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Scaling and universality of thermodynamics and correlations of an ideal relativistic Bose gas with pair production

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An ideal relativistic Bose-gas model with pair production is solved exactly in d dimensions. The critical behavior is shown to be the same as the usual nonrelativistic Bose gas. Scaling functions (including nonuniversal parameters) for the equation of state and the correlation functions are obtained in a closed form. The scale-factor universality is verified by the formation of the expected universal ratios. Various limiting cases of the model depending on the relative magnitudes of the three basic length scales, i.e., Compton wavelength, thermal wavelength, and interparticle distance are discussed. In these limiting cases, previously known results are recovered, wherever these are available.

I. INTRODUCTION

Recently, several authors[1–3] have discussed the properties of an ideal relativistic Bose gas with the nonzero chemical potential μ. The thermodynamical behavior of the gas has been studied in d dimensions with particular reference to Bose-Einstein condensation. Beckmann, Karsch, and Miller[1] have discussed the dependence of critical indices on the spatial dimensionality of the system, whereas Goulart Rosa, Jr., and co-workers[2] have discussed in detail the various interesting limiting cases of this model and their interrelationships. Haber and Weldon[3] have emphasized the importance of taking the particle-antiparticle pair production into account at high temperatures. These different works can be classified with the help of the three basic lengths in this problem.[2] These are, respectively, the thermal wavelength λ_T, the mean interparticle spacing $\bar{\lambda}$, and the Compton wavelength λ_C. From these, one can form two independent ratios

$$R_1=\bar{\lambda}/\lambda_T, \quad R_2=\lambda_C/\lambda_T . \tag{1}$$

Clearly, we will have a quantum (classical) gas if $R_1 \ll 1$ ($\gg 1$) and the gas (whether classical or quantum) will be nonrelativistic (NR) or ultrarelativistic (UR) depending on whether $R_2 \ll 1$ or $R_2 \gg 1$. The four limiting cases are shown in Fig. 1 schematically. The standard NR Bose gas[4] falls in region A, whereas the standard Maxwell-Boltzmann distribution[4] is obtained in region B. Region C, being that of a classical UR gas, is perhaps not of much interest. The recent work mentioned above[1–3] has mainly focused on region D and its relationship with region A, i.e., on the quantum UR-NR gas.

In this paper, we solve the model exactly, in general, without neglecting pair production[1,2] and without making a high-temperature expansion.[3] We show that, in the quantum case, the critical exponents and the scaling functions of the model are the same as those of the standard NR Bose gas.[5] The UR and the NR gases differ only in the values of the critical amplitudes. However, we show that even this difference can be eliminated by employing the scale-factor universality.[6]

In Sec. II, we introduce and solve the model, obtaining expressions for the equation of state and the correlation function in the critical region. Scaling and universality of the model are verified in Sec. III, where we also give expressions for the scaling functions and the various universal amplitude ratios. All the special cases, with special reference to massless bosons, are discussed in Sec. IV, and our concluding remarks follow in Sec. V.

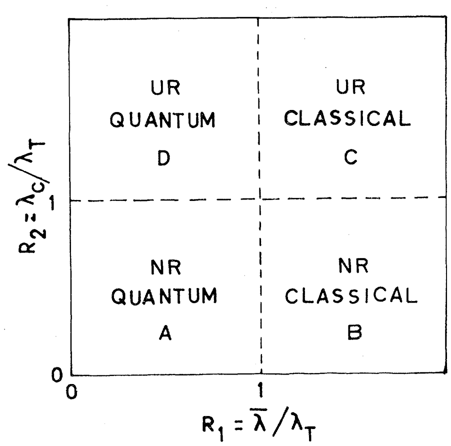

FIG. 1. Regions of interest of the model. Here R_1 is the ratio of interparticle spacing (mean) to the thermal wavelength, and R_2 is the ratio of the Compton wavelength to the thermal wavelength.

II. MODEL AND SOLUTION

We deal with a system of N bosons and $\bar{N}$ antibosons, each of mass m, enclosed in a hypercubical box of volume L^d, where L is the length of an edge in each of the d spatial dimensions. All the interactions are neglected, but pair production is allowed. Because of this, the particles and antiparticles are no longer independent. This implies that instead of the usual conservation of bosons and antibosons separately, we will have a conservation[3] of $Q=N-\bar{N}$. Here Q is any additive, conserved, quantum number, which may be likened to a generalized "charge." The constraint may be satisfied in the mean by going to a grand-canonical ensemble, as usual.[4] Starting from the Hamiltonian,

$$\mathscr{H}=\sum_{i=1}^{N}(\vec{k}_i^2+m^2)^{1/2}+\sum_{i=1}^{\bar{N}}(\vec{k}_i^2+m^2)^{1/2} \tag{2}$$

