundle N of E, define $F_j(N) = F_j(E) \cap (N_{x_j} \oplus N_{x'_j})$ and $f_j(N) = \dim F_j(N)$.

DEFINITION 2.2

Fix a rational number $\alpha \in (0, 1]$. A GPB $(E, F_j(E))$ is α-stable (resp. α-semistable) if for every proper subbundle N of E, one has $(d(N) + \alpha \Sigma_j f_j(N))/r(N) < (\text{resp.} \leq)$ $(d(E) + \alpha r m)/r$.

DEFINITION 2.3

Let $p_j\colon F_j(E) \to E_{x_j},\ p'_j\colon F_j(E) \to E_{x'_j}$ be the projections. Assume that for each j, at least one of p_j, p'_j is an isomorphism. The subspace $F_j(E)$ determines an element $F_j(E)$ of $\mathrm{Gr}(r, E_{x_j} \oplus E_{x'_j}) \subset \mathbf{P}(\Lambda^r(E_{x_j} \oplus E_{x'_j}))$. One has a (rational) morphism $\delta\colon \mathbf{P}(\Lambda^r(E_{x_j} \oplus E_{x'_j})) \to \mathbf{P}(\Lambda^r E_{x_j} \oplus \Lambda^r E_{x'_j})$. Let $\det F_j(E)$ denote the one-dimensional subspace of $\Lambda^r E_{x_j} \oplus \Lambda^r E_{x'_j}$ determined by $\delta(F_j(E))$. Define the determinant of $(E, F_j(E))$ to be the generalized parabolic line bundle $(\det E, \det F_j(E))$.

DEFINITION 2.4

A family of GPBs of rank r, degree d parametrized by a scheme T is a tuple $(\mathcal{E}, F_j(\mathcal{E})_j)$ where $\mathcal{E} \to T \times X$ is a family of vector bundles of rank r, degree d on X which is flat over T and $F_j(\mathcal{E})$ is a rank r subbundle of $\mathcal{E}\mid_{T\times x_j} \oplus \mathcal{E}\mid_{T\times x'_j}$. The notion of equivalence of families is the obvious one.

We fix a generalized parabolic line bundle $\bar{L} := (L, F_j(L))$. Fix isomorphisms $h_j\colon L_{x_j} \to k$, $h'_j\colon L_{x'_j} \to k$. Then $F_j(L)$ can be identified to a point $F_j(L)$ of $\mathbf{P}^1$ of the form $(1:0)$, $(0:1)$ or $(1:\lambda_j)$, $\lambda_j \in k^*$.

2.2 *The moduli functor*

For simplicity, let us assume that there is only one divisor $D = x_1 + x_2$. Let $(\mathcal{E}, F(\mathcal{E})) \to T \times X$ be a family of GPBs of rank r, degree d on X with $\mathcal{E}_t$, $t \in T$, of fixed determinant L. For $i = 1, 2$ we have vector bundles

$$\mathcal{E}_{x_i} = \mathcal{E}\,|_{T\times x_i} \to T.$$

Let $\mathcal{G}r \to T$ denote the Grassmannian bundle of rank r subbundles of $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}$. It is embedded as a closed subvariety in $\mathbf{P}(\Lambda^r(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}))$ by Plücker embedding. Note that $F(\mathcal{E})$ defines a section of $\mathcal{G}r$. Since $\det \mathcal{E}\,|_{t\times X} = L$, it follows that $\det \mathcal{E} = p_T{}^*N \otimes p_X^* L$ for some line bundle N on T. Hence for $i = 1, 2$ one has $\det \mathcal{E}_{x_i} = N \otimes L_{x_i} = N$, using the isomorphism h_i. Let $q_i\colon \Lambda^r(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}) \to \det \mathcal{E}_{x_i} = N$ be the projections, $i = 1, 2$. Define a hyperplane subbundle $\mathcal{H}$ of $\mathbf{P}(\Lambda^r(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}))$ by $q_2 = 0$ if $F(L) = (1:0)$, by $q_1 = 0$ if $F(L) = (0:1)$ and by $q_2 - \lambda_j q_1 = 0$ if $F(L) = (1:\lambda)$. Let $H_T := \mathcal{G}r \cap \mathcal{H}$. It is a closed reduced subscheme of $\mathcal{G}r \subset \mathbf{P}(\Lambda^r(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}))$. Note that H_T is independent of the choice of h_1, h_2.

