PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 55, No. 4
October 2000
pp. 609–615

Singularity free non-rotating cosmological solutions for perfect fluids with $p = k\rho$

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Abstract. It is an attempt to explore non-singular cosmological solutions with non-rotating perfect fluids with $p = k\rho$. The investigation strongly indicates that there is no solution of the above type other than already known. It is hoped that this result may be rigorously proved in future.

1. Introduction

Singularity free cosmological solutions of the type stated in the title known so far are of a very special class and have the following characteristics:

(a) The space time is cylindrically symmetric.
(b) In case the metric is diagonal, the $g_{\mu\nu}$'s are of the form $g_{\mu\nu}$ = a function of time multiplied by a function of the radial coordinate.

However (a) does not necessarily require that $g_{\mu\nu}$ is diagonal and if the metric has a non-diagonal element for the two spaces which are the orbits of the group of isometries which exist because of (a), only g_{tt}, g_{rr} and γ = det$|g_{ab}|$(where $a, b = 2, 3$) are expressible in the product form.

(c) There is only one independent time function namely cosh (αt).
(d) Only two values of k are permissible namely $k = 1/3$ or 1.

It may be mentioned that the authors of these papers assumed only the existence of a G_2 and not full cylindrical symmetry (cylindrical symmetry requires that one of the isometrics is a rotation. They did assume the separability as stated in (b). Neither (c) nor (d) was assumed; indeed nonsingular solutions in which there is no equation of state of the form $p = k\rho$ were also exhibited.

The situation seemed intriguing and one felt tempted to ask the question – Do these known solutions exhaust the class of non-singular solutions of the type stated in the title? Or are they just a subset of measure zero amongst the whole class of non-singular solutions? At first sight the second alternative may seem appealing for it is quite likely that the other solutions which exist have escaped discovery because of their complicated nature.

In the present paper, an attempt is made to plug in the condition of non-singularity at the beginning of the investigation and then try to solve the relevant equations. This has been facilitated by the discovery of two results by the present author, namely

(1) For any non-rotating cosmological solution which is singularity free, the space – average of each of the Raychaudhuri scalars must vanish provided only that the strong energy condition is obeyed.
(2) If in the above non-singular solution the cosmic matter be a perfect fluid, then the time average of each of the Raychauduri scalar also vanishes.

The result of the investigation gives a strong indication that the already known solutions are the only non-singular solutions of the stated type although at two points we have to introduce somewhat ad-hoc assumption which seem provably true but not proven to be such.

2. The coordinate system

The non-rotating condition allows us to choose a coordinate system which is both comoving and the time coordinate orthogonal to the three space

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 + g_{ik}\mathrm{d}x^i\mathrm{d}x^k. \tag{1}$$

The fluid velocity vector has the components

$$v^1 = 0, v^0 = \frac{1}{\sqrt{g_{00}}}. \tag{2}$$

The acceleration vector has the components

$$\dot{v_i} = \frac{1}{2}(\ln g_{00})_{,i}\ ;\ \dot{v_0} = 0. \tag{3}$$

Thus the acceleration vector is orthogonal to v^{α} and in the three space is a gradient vector. We can choose therefore a space-like coordinate say x^1 along $\dot{v}$, which is orthogonal to the other two coordinates. Thus the metric now assumes the form

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 + g_{rr}\mathrm{d}r^2 + g_{ab}\mathrm{d}x^a\mathrm{d}x^b. \tag{4}$$

The indices a, b run over the values 2, 3 and we have written r in place of x^1. The divergence relation $T^{\mu}_{\nu;\mu} = 0$ gives with $p = k\rho$:

$$pg_{00}^{(k+1)/2k} = \text{ function of } t \text{ alone.} \tag{5}$$

$$p\left(\sqrt{|^3g|}\right)^{1+k} = \text{ function of space co-ordinates only} \tag{6}$$

and combining the above two relations,

$$\left(\sqrt{|^3g|}\right)^{2k} g_{00}^{-1} = \text{ function of } t \times \text{ function of space co-ordinates only.} \tag{7}$$