(we are using units such that $\hbar=c=k_B=1$), it is easy to obtain the partition function.[4] We get

$$\ln Z=-g\sum_{\vec{k}}\ln[1-\exp(\beta\mu-\beta E_k)]$$
$$-g\sum_{\vec{k}}\ln[1-\exp(-\beta\mu-\beta E_k)]\ , \tag{3}$$

where

$$E_k=(k^2+m^2)^{1/2},\quad \beta=T^{-1}\ , \tag{4}$$

and g is the spin degeneracy factor. Because of their structure, one can think of the two terms in (3) as being due to particles and antiparticles, respectively. We now convert sums into integrals using periodic boundary conditions[7] and do the "angular" integrals. Introducing dimensionless variables, one obtains, in the thermodynamic limit,

$$\ln Z=-Va_d^{-1}\int_0^\infty x^{d-1}dx\{\ln 2-a(x)+\ln[\cosh a(x)-\cosh(\alpha-\phi)]\}\ , \tag{5}$$

where

$$a_d=2^{d-1}\pi^{d/2}\Gamma(d/2)m^{-d}g^{-1}\ , \tag{6}$$

$$\alpha=\beta m\ , \tag{7}$$

$$a(x)=\alpha(1+x^2)^{1/2}\ , \tag{8}$$

and

$$\phi=\begin{cases}-\beta(\mu-m), & Q>0 \quad (9)\\ +\beta(\mu+m), & Q<0. \quad (10)\end{cases}$$

We note that the parameter ϕ has been so defined as to reduce to its NR counterpart in that limit.[8] Since $\ln Z=\beta pV$, where p is the pressure of the system, one can get all the thermodynamic quantities from (5). We write down the expressions for the charge density $\rho=Q/V$ and the energy density $u=U/V$ for future reference,

$$\pm\rho=a_d^{-1}W_d(\alpha,\phi),\quad Q\lessgtr 0 \tag{11}$$

$$u=ma_d^{-1}Y_d(\alpha,\phi)\ , \tag{12}$$

where

$$W_d(\alpha,\phi)=\int_0^\infty x^{d-1}dx\frac{\sinh(\alpha-\phi)}{\cosh a(x)-\cosh(\alpha-\phi)}\ , \tag{13}$$

$$Y_d(\alpha,\phi)=\int_0^\infty x^{d-1}dx\frac{\alpha^{-1}a(x)\cosh(\alpha-\phi)-\sinh(\alpha-\phi)-\alpha^{-1}a(x)e^{-a(x)}}{\cosh a(x)-\cosh(\alpha-\phi)}\ . \tag{14}$$

One can now study the critical behavior of the model[9] by looking at the behavior of the integrals in (5), (13), and (14). We prefer to do the integrals in a closed form first by expanding the integrands in infinite series and introducing modified Bessel functions in the resulting integrals.[1,10] We get

$$a_d\beta p=b_d\sum_{r=1}^{\infty}r^{-d'}\cosh(r\alpha-r\phi)K_{d'}(r\alpha)\ , \tag{15}$$

$$W_d(\alpha,\phi)=b_d\sum_{r=1}^{\infty}r^{-(d'-1)}\sinh(r\alpha-r\phi)K_{d'}(r\alpha)\ , \tag{16}$$

$$Y_d(\alpha,\phi)=b_d\sum_{r=1}^{\infty} r^{-(d'-1)}[\cosh(r\alpha-r\phi)K_{d'+1}(r\alpha)-\sinh(r\alpha-r\phi)K_{d'}(r\alpha)-(r\alpha)^{-1}\cosh(r\alpha-r\phi)K_{d'}(r\alpha)] \ , \tag{17}$$

where

$$b_d=\pi^{-1/2}2^{d'}\Gamma(d/2)\bar{\alpha}^{(d'-1)}, \quad d'=\tfrac{1}{2}(d+1) \tag{18}$$

and $K_{d'}(r\alpha)$ is a modified Bessel function of order d' and real argument $r\alpha$.[11] The critical behavior is obtained by studying the variation of μ, or equivalently, ϕ as a function of T. First, we note from (5) that to keep $\ln Z(=\beta bV)$ real, μ must satisfy the condition $-m\le\mu\le m$. The parameter ϕ is therefore defined to be positive [see (9) and (10)]. (We assume $Q>0$ from now on, $Q<0$ being similar to this.) It is easy to see from (13) or (16) that as $T\to\infty$, $\mu\to 0$ or $\phi\to\beta m$. As T is lowered, μ increases until it hits m. The critical temperature T_c is given by (11) for $\phi=0$, i.e.,

$$a_d\rho=W_d(\alpha_c,0) = b_{dc}\sum_{r=1}^{\infty} r^{-(d'-1)}\sinh(r\alpha_c)K_{d'}(r\alpha_c) \ , \tag{19}$$

where b_{dc} is the value of b_d at $\alpha_c=\beta_c m=m/T_c$. Now, by using the large argument form[12] of the function $K_\nu(z)$, viz.,

$$K_\nu(z)\approx\left[\frac{\pi}{2z}\right]^{1/2}e^{-z}\left[1+\frac{4\nu^2-1}{8z}+\cdots\right], \quad z\gg 1 \tag{20}$$

it is seen that the ratio of the successive terms of the series (19) is given by

$$\frac{u_{r+1}}{u_r}=1-\frac{d}{2r}+O\left[\frac{1}{r^2}\right], \quad r\to\infty \ .$$

So, by standard ratio tests,[13] the series converges only if $d>2$. We conclude that a nonzero T_c exists only for $d>2$. It is remarkable that this result is independent of α_c $(=mc^2/k_BT_c$ in conventional units). Of course, to get a value of T_c for $d>2$ one would have to evaluate (19) numerically; see Sec. IV. We shall now work in dimensions $d>2$.