More generally, if we consider parabolic structures over finitely many disjoint divisors $D_j = x_j + x'_j$, for each j one constructs the hyperplane bundle $\mathcal{H}_j$ and Grassmannian bundle $\mathcal{G}r_j$ over T. Let $\mathcal{G}r$ be the fibre product of $\mathcal{G}r_j$ over T and H_T the fibre product of $\mathcal{G}r_j \cap \mathcal{H}_j$ over T.

DEFINITION 2.5

Let $F^{ss}_{\bar{L}}$ be the functor $F^{ss}_{\bar{L}}$: Schemes $\to$ Sets which associates to a scheme T the set of equivalence classes of families $(\mathcal{E}, F(\mathcal{E})) \to T \times X$ of α-semistable GPBs of rank r and degree d with $\det \mathcal{E}_t \cong L$ for all $t \in T$ such that the section of $\mathcal{G}r$ defined by $(F_j(\mathcal{E})_j)$ maps into H_T.

One similarly defines a full subfunctor $F^s_{\bar{L}}$ of $F^{ss}_{\bar{L}}$ with semistable replaced by stable.

2.3 *Construction of the moduli space*

Let S denote the set of semistable GPBs $(E, F(E))$, where E is a vector bundle of rank r, degree d with fixed determinant L and $F(E)$ is a subspace of $E_{x_j} \oplus E_{x'_j}$ of dimension r with fixed weights $(0, \alpha)$, $0 < \alpha < 1$. For $m \gg 0$, all GPBs in S satisfy the condition

$$(*) \qquad H^1(E(m)) = 0,\ H^0(E(m)) \cong \mathbf{C}^n,\ H^0(E(m)) \to H^0(E(m) \otimes (\oplus_j \mathcal{O}_{D_j}))$$

is surjective.

Let Q denote the quot scheme of coherent quotients of $\mathcal{O}^n_X \otimes \mathcal{O}_X(-m)$ with fixed Hilbert polynomial determined by r, d. Let R be the nonsingular subvariety of Q corresponding to quotient vector bundles E satisfying condition $(*)$. Denote by R_0 the nonsingular closed subvariety in R corresponding to E with $\Lambda^r(E) = L$. Let $\mathcal{E} \to R \times X$ be the universal quotient bundle. Over R, we have the fibre bundle $\mathcal{G}r$ with each fibre an m-fold product of $\mathrm{Gr}(r, 2r)$ as in §2.2. Over R_0, we have the fibre bundle $\tilde{R}_0 := H_{R_0}$ whose fibres are

m-fold products of hyperplane sections of $\mathrm{Gr}(r, 2r)$. Then $\tilde{R_0}$ is a closed subvariety of $\mathcal{G}r$. Let $\tilde{R_0}^s$ (resp. $\tilde{R_0}^{ss}$) be the open set of stable (resp. semistable) points in $\tilde{R_0}$.

The GIT quotient M of $\mathcal{G}r$ by $\mathrm{PGL}(n)$ for a suitable polarization (depending on α) is the coarse moduli space for GPBs [1, 2]. It is a normal projective variety. Since $\tilde{R_0}$ is a $\mathrm{PGL}(n)$-invariant closed subscheme (subvariety) of $\mathcal{G}r$, the GIT quotient $M_{\bar{L}}$ of $\tilde{R_0}^{ss}$ by $\mathrm{PGL}(n)$ is a closed subvariety of M (with a natural reduced subscheme structure). The GIT quotient $M^s_{\bar{L}} = \tilde{R_0}^s//\mathrm{PGL}(n)$ is an open subscheme of $M_{\bar{L}}$. It is easy to see that $M_{\bar{L}}$ (resp. $M^s_{\bar{L}}$) is the coarse moduli space for the functor $F^{ss}_{\bar{L}}$ (resp. $F^s_{\bar{L}}$).