From (5) and (6) it follows that p, θ just like g_{00} are functions of r and t only. Consider now the following three field equations

$$2\sigma^2 + \frac{1}{3}\theta^2 + \frac{4\pi}{3}(\rho + 3p) = \dot{v}^{\mu}_{;\mu} - \dot{\theta}, \tag{8}$$

$$\sigma^{ik}|k = \frac{2}{3}\theta'^{i}, \tag{9}$$

$${}^{3}R = -\frac{2}{3}\theta^2 + 2\sigma^2 + 16\pi\rho. \tag{10}$$

In the above the vertical bar followed by an index (e.g. $\sigma^{ik}|k$) indicates covariant derivative with respect to the three space metric and ${}^{3}R$ is the scalar curvature of the three space.

Plugging the condition that p, ρ, θ, g_{00} are functions of r and t only, we get from eqs (8)–(10).

$$[2\sigma^2 - \dot{v}^{\mu};\mu]_{,a} = 0 \tag{11}$$

$$[{}^{3}R - 2\sigma^2]_{,a} = 0 \tag{12}$$

$$\sigma^{ak}|k = 0 \tag{13}$$

where $a = 2, 3$.

A detailed analysis of the eqs (11)–(13) leads to the conclusion that the metric tensor components may be independent of x^2 and x^3 i.e we prove the existence of G^2.

A look at the eqs (5)–(7) allow us to write

$$\begin{aligned} p g_{00}^{(k+1)/2k} &= \text{function of } t \text{ alone.} \\ p\sqrt{|{}^{3}g|}^{\,1+k} &= \text{function of } r \text{ alone.} \\ \left(\sqrt{|{}^{3}g|}\right)^{2k} g_{00}^{-1} &= (\text{function of } t) \times (\text{function of } r). \end{aligned}$$

Although these equations do not lead to the separability for $p, \sigma,\ g_{00}$ and $|{}^{3}g|$ but one feels inclined to conjecture that such indeed is the case. In fact the assumption that any of them is of the form $T(t)$. $R(r)$ makes the others of the same form. This is the first assumption to which we referred in the introduction.

We shall conclude this section by making a specific choice of the origin of our coordinate system. As we have noted in the introduction, in this case, the space average and time average of Raychaudhuri scalars vanish. This would require in particular that they vanish at $r \to \pm\alpha$ and $t \to \pm\alpha$. Consequently these scalars like $p, \rho,\ \sigma^2,\ \theta^2$ which are positive definite must have at least one maximum. Take the case of ρ- with $\rho = \rho t$ (function of t alone) $\times$ ρr (function of r alone).

$(\partial \ln\rho/\partial r) = 0$ would require $(\partial\rho_r/\partial r) = 0$ $(\rho_r = 0)$ and this would give some constant values of r. In case of p (or ρ) it is easy to see that there will be only one such constant. We choose this constant to be such that the maximum occurs at $r = 0$. Similarly the maximum of ρ_t occurs at $t = 0$. Our origin is thus a very special point where

$$\frac{\partial\rho}{\partial r}=\frac{\partial\rho}{\partial t}=\frac{\partial g_{00}}{\partial r}=\frac{\partial g_{00}}{\partial t}=\frac{\partial\sqrt{^3g}}{\partial r}=\frac{\partial\sqrt{^3g}}{\partial t}=0$$

$$\text{as also } \frac{\partial\sigma^2}{\partial r}=\frac{\partial\theta^2}{\partial r}=0.$$

Presumably an investigation of these relations will lead to the vanishing of the first order derivatives of all the metric tensor components with respect to both r and t. This our second assumption.

3. Proof of cylindrical symmetry

By direct calculation

$$^3R={}^3R_1^1+\frac{1}{\sqrt{|g_{rr}|}}\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{|g_{rr}|}}\frac{\partial\ln\sqrt{\gamma}}{\partial r}\right)+\left(\frac{1}{\sqrt{|g_{rr}|}}\frac{\partial}{\partial r}\ln\sqrt{\gamma}\right)^2, \quad (14)$$

where γ = det$|g_{ab}|$.