From (11) and (16) we determine the behavior of ϕ in the critical region. For this, we need an expansion of $W_d(\alpha,\phi)$ near $\phi=0$. We saw above that $W_d(\alpha_c,0)$ is finite for $d>2$. Similarly, one can see that $W_d(\alpha,0)$ is also finite. For $\partial W_d(\alpha,\phi)/\partial\phi$ at $\phi=0$, we get

$$\left.\frac{\partial W_d(\alpha,\phi)}{\partial\phi}\right|_{\phi=0}=-b_d\sum_{r=1}^{\infty} r^{-(d'-2)}\cosh(r\alpha)K_{d'}(r\alpha) \ .$$

By using the same techniques as before, we see that this sum converges only for $d>4$. To obtain the manner in which it diverges for $2<d<4$, as $\phi\to 0$, we proceed as follows. We calculate the derivative

$$\frac{\partial W_d(\alpha,\phi)}{\partial\phi}=-b_d\sum_{r=1}^{\infty} r^{-(d'-2)}\cosh(r\alpha-r\phi)K_{d'}(r\alpha) \ .$$

Since the divergence is coming from $\phi\to 0$, the terms $r\gg 1$ are important. We get, asymptotically,

$$\frac{\partial W_d(\alpha,\phi)}{\partial\phi}\approx-\tfrac{1}{2}(2/\alpha)^{d/2}\Gamma(d/2)F_{(d-2)/2}(\phi) \ , \quad \phi\to 0$$

where $F_n(\phi)$ are the standard Bose functions,[14] defined by

$$F_n(\phi)=\sum_{r=1}^{\infty} r^{-n}e^{-r\phi} \ . \tag{21}$$

For $n<1$, these functions diverge[15] near $\phi=0$ as

$$F_n(\phi)\approx\Gamma(1-n)\phi^{n-1}, \quad n<1, \quad \phi\to 0 \ . \tag{22}$$

Using this, we get

$$\frac{\partial W_d(\alpha,\phi)}{\partial\phi}\approx-\tfrac{1}{2}(2/\alpha)^{d/2}\Gamma(d/2)\times\Gamma(2-d/2)\phi^{(d-4)/2} \tag{23}$$

for $2<d<4$. On the other hand, the quantity $\partial W_d(\alpha,0)/\partial\alpha$ is found to be finite and negative at α_c. So we get

$$W_d(\alpha,0)\approx W_d(\alpha_c,0)-(\alpha-\alpha_c)W' \ , \tag{24}$$

where we have written

$$-W'=\left[\frac{dW_d(\alpha,0)}{d\alpha}\right]_{\alpha=\alpha_c} \ . \tag{25}$$

Using (23) and (24) in (11), we get

$$\phi\approx(C^+)^{-1}t^{2/(d-2)}, \quad 2<d<4, \quad t>0$$
$$C^+=\left[\frac{2^{d/2}\Gamma(d/2)\Gamma(2-d/2)}{(d-2)\alpha_c^{(d+2)/2}W'}\right]^{2/(d-2)} , \tag{26}$$

where we have introduced the reduced temperature variable,

$$t=(T-T_c)/T_c \ . \tag{27}$$

The notation C^+ for the amplitude conforms to the usual notation, see (35).

Equation (26) is the key equation of this paper. Comparison of this with Eqs. (23) and (25) of Ref. 5 shows that the exponent $2/(d-2)$ is the same. This is the basic reason why the critical behavior of this model is the same as that of the usual NR gas. The only difference will come in the amplitudes; see Sec. IV.

Turning now to the behavior for $T<T_c$, we note that, as usual,[4]

$$\phi=0, \quad t\leq 0 \ . \tag{28}$$

One could proceed in the standard way[4] by splitting off a term from the sums in (3). That will give us only the order parameter. To get the full equation of state, we proceed as in Ref. 5. Since the details of the analysis are very similar, we quote only the final results. For the order parameter Ψ, we get

$$\Psi(H,T)=H/\phi(H,T) \ , \tag{29}$$

where $H=\nu\beta$ and the ν is the symmetry-breaking field for particles.[16] Equation (11) is modified to

$$\rho-(\Psi)^2=a_d^{-1}W_d(\alpha,\phi\{H,T\}) \ . \tag{30}$$

By using exactly the same methods as before, it is easy to show that in the critical region, one gets

$$H=[D(\alpha_c)]^{-2/(d-2)}\Psi(\Psi^2+B^2t)^{2/(d-2)} \ , \tag{31}$$

where

$$D(\alpha_c)=\left[\frac{2}{\alpha_c}\right]^{d/2}\frac{\Gamma(d/2)\Gamma(2-d/2)}{a_d(d-2)} \tag{32}$$

and

$$B=(\alpha_c W'/a_d)^{1/2} \ . \tag{33}$$

The correlation function for the particles may be defined and calculated as in Ref. 5. We get, in $\vec{k}$ space,

$$G(\vec{k},H,T)=[(\alpha^2+\beta^2k^2)^{1/2}-\alpha+\phi]^{-1} \ , \tag{34}$$

where ϕ is to be determined from (29) and (31). With this we have obtained all the information necessary for studying the scaling and the universality properties of the system. We turn to these questions in Sec. III.