Theorem 1. *Let $\alpha \in (0,1)$. Then there is a coarse moduli space $M_{\bar{L}}$ (resp. $M^s_{\bar{L}}$) for the functor $F^{ss}_{\bar{L}}$ (resp. $F^s_{\bar{L}}$). The moduli space $M_{\bar{L}}$ is a projective (irreducible) variety, containing $M^s_{\bar{L}}$ as an open subvariety.*

Let M' (resp. $M'_{\bar{L}}$) be the open subscheme of M (resp. $M_{\bar{L}}$) consisting of GPBs $(E, F_j(E))$ such that the projections p_j, p'_j are isomorphisms for all j. Then $M'_{\bar{L}}$ corresponds to GPBs in M with fixed determinant $\bar{L}$ with $F_j(L) = (1 : \lambda_j), \lambda_j \in k^*$ and $M_{\bar{L}}$ is the closure of $M'_{\bar{L}}$.

3. Application to nodal curves

3.1.

Let Y be an irreducible projective nodal curve with nodes y_j, $j = 1, \ldots, m$ and X its desingularization with D_j, the inverse image of y_j. Then there is a correspondence from GPBs on X of rank r, degree d to torsion-free sheaves on Y of the same rank and degree [1, 2]. The correspondence induces a surjective morphism f from M onto U, where U is the moduli space of torsion-free sheaves of rank r, degree d on Y. The restriction of the morphism f to M' is an isomorphism onto the open subvariety U' of U corresponding to vector bundles on Y. One has $f^{-1}U' = M'$.

For $r = 1$, a GPB $\bar{L}$ corresponds to a torsion-free sheaf $\mathcal{L}$ on Y. Then $\mathcal{L}$ is a line bundle if and only if $F_j(L) = (1, \lambda_j)$, $\lambda_j \in k^*$, $\forall j$. Suppose that $\mathcal{L}$ is a line bundle and let $U'_{\mathcal{L}}$ be the closed subset of U' corresponding to vector bundles with fixed determinant $\mathcal{L}$. Then $f^{-1}U'_{\mathcal{L}} = M'_{\bar{L}}$ and the morphism f maps $M'_{\bar{L}}$ isomorphically onto $U'_{\mathcal{L}}$. Hence, if $\mathcal{L}$ is a line bundle, then $f(M_{\bar{L}})$ contains $U'_{\mathcal{L}}$ as an open subset. Since $M_{\bar{L}}$ is an irreducible, closed subscheme of M, the image $f(M_{\bar{L}})$ is a closed, irreducible subscheme of U and $U'_{\mathcal{L}}$, being open, is dense in it. It follows that $f(M_{\bar{L}})$ is the closure of $U'_{\mathcal{L}}$.

Remark 3.1. The projective scheme $M_{\bar{L}}$ can be regarded as the compactified moduli space of vector bundles of rank r with determinant $\mathcal{L}$ on the nodal curve Y.

Remark 3.2. In fact, for any torsion-free sheaf $\mathcal{L}$ of rank 1, the image $f(M_{\bar{L}})$ is a closed, irreducible subscheme of U containing the subset of U consisting of torsion-free sheaves with fixed determinant $\mathcal{L}$ as an open dense set.

3.2 *Relation to Seshadri–Nagaraj conjecture*

For a torsion-free sheaf F of rank r on Y, let $N := (\Lambda^r F)/(\text{torsion})$, where (torsion) denotes the maximum subsheaf with proper support. Denote by I_j the ideal sheaf of the node y_j. Define $U_{\mathcal{L}}$ as the set

$$U_{\mathcal{L}} = \{F \in U \colon I_j^r \mathcal{L} \subset N \subset \mathcal{L}, \quad \forall j\}.$$

Seshadri and Nagaraj had defined this set for Y with one node and conjectured that if $\mathcal{L}$ is a line bundle, then $U_{\mathcal{L}}$ is the closure of $U'_{\mathcal{L}}$ (Conjecture (a), page 136 of [3]). We prove this conjecture.

Let $\tilde{R}_0^{1-ss}$ denote the subset consisting of 1-semistable points, then $\tilde{R}_0^{ss} \subset \tilde{R}_0^{1-ss}$. Let P be the moduli space of 1-semistable GPBs [5]. One has morphisms $f \colon \tilde{R}_0^{ss} \to U$ inducing $f \colon M_{\bar{L}} \to U$ and $f_1 \colon \tilde{R}_0^{1-ss} \to U$ inducing $f_1 \colon P \to U$.