We evaluate $^3R_1^1$, by using an analogue of the procedure leading to Raychaudhuri equation. Consider the three space defined by $t=$ constant. In this space, the unit vector along r is geodetic and hypersurface orthogonal. Hence if we consider the Raychauduri equation in 3-dimensions with this unit vector taking the place of the velocity vector, only its 'expansion' and shear will appear. They are defined as

$$\xi^i=\frac{1}{\sqrt{g_{rr}}}\delta_1^i,$$

$$\text{'Expansion' } \Theta=\xi^i_{|i}=\frac{1}{\sqrt{g_{rr}}}(\ln\sqrt{\gamma})_{,r},$$

$$\text{'Shear' } \Sigma_{ik}=\frac{1}{2}(\xi_{i|k}+\xi_{k|i})-\frac{1}{2}(g_{ik}+\xi_i\xi_k)\xi^1_{|1}.$$

Note that in the last term $1/2$ occurs in place of $1/3$ in 4 dimensions. This is to satisfy the trace free condition of Σ^i_k. Because of the definition the 'shear tensor' Σ_{ik} is orthogonal to the vector ξ^i. Thus the only non-vanishing components are $\Sigma^2_2(=-\Sigma^3_3)$ and Σ_{23}. The Raychauduri analogue three dimensional equation is thus

$$^3R_1^1={}^3R_{ik}\xi^i\xi^k=-\frac{1}{\sqrt{|g_{rr}|}}\frac{\partial}{\partial r}\Theta-\frac{1}{2}\Theta^2-2\Sigma^2. \quad (15)$$

Combining (10), (14) and (15) we get

$$-\frac{2}{3}\theta^2+2\sigma^2+16\pi\rho-\frac{1}{2}\Theta^2+2\Sigma^2=0. \quad (16)$$

With our assumption of the vanishing of 1st order derivatives at the origin, one may think that eq. (16) would require the vanishing of ρ at the origin and consequently lead to the trivial case of ρ, p etc. vanishing everywhere. However there is a catch, if there is an angular coordinate, then 'elementary flatness' condition would require that as $r \to 0, g_{22} = r^2 g_{rr}$ where x^2 is the angular coordinate and consequently although $g_{22}, r = 0$, (ln g_{22}), $_r \to (2 \ln r),_r = 2/r$ as $r \to 0$ and thus blow up as r^{-1}. In (16) therefore ρ need not be zero at the origin.

We are thus led to the conclusion that at least one of the coordinate x^2, x^3 must be an angular coordinate. Both of them cannot be angular coordinates for the last two terms in (16), i.e.

$$-\frac{1}{2}\Theta^2 + 2\Sigma^2 = \frac{1}{2}(2\Sigma - \Theta)(2\Sigma + \Theta)$$

would blow up as $r \to 0$ and ρ would become singular. In case only one angular coordinate is present, the term $0(r^{-1})$ would cancel out either in $(2\,\Sigma - \Theta)$ or in $(2\,\Sigma + \Theta)$ so that the product gives a finite non-zero contribution to ρ. With two angular coordinates, both the factors blow up.

4. The final form of the metric

So far, the assumption of separability of ρ (i.e. $\rho = \rho_t \rho_r$) has led to the separability of g_{00} and ${}^3g$. A reference to any of the eqs (8), (9) or (10) now shows that g_{11} and Y also have the separability property.

Once g_{11} and g_{00} are both found to be separable, we may make scale transformations of r and t to make $|g_{11}| = g_{00}$ and thus the line element reduces to the final form

$$\mathrm{d}z^2 = g_{00}(\mathrm{d}t^2 - \mathrm{d}r^2) + g_{ab}\mathrm{d}x^a\mathrm{d}x^b$$

with the restriction (g_{22}) = $g_{11}r^2$ as $r \to 0$. So that (ln $|g_{22}|)^{\cdot}$ = (ln $g_{11})^{\cdot}$.