III. SCALING AND UNIVERSALITY

In this section, we obtain the various critical exponents, amplitudes, and the universal scaling functions. We shall follow Tarko and Fisher[17] with regard to notation, especially for the critical amplitudes. For the sake of completeness, we list the standard definitions.

(a) *Critical isochore:* $H=0$, $T>T_c$,

$$\chi_0(T)=\left[\frac{\partial\Psi}{\partial H}\right]_{H=0}\approx C^+t^{-\gamma} \ , \tag{35}$$

$$\frac{\xi_0(T)}{m}=\left[\frac{d\ln G(\vec{k})}{d(k^2/m^2)}\right]_{k=0}\approx f^+t^{-\nu} \ , \tag{36}$$

$$C_0(T)=\left[\frac{\partial u}{\partial T}\right]_\rho\approx E^+t^{-\alpha} \ . \tag{37}$$

(b) *Phase boundary:* $H=0$, $T<T_c$,

$$\chi_0(T)\approx c^-|t|^{-\gamma'} \ , \tag{38}$$

$$\frac{\xi_0(T)}{m}\approx f^-|t|^{-\nu'} \ , \tag{39}$$

$$C_0(T)\approx E^-|t|^{-\alpha'} \ , \tag{40}$$

$$\Psi_0(T)\approx B|t|^{\beta} \ . \tag{41}$$

(c) *Critical isotherm:* $T=T_c$, $H>0$,

$$\chi(H)=\left[\frac{\partial\Psi}{\partial H}\right]_{t=0}\approx C^cH^{-\gamma^c} \ , \tag{42}$$

$$\frac{\xi(H)}{m}\approx f^cH^{-\nu^c} \ , \tag{43}$$

$$C(H)\approx E^cH^{-\alpha^c} \ . \tag{44}$$

(d) *Correlation decay:* $T=T_c$, $H=0$,

$$G(\vec{k},0,T_c)\approx\hat{D}/(k/m)^{2-\eta}, \quad k\to 0 \ . \tag{45}$$

All of these exponents and amplitudes except those of the specific heat can be easily obtained from (31)–(34). The calculation of the specific heat is somewhat lengthy and may be found in the Appendix, where it is discussed in detail. The results for all the exponents and the amplitudes are listed in Table I. The following remarks are in order about this table. Because of the condition (28), the correlations stay long ranged below T_c, and, so, many of the low-temperature exponents are undefined. The remaining exponents, listed here, are universal in that they are independent of α_c for a given d. They are actually the same as those of the NR case, $\alpha_c\gg 1$. The physical reason for this is that, near T_c (high or low), the correlation length diverges, and therefore only the properties of the Hamiltonian near $k=0$ are important. In that limit, the Hamiltonian (2) reduces to that of two species of ideal NR gases. The critical amplitudes, also listed in Table I, clearly depend on α_c.

The equation of state (31) can be easily rewritten in the scaled form[18]

$$H=l_1\Psi^\delta f(l_2t/\Psi^{1/\beta}) \tag{46}$$

with

TABLE I. Critical exponents and the amplitudes defined in (35)—(45). The listed values apply for $2<d<4$. For $d>4$, the usual mean-field values apply for exponents.

Critical exponent	Value	Critical amplitude	Value
γ	$\frac{2}{d-2}$	C^+	Eq. (26)
ν	$\frac{1}{d-2}$	f^+	$(\frac{1}{2}\alpha_c C^+)^{1/2}$
α	$\frac{d-4}{d-2}$	E^+	$\frac{2\alpha_c W'}{(d-2)a_d C^+}$
β	$\frac{1}{2}$	B	$\left[\frac{\alpha_c W'}{a_d}\right]^{1/2}$
γ^c	$\frac{4}{d+2}$	C^c	$[D(\alpha_c)]^{2/(d+2)}$
ν^c	$\frac{2}{d+2}$	f^c	$(\frac{1}{2}\alpha_c)^{1/2}[D(\alpha_c)]^{1/(d+2)}$
α^c	$\frac{2d-8}{d+2}$	E^c	$\frac{\alpha_c B^2 W'}{2a_d[D(\alpha_c)]^{6/(d+2)}}$
η	0	$\hat{D}$	$\frac{2}{\alpha_c}$

$$\delta=(d+2)/(d-2)\,, \tag{47}$$

$$l_1=[D(\alpha_c)]^{-2/(d-2)}\,, \tag{48}$$

$$l_2=B^2\,, \tag{49}$$

and

$$f(x)=(1+x^2)^{2/(d-2)}\,. \tag{50}$$

This scaling form implies the following four relations[18] between the six thermodynamic exponents $(\gamma,\alpha,\beta,\gamma^c,\alpha^c,\delta)$,