PROPOSITION 3.3

Let $\bar{L}$ be any GPB of rank 1 *on X and $\mathcal{L}$ the corresponding torsion-free sheaf of rank* 1 *on Y.*

(1) *If $(E, F_j(E)) \in \tilde{R}_0^{1-ss}$, then $F = f_1((E, F_j(E))) \in U_{\mathcal{L}}$ and hence $f(M_{\bar{L}}) \subset U_{\mathcal{L}}$.*
(2) *The morphism $\tilde{R}_0^{1-ss} \to U$ surjects onto $U_{\mathcal{L}}$.*
(3) *$f(M_{\bar{L}}) = U_{\mathcal{L}}$ for α sufficiently close to* 1.

Proof. We may assume that Y has only one node y. It is easy to see (from the proof) that the general case follows exactly on same lines. Consider a GPB $(E, F(E))$. Let $p_i \colon F(E) \to E_{x_i}$, $i = 1, 2$ be the projections and $a_i = \dim \ker p_i$. Let $E_0 = p^*(F)/(\text{torsion})$, then $E_0 \subset E$. Since $F|_{Y-y} = p_* E\,|_{Y-y}$, one has $N|_{Y-y} = (p_* L)|_{Y-y} = \mathcal{L}\,|_{Y-y}$. Hence to check that $I^r \mathcal{L} \subset N \subset \mathcal{L}$, we have only to check it locally at the node y.

Let (A, m) be the local ring at y. Its normalization $\bar{A}$ is a semilocal ring with two maximal ideals m_1, m_2. The inclusion

$$F_y \subset (p_* E)_y$$

may be identified with the inclusion

$$(r - a_1 - a_2)A \oplus a_1 m_1 \oplus a_2 m_2 \subset r\bar{A}$$

(Proposition 4.2 of [2]).

(1) We consider the following cases separately.

Case (*i*). Suppose that p_1, p_2 are both isomorphisms. Then $\det(E, F(E)) = \bar{L}$ corresponds to a torsion-free sheaf $\mathcal{L}$ which is locally free at y. In this case, F is locally free at y with $(\det F)_y = \mathcal{L}_y$ so that $N = \mathcal{L} \supset I^r \mathcal{L}$.

Case (*ii*). Assume that p_1 is an isomorphism, p_2 is not an isomorphism (the opposite case can be dealt with similarly). Then $\det(E, F(E)) = (L, L_{x_1}) = \bar{L}$ corresponds to $\mathcal{L}$ which is not locally free at y. One always has $N \subset p_* L$ and $N_y \subset \mathcal{L}_y$ if and only if $N_y \otimes k(y) \subset F(L)$. Locally, $a_1 = 0$ so that $F_y = (r - a_2)A \oplus a_2 m_2$. Then $N_y = (m_2)^{a_2}$ so that $N_y \otimes k(y) \subset L_{x_1} = F(L)$. Hence $N \subset \mathcal{L}$. Since $m^r \subset m_2^{a_2}$, it follows that $I^r \mathcal{L} \subset N \subset \mathcal{L}$.

Case (*iii*). If both p_1 and p_2 are not isomorphisms, one has $a_1 \geq 1$, $a_2 \geq 1$. Then locally, $N_y = m_1^{a_1} m_2^{a_2} \subset m_1 m_2 = m$. It follows that N_y maps to zero in $p_* L \otimes k(y)$ so that $N \subset I\mathcal{L} \subset \mathcal{L}$. Since $m^r \subset m_1^{a_1} m_2^{a_2}$, one has $I^r \mathcal{L} \subset N \subset \mathcal{L}$.

Note that any $(E, F(E)) \in M_{\bar{L}}$ with $F(L) = (1 : \lambda), \lambda \in k^*$ (i.e. $\mathcal{L}$ locally free at y) occurs only in cases (i) or (iii). For $(E, F(E)) \in M_{\bar{L}}$ with $F(L) = (1 : 0)$ (or $F(L) = (0 : 1)$) only cases (ii) and (iii) occur. Part (1) now follows. Note that $(E, F_j(E))$ is 1-semistable if and only if F is semistable [1, 2].