5. The three cases

Multiplying eq. (8) throughout by g_{00} and replacing $\dot{v}^{\mu}_{;\mu}$ by derivatives $\partial/\partial r \ln p$, we get

$$\left(\frac{1}{3}\theta^2 + 2\sigma^2\right) g_{00} + \frac{4\pi}{3}(\rho + 3p)g_{00} = -g_{00}\dot{\theta} - \frac{1}{\sqrt{\gamma}} \times \frac{\partial}{\partial r}\left[\frac{k}{1+k}(\ln p)_{,r}\sqrt{\gamma}\right]. \tag{17}$$

With the results already deduced, we have

$$L_1 \text{ and } R_1 \text{ are function of } t \text{ alone AND } R_2 \text{ is function of } r \text{ alone}, \tag{18}$$

where the terms in (17) are indicated as follows: The capital letters L and R refer to the left side and right side respectively; the subscripts 1 and 2 indicate position of the term beginning from the left.

From (18) and (17) we conclude that L_2 is either a function of only one variable r or t or may be just a constant. Thus we have three cases:

(A) $\rho g_{00} =$ Const. > 0

(B) $\rho g_{00} = f(t)$

(C) $\rho g_{00} = \Phi(r)$

For both (A) and (B) the divergence integral $\rho g_{00}{}^{(k+1)/2k}$ = function of t alone gives (as g_{00} involves r as well), $k = 1$ or $p = \rho$. We take up the case (A)

Case A. ρg_{00} = constant, $k = 1$. The other divergence integral $\rho \left(\sqrt{(^3 g|}\right)^{1+k}$ is a function of r alone given this case. $\rho g_{00} \gamma$ is a function of r alone, consequently γ is a function of r alone. Using this condition,

$$\theta = \frac{1}{2\sqrt{g_{00}}} \frac{\partial}{\partial t} \left[\log |^3 \sqrt{g}|\right] = \frac{1}{2\sqrt{g_{00}}} (\ln g_{00}).$$

Hence, the shear components are

$$S_1 \equiv \sigma_1^1 = \frac{1}{2\sqrt{g_{00}}} (\ln g_{00})^{\cdot} - \frac{1}{3}\theta = \frac{2}{3}\theta,$$
$$S_2 = S_1 = \frac{2}{3}\theta,$$
$$S_3 = -\frac{4}{3}\theta$$

and

$$\sigma^2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2) = \frac{4\theta^2}{3}. \tag{19}$$

Eliminating σ^2 from (17) with the help of the above relation we get,

$$3 g_{00} \theta^2 = -g_{00} \dot{\theta} + \text{ constant.}$$

At the origin $\theta = 0$ and $\dot{\theta}$ is positive, so the constant is positive. Writing this as b^2 and putting in the value of θ, we get,

$$\left(\frac{\dot{g_{00}}}{g_{00}}\right)^2 = -\frac{\partial}{\partial t} \left(\frac{\dot{g_{00}}}{g_{00}}\right) + 4b^2$$

which gives on integration $g_{00} = \cosh 2bt \times$ a function of r alone.

However it is easy to see that in this case g_{ab} cannot be diagonal. To see this use (19) in (16) to get

$$2\theta^2 g_{00} + 16\pi\rho g_{00} + \left[2\Sigma^2 - \frac{1}{2}\Theta^2\right] g_{00} = 0.$$

If g_{ab} be diagonal, the last term is a function of r alone, hence with ρg_{00} = constant, one would have, $\theta^2 \ g_{00}$ = function of r alone whereas $\theta^2 \ g_{00}$ is a function of t alone.

Further analysis of this non-diagonal case leads us to the solution given by Mars. We omit the details of this analysis.

Case B. $\rho\, g_{00} =$ function of t alone, $k = 1, p = \rho$. A rather painstaking analysis gives in this case the group of solutions given by Dadhich *et al*.

Case C. $\rho g_{00} = \phi(r)$. Again an analysis leads to the Senovilla solution with $k = \frac{1}{3}$ i.e. $p = \frac{1}{3\rho}$.

6. Conclusion

Our motivation was to examine whether non-singular non-rotating perfect fluid (with $p = k\rho$) cosmologies exist besides those already discovered and presented in the literature. We have not been able to give an unequivocal answer but have found that with some almost imperative assumptions, one is led to the already known solutions.