$$\gamma^c=(\delta-1)/\delta\,, \tag{51}$$

$$\gamma=\beta(\delta-1)\,, \tag{52}$$

$$2=\alpha+2\beta+\gamma\,, \tag{53}$$

$$\alpha^c=\alpha/\beta\delta\,. \tag{54}$$

From Table I and Eq. (47), we see that all of these are satisfied in the present case. The universality of the scaling function $f(x)$ in (46) implies the universality of the following combinations[6,17]:

$$Q_0=E^+C^+/B^2\,, \tag{55}$$

$$Q_1=\delta C^c/(B^{\delta-1}C^+)^{1/\delta}\,, \tag{56}$$

$$Q_2=[(2-\alpha)E^c/B](C^+/B)^{1+\alpha^c}\,. \tag{57}$$

In our case, the universal values of these ratios are easily seen to be $2/(d-2)$, $(d+2)/(d-2)$, and $d/(2d-4)$, respectively.

To get the correlation function in the scaling form, we write (34) near $k=0$ to get

$$G(\vec{k},H,T)\approx(\phi+\tfrac{1}{2}\alpha_c k^2/m^2)^{-1},\quad k\to 0\,.$$

Using (29) and (46), we can write it as

$$G(\vec{k},H,T)\approx l_3(k/m)^{-(2-\eta)}f_1\left[\frac{l_4 k}{m\Psi^{\nu/\beta}},\frac{l_2 t}{\Psi^{1/\beta}}\right] \tag{58}$$

with

$$l_3=\hat{D}\,, \tag{59}$$

$$l_4=(\alpha_c/2l_1)^{1/2}\,, \tag{60}$$

$$f_1(x,y)=[1+x^{-2}f(y)]^{-1}\,, \tag{61}$$

and the other quantities as defined earlier. This implies the following relations[18] between the exponents,

$$\gamma=(2-\eta)\nu\,,$$

$$\nu^c=\nu/\beta\delta\,,$$

which are easily seen to be satisfied. The additional universal ratios implied[17] by the universality of $f_1(x,y)$ are

$$Q_2=(C^+/C^c)(f^c/f^+)^{2-\eta}\,, \tag{62}$$

$$Q_3=(\hat{D}/C^+)(f^+)^{2-\eta}\,. \tag{63}$$

We find by using Table I that $Q_2=Q_3=1$. For convenience, we list the definitions and the values of all the ratios in Table II. The universal ratios are independent of α_c for a given d, and, naturally, are the same as would be obtained for the NR-UR cases.

TABLE II. Definitions and values of the expected universal ratios. For notation, see Sec. III.

Ratio	Definition	Value
Q_0	E^+C^+/B^2	$\frac{2}{d-2}$
Q_1	$\delta C^c/(B^{\delta-1}C^+)^{1/\delta}$	$\frac{d+2}{d-2}$
Q_2	$(C^+/C^c)(f^c/f^+)^{2-\eta}$	1
Q_3	$(\hat{D}/C^+)(f^+)^{2-\eta}$	1
Q_4	$[(2-\alpha)E^c/B](C^+/B)^{1+\alpha^c}$	$\frac{d}{2d-4}$
X	$E^+(f^+/m)^d$	$\frac{g\Gamma(2-d/2)}{(d-2)^2\pi^{d/2}2^{d-2}}$

In addition to the above three-exponent and three-amplitude scaling and universality, one finds in the literature two-exponent[18] and two-amplitude[19] universality. These imply the relation

$$d\nu=2-\alpha \; , \tag{64}$$

between the exponents, and the universal ratio

$$X=E^+(f^+/m)^d \; , \tag{65}$$

between the amplitudes. It is easily seen that (64) is satisfied, and

$$X=\frac{g\Gamma(2-d/2)}{(d-2)^2\pi^{d/2}2^{d-2}} \tag{66}$$

in our case.

To summarize, we have obtained the universal scaling functions for thermodynamics and correlations and verified scale-factor universality, in general. In Sec. IV we discuss the model in various special cases and recover many familiar results.

IV. SPECIAL CASES

Qualitatively, there are two types of special cases to consider, the quantum and the classical. The general quantum case has been discussed thoroughly in Sec. III. Here we work out its NR and UR limits with special reference to the $m=0$ limit. The NR and the UR limits are obtained from our results for $\alpha_c \gg 1$ and $\alpha_c \ll 1$, respectively. Since the universal quantities do not depend on α_c, only the nonuniversal ones need to be calculated. It can be seen from Sec. III that all such quantities depend on $W_d(\alpha_c,0)$, W', and C^+, so it is sufficient to consider only these.