(2) Let $F \in U_{\mathcal{L}}$. Since $I^r\mathcal{L} \subset N \subset \mathcal{L}$ it follows that $L|_{X-D} = p^*N|_{X-D}$ where $D = \sum_j (x_j + x'_j)$. Since det $E_0 = p^*N$ outside D, one has $L =$ det E_0 outside D. It follows that $L =$ det $E_0 \otimes \mathcal{O}_X(\sum_j (a_j x_j + a'_j x'_j))$, $a_j + a'_j \leq r$. Let E be given by an extension

$$0 \to E_0 \to E \to \oplus_j (k(x_j)^{a_j} \oplus k(x'_j)^{a'_j}) \to 0.$$

The composite $F \hookrightarrow p_*E_0 \hookrightarrow p_*E$ induces a linear map $F \otimes k(y_j) \to p_*(E) \otimes k(y_j)$. Let $F_j(E)$ be the image of this linear map. Then $(E, F_j(E))$ maps to F and it is 1-semistable as F is semistable [2]. By construction, det $E =$ det $E_0 \otimes \mathcal{O}_X(\sum_j (a_j x_j + a'_j x'_j)) = L$. It follows that $(E, F_j(E)) \in \tilde{R}_0^{1-ss}$.

(3) Let $P_{\bar{L}}$ denote the closure of $\tilde{R}_0^{1-ss}/\mathrm{PGL}(n)$ in P. For α close to 1, there is a surjective birational morphism ϕ: $M \to P$ with $f = f_1 \circ \phi$. It maps $M_{\bar{L}}$ birationally into $P_{\bar{L}}$. Since both these spaces are irreducible and of the same dimension, it follows that $\phi(M_{\bar{L}}) = P_{\bar{L}}$. Since $f(\tilde{R}_0^{1-ss})$ surjects onto $U_{\mathcal{L}}$, it follows that $f_1(P_{\bar{L}}) \supset U_{\mathcal{L}}$ and hence $f(M_{\bar{L}}) \supset U_{\mathcal{L}}$. From (1), it follows that $U_{\mathcal{L}} = f(M_{\bar{L}}) = f_1(P_{\bar{L}})$. □

COROLLARY 3.4

If $\mathcal{L}$ is a line bundle, then $U_{\mathcal{L}}$ is the closure of $U'_{\mathcal{L}}$.

Proof. From Proposition 3.3, one has (as sets) $f(M_{\bar{L}}) = U_{\mathcal{L}}$. Since $f(M_{\bar{L}})$ is the closure of $U'_{\mathcal{L}}$ if $\mathcal{L}$ is a line bundle, the result follows. □

Remark 3.5. The proof of Proposition 3.3(1) easily generalizes to $(E, F_j(E))$ replaced by a family $(\mathcal{E}, F_j(\mathcal{E})) \to T \times X$.

Remark 3.6. Sun [6] had proved the conjecture (a) of Seshadri–Nagaraj by considering a smooth curve X degenerating to an irreducible nodal curve Y. However he does not get a moduli functor or a scheme structure on $U_{\mathcal{L}}$ except in some cases (e.g. one node, rank 2, degree 1).

Remark 3.7. Schmitt [4] has constructed a moduli space $\mathcal{M}$ of α-semistable descending singular SL(r)-bundles (A, q, τ) where (A, q) is a GPB on X and τ: $\mathrm{Sym}^*(A \otimes \mathbf{C}^r)^{\mathrm{SL}(r,\mathbf{C})} \to \mathcal{O}_X$ a nontrivial homomorphism. It is shown that det $A = \mathcal{O}_X$ and for $\alpha \in (0, 1) \cap \mathbf{Q}$, there is a forgetful morphism h: $\mathcal{M} \to M$ (§5.1 of [4]). For α close to 1, one has a forgetful morphism $\mathcal{M} \to U = U(r, 0)$ whose set theoretic image is $U_{\mathcal{L}}$ (Proposition 5.1.1 of [4]). Then, since det $A = \mathcal{O}_X$, it follows that $h(\mathcal{M}) = M_{\bar{L}}$ (as sets).

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