The quantity $W_d(\alpha_c,0)$ is given by [see (19)],

$$W_d(\alpha_c,0)=b_{dc}\sum_{r=1}^{\infty} r^{-(d'-1)}\sinh(r\alpha_c)K_d(r\alpha_c) \; .$$

For $\alpha_c \gg 1$, one can use (20) to get

$$W_d(\alpha_c,0)=\frac{1}{2}\left[\frac{2}{\alpha_c}\right]^{d/2}\Gamma(d/2)\zeta(d/2) \quad (\mathrm{NR}) \; . \tag{67}$$

For $\alpha_c \ll 1$, we use the low-z expansion[20] of $K_\nu(z)$, viz.,

$$K_\nu(z)\approx\frac{2^{\nu-1}\Gamma(\nu)}{z^\nu}, \quad z\ll 1 \tag{68}$$

to get

$$W_d(\alpha_c,0)=\frac{2}{\alpha_c^{d-1}}\Gamma(d)\zeta(d-1) \quad (\mathrm{UR}) \; , \tag{69}$$

where we have also used the duplication formula[21] for gamma functions, viz.,

$$\Gamma(2z)=2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\tfrac{1}{2}) \; . \tag{70}$$

Exactly similar techniques may be used to show that

$$W'=\frac{1}{2\alpha_c}\left[\frac{2}{\alpha_c}\right]^{d/2}\Gamma\left[\frac{d+2}{2}\right]\zeta\left[\frac{d}{2}\right] \quad (\mathrm{NR}) \; , \tag{71}$$

$$W'=\frac{2}{\alpha_c^d}(d-1)\Gamma(d)\zeta(d-1) \quad (\mathrm{UR}) \; . \tag{72}$$

From (26), we can easily get C^+ in these limits. Using (68), (69), (71), and (72), the values of all the amplitudes in Table I can be determined. We list these for $d=3$ in Table III. These results agree with the known values, wherever results were available before. For example, in the NR case, E^+ gives the correct discontinuity in the specific-heat derivative.[22] For the UR case, all the amplitudes are being presented here for the first time except E^+, which was calculated by Haber and Weldon.[3] It may be mentioned here that in deriving these results, use has been made of the following expressions for the critical temperatures in the two limits:

TABLE III. Critical amplitudes defined in (35)—(45) for $d=3$ in the NR and the UR limit.

Critical amplitude	NR	UR
C^+	$16\pi/9[\zeta(\frac{3}{2})]^2$	$9\alpha_c/8\pi^2$
f^+	$(8\pi\alpha_c)^{1/2}/3\zeta(\frac{3}{2})$	$3\alpha_c/4\pi$
E^+	$27\rho[\zeta(\frac{3}{2})]^2/16\pi$	$32\pi^2\rho/9\alpha_c$
B	$(3\rho/2)^{1/2}$	$(2\rho)^{1/2}$
C^c	$\{4\pi\rho^2/[\zeta(\frac{3}{2})]^2\}^{1/5}$	$(3g^{-1}\rho m^3/2\pi^2\alpha_c)^{1/5}$
f^c	$(\frac{1}{2}\alpha_c)^{1/2}[2\pi^{1/2}\rho/\zeta(\frac{3}{2})]^{1/5}$	$(\frac{1}{2}\alpha_c)^{1/2}(2\pi^2\alpha_c/3g^{-1}\rho m^3)^{-1/10}$
E^c	$\frac{9}{8}\rho^2[\zeta(\frac{3}{2})/2\pi^{1/2}\rho]^{6/5}$	$2\rho^2(2\pi^2\alpha_c/3g^{-1}\rho m^3)^{3/5}$
$\hat{D}$	$2/\alpha_c$	$2/\alpha_c$

$$T_c(\mathrm{NR})=\frac{2\pi}{m}\left[\frac{g^{-1}\rho}{\zeta(d/2)}\right]^{2/d},$$

$$=\frac{2\pi}{m}\left[\frac{g^{-1}\rho}{\zeta(\frac{3}{2})}\right]^{2/3},\quad d=3\ ; \tag{73}$$

$$T_c(\mathrm{UR})=\left[\frac{g^{-1}\rho\pi^{(d+1)/2}}{2m\Gamma\left[\frac{d+1}{2}\right]\zeta(d-1)}\right]^{1/(d-1)},$$

$$=(3g^{-1}\rho/m)^{1/2},\quad d=3\ . \tag{74}$$

Equations (73) and (74) can be easily obtained using (19), (67), and (69). The variation of T_c between these two limits can be obtained by doing the integral for $W_d(\alpha_c,0)$ numerically [see (13)]. In Fig. 2, we show a graph of T_c/m vs ρ/gm^d for various dimensions. The two limits can be clearly seen. They cross into each other near

$$\rho/gm^d=1$$

for all d. The variation of all the amplitudes can be obtained by getting W' in addition to $W_d(\alpha_c,0)$. Since the amplitude E^+, being related to the specific-heat anomaly, is of particular interest, we show the variation of the dimensionless combination E^+/gm^3 vs T_c/m ($d=3$) in Fig. 3. A smooth variation is seen as expected.

Now we obtain the results for the classical regime. This is valid for high temperatures and low densities. Naturally, there is no pair production in this limit, so, in (15)—(17), we write $\frac{1}{2}\exp(r\alpha-r\phi)$ for $\cosh(r\alpha-r\phi)$ and $\sinh(r\alpha-r\phi)$. Also since $\alpha-\phi=\beta\mu$ is going to zero in this limit, we keep only the first term in the sum.[2] This gives us the following expressions for βp, ρ, and u:

$$a_d\beta p=\tfrac{1}{2}b_d e^{\beta\mu}K_{d'}(\alpha)\ , \tag{75}$$

$$a_d\rho=\tfrac{1}{2}b_d e^{\beta\mu}K_{d'}(\alpha)\ , \tag{76}$$

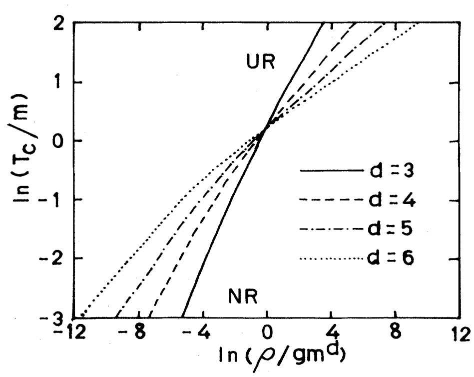

FIG. 2. Variation of T_c/m vs ρ/gm^d for $d=3$—6.

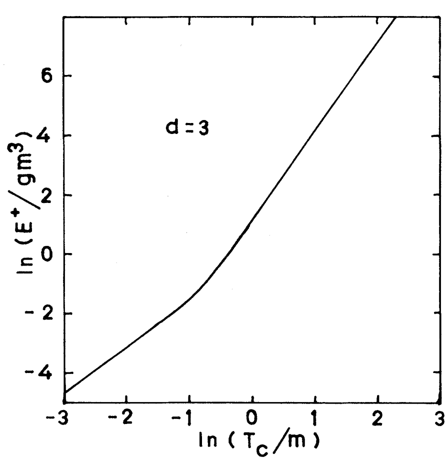

FIG. 3. Variation of the specific-heat amplitude E^+/gm^3 (related to the specific-heat anomaly) vs T_c/m for $d=3$.

$$a_d u/m = \tfrac{1}{2} b_d e^{\beta\mu} \left[K_{d'+1}(\alpha) - K_{d'}(\alpha) - \frac{1}{\alpha} K_{d'}(\alpha) \right] . \tag{77}$$

From (75) and (76), we get the classical equation of state

$$\beta p = \rho , \tag{78}$$

whereas (76) and (77) yield

$$u = m\rho \left[\frac{K_{d'+1}(\alpha)}{K_{d'}(\alpha)} - 1 - \frac{1}{\alpha} \right] . \tag{79}$$

The energy density u is a transcendental function of T, showing that the specific heats are temperature dependent in general. However, using (20) and (68), we find that

$$u = \frac{d}{2} \rho T \quad (\mathrm{NR}) , \tag{80}$$

$$u = d\rho T \quad (\mathrm{UR}) , \tag{81}$$

as expected. Of course, it is clear from (78) and (79) that there is no phase transition in the classical limit.

Finally, we discuss the $m=0$ limit. This may be thought of as the extreme case of the UR limit. So, from (74), we see that

$$T_c(m=0) = \infty, \quad d > 2 .$$

It means that a gas of charged, massless bosons is always in the condensed state[3] for $d > 2$. The pressure and energy can be obtained from (15) and (17) by setting $\alpha = m = \phi = 0$ and using (68). We get

$$p = \frac{2g}{\pi^{(d+1)/2}} \Gamma \left[\frac{d+1}{2} \right] \zeta(d+1) T^{(d+1)} , \tag{82}$$

$$u = dp . \tag{83}$$

These agree with the usual photon results in three dimensions.[4] The charge density is obtained from (30) by noting that

$$[\Psi(0,T)]^2 = \rho_0(T) ,$$

in the usual notation. It is seen that $\rho_0(T) = \rho$, which means that the charge density always resides in the ground state.[3]

For $d=2$, the value of T_c depends on the order of the limits $m \to 0$ and $d \to 2$. To see this, expand (74) in powers of $\epsilon = d-2$ using

$$\zeta(1+\epsilon) \approx \epsilon^{-1}, \quad \epsilon \to 0 .$$

One gets

$$T_c(\mathrm{UR}) \approx \pi g^{-1} \rho\epsilon/m, \quad d \to 2 .$$

So

$$\lim_{m\to 0} \lim_{d\to 2} T_c(\mathrm{UR}) = 0 ,$$

but

$$\lim_{d\to 2} \lim_{m\to 0} T_c(\mathrm{UR}) = \infty .$$

Clearly, if one starts with charged, massless bosons in two dimensions, $T_c(\mathrm{UR}) = \infty$.[3]

With this we have completed the study of the special cases of our general results. It may be noted that we have mostly concentrated on the leading specialized behavior. By carrying out the approximation further, one can get corrections to this behavior. Our procedure will give these corrections in a precise way without any spurious terms such as found in Ref. 23, because we do not make any approximation for the density of states (second paper in Ref. 2). In our case, the approximations are made at the last stage. However, it may be noted that in Ref. 23, the approximation was made only for the mean number of particles and the graph of T_c vs m was based on the exact expressions.[24]

V. CONCLUDING REMARKS

We have solved the ideal, relativistic, charged Bose-gas model exactly including the effects of pair production. We have shown that in the quantum case, universal quantities like the exponents, scaling functions, and amplitude ratios are the same for all values of the parameter $\alpha_c = m/T_c$ and are the same as those of the usual NR Bose gas. Our general analysis gives correct results in all the previously known special cases.

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APPENDIX: THE SPECIFIC HEAT

In this appendix, we study the critical behavior of the specific heat in detail. Although many specific heats may be defined, we shall consider only the specific heat at constant ρ, defined by

$$C_\rho = \left[\frac{\partial u}{\partial T} \right]_\rho , \tag{A1}$$

where the internal energy density is given by (12). This differs from the usual energy density defined by

$$\bar{u} = - \left. \frac{\partial(\beta p)}{\partial \beta} \right|_{\beta\mu} ,$$

by the term $m\rho$ which is temperature independent. So C_ρ obtained from (A1) will be the same as the usual specific heat. Now, since u is an explicit function of ϕ and T, we write

$$\left[\frac{\partial u}{\partial T}\right]_\rho = \left[\frac{\partial u}{\partial T}\right]_\phi + \left[\frac{\partial u}{\partial \phi}\right]_T \left[\frac{\partial \phi}{\partial T}\right]_\rho . \tag{A2}$$

It is clear that the singularities in C_ρ will come from the nonanalytic behavior of ϕ,

$$\phi \approx (C^+)^{-1} t^{2/(d-2)}, \quad 2<d<4, \quad t\to 0+ \tag{A3}$$

$$=0, \quad t \le 0 . \tag{A4}$$

So the second term in (A2) is the singular part. Now, from (12) and (17), we get

$$\left[\frac{\partial u}{\partial \phi}\right]_T = m a_d^{-1} Z_d(\alpha,\phi) ,$$

$$Z_d(\alpha,\phi) = b_d \sum_{r=1}^{\infty} r^{-(d'-2)} [\cosh(r\alpha - r\phi) K_{d'}(r\alpha) - \sinh(r\alpha - r\phi) K_{d'+1}(r\alpha) + (r\alpha)^{-1} \sinh(r\alpha - r\phi) K_{d'}(r\alpha)] . \tag{A5}$$

By the methods used in Sec. II, it can be seen that $Z_d(\alpha,0)$ is convergent for $d>2$. In fact, by calculating $\partial W_d(\alpha,0)/\partial\alpha$ from (16) and comparing it with $Z_d(\alpha,0)$ from (A5), one finds that

$$Z_d(\alpha,0) = \frac{\partial W_d(\alpha,0)}{\partial \alpha} . \tag{A6}$$

Therefore the singular part of the specific heat is given by

$$C_\rho(\text{sing}) = -\frac{mW'}{a_d} \left[\frac{\partial \phi}{\partial T}\right]_\rho .$$

Using (A3) and (A4) in this equation, we get

$$C_\rho(\text{sing}) = \begin{cases} -E^+ t^{(4-d)/(d-2)}, & t>0 \quad \text{(A7)} \\ 0, \quad t<0 & \quad \text{(A8)} \end{cases}$$

where

$$E^+ = \frac{2\alpha_c W'}{(d-2)a_d C^+} .$$

By using the standard definitions of exponents [(37) and (40)], we find that

$$\alpha' = 0, \quad \alpha = (d-4)/(d-2) . \tag{A9}$$

Clearly, the specific heat is continuous across T_c for $2<d<4$. For $3<d<4$, $C'_\rho = (dC_\rho/dT)$ diverges like $t^{(6-2d)/(d-2)}$. For $d=3$, C_ρ goes like t so that there is only a jump in the derivative. The magnitude of the jump is given by

$$\Delta = \left[\frac{dC_\rho}{dT}\right]_+ - \left[\frac{dC_\rho}{dT}\right]_- = -\frac{E^+}{T_c}, \quad d=3 .$$

Using (6), (26), and (A9) for $d=3$, we get

$$\Delta = -(2\pi^4)^{-1} m^2 \alpha c^7 (W')^3 g .$$

In Fig. 3, we have plotted E^+/gm^3 $(=-T_c\Delta/gm^3)$ as a function of T_c/m. It is seen to interpolate between the limiting values,[3,4]

$$-\frac{T_c\Delta}{gm^3} = \frac{27}{16\pi g}[\zeta(\tfrac{3}{2})]^2 \left[\frac{\rho}{m^3}\right] \quad (\text{NR})$$

and

$$-\frac{T_c\Delta}{gm^3} = \frac{32\pi^2}{9\alpha_c g}\left[\frac{\rho}{m^3}\right] \quad (\text{UR}) .$$

The behavior of C_ρ for $2<d<3$ is more complex. As one goes to lower dimensionalities, one has to differentiate C_ρ a greater number of times to make it diverge. In general, all the $(n-1)$th derivatives of C_ρ are finite, but the nth derivative diverges if the condition $(6+2n)/(2+n)<d<3$, $n=1,2,3,\ldots$ is satisfied. Of course, the exponent α is still given by (A10).[18]

Finally, we mention that if pair production is neglected in the UR case, as is done in Ref. 1, for example, C_ρ itself would be found to be discontinuous for $d=3$ in that limit.

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