A New Singularity Theorem in Relativistic Cosmology

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It is shown that if the timelike eigenvector of the Ricci tensor be hypersurface orthogonal so that the space time allows a foliation into space sections then the space average of each of the scalar that appear in the Raychaudhuri equation vanishes provided the strong energy condition holds good. This result is presented in the form of a singularity theorem.

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Quite a number of theorems on the singularity in cosmological solutions exist in the literature. These are concerned both with the definition of singularity as well as the condition leading to its occurrence. The intuitive definition of singularity is an unacceptable behaviour of physical variables like their blowing up or abrupt discontinuity involving some breakdown of conservation principles. Of course such peculiarities will be reflected in the geometry of space time. However it has been argued that such “singularities” may be removed out of sight by introducing coordinate systems whose domain do not include the ”singular regions”. To take care of such situations and also for mathematical convenience a definition of singularity has emerged which identifies singular space times as those in which some null or time like geodesic is incomplete. Without making a critical discussion on this definition, we reproduce a formulation of Hawking and Penrose as a standard singularity theorem. It states [1]

Space time is not timelike and null geodetically complete if (1) $R_{\alpha\beta}k^{\alpha}k^{\beta} \geq 0$ for every non space like vector k^{α}; $R_{\alpha\beta}$ being the Ricci tensor (2) Every non space-like geodesic contains a point at which $k_{[\alpha}R_{\beta]\delta\gamma[\rho}K_{\mu]}K^{\gamma}k^{\delta} \neq 0$ where k^{α} is the tangent vector to the geodesic (3) There are no closed timelike curves (4) There exists at least one closed trapped surface. With the field equations of general relativity, the first condition reduces to the strong energy condition (along with an attractive gravitation). Although there exists situations like the false vacuum where the strong energy condition is violated, we shall retain condition (1) in our discussion. Any violation of condition (3) would mean a breakdown of causality. Thus the conditions (1) and (3) may be considered to be fundamental elements of standard physics. Not so however are the other two. Indeed it seems difficult to reconcile the presence of the rather awkward and complicated condition (2) in the statement of a fundamental theorem. Regarding condition(4) we may recall that we usually believe that any realistic model of the universe should develop a variety of structures at least some of which would eventually undergo gravitational collapse leading to the formation of trapped surfaces. To eliminate trapped surfaces from our consideration is to effectively restrict to structureless universes, unless of course our understanding of stellar evolution and gravitational collapse is basically wrong.

In this background came the solution of Senovilla [2]. The solution is free of any curvature or physical singularity and as shown somewhat later by Chinea et al [3] is also geodetically complete. Of the four conditions in the Hawking Penrose theorem, only the condition regarding the trapped surfaces did not hold good in Senovilla's solution. It thus raised the intriguing question of a more useful singularity theorem which will spell out the positive characteristic properties (physical and/or mathematical) of nonsingular solutions. An attempt in this direction was by the present author [4] where it was shown that for any nonsingular spatially open non-rotating universe, the space time averages of each of the scalars that appear in Raychaudhuri equation must vanish.

Somewhat later this work was criticised by Saa and Senovilla [5] on the ground that for spatially open Friedmann universes with a big bang these scalars have zero space time averages. This did not contradict the theorem itself but merely indicated that the converse is not true. However even this criticism can be easily met by demanding that the space time average for both the halves of space time- one containing future infinity and the other past infinity-must separately vanish. Senovilla [5] further made a conjecture that for a still further restricted class of singularity free cosmological solution, the spatial average of the energy density shall vanish. However the arguments that he advanced leading to the conjecture were fallacious. In the present paper we consider that the universe is non-rotating in the sense that the timelike eigenvector of the Ricci tensor is hypersurface orthogonal and give a proof that the spatial averages of each of the Raychaudhuri scalars indeed vanish for singularity free solution. Of course for perfect fluids, the timelike eigenvector of the Ricci tensor coincides with the velocity vector of the fluid which will be non-rotating because of our assumed condition. For general imperfect fluids however, there maybe rotation of the matter present even though the eigenvector of the Ricci tensor is hypersurface orthogonal. Our present assumption is thus somewhat weaker than in [4].

We shall use this hypersurface orthogonal unit timelike eigenvector v^α to set up the Raychaudhuri equation which now reads,

$$\frac{1}{3}\theta^2 + 2\sigma^2 + \kappa(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)v^\alpha v^\beta = -\dot{v}^\alpha_{;\alpha} - \dot{\theta} \quad (1)$$

The metric with the time coordinate along this vector will have the form

$$ds^2 = g_{00}dt^2 + g_{ik}dx^i dx^k \quad (2)$$

The scalars $\theta, \sigma, \dot{v}^\alpha_{;\alpha}, \dot{\theta}$ are built up from v^α and its covariant derivatives. Thus with our choice of v^α, these scalars will be algebraic combination of scalars formed from the Ricci tensor and its covariant derivatives. Hence a blow up of any of these scalars would indicate a blow up of some Ricci scalars and hence signal the presence of a singularity.

We now enunciate and prove the following theorem:

The space time will be singular in the sense that some scalar built from the Ricci tensor will blow up if
(a) the strong energy condition is satisfied.
(b) the timelike eigenvector of the Ricci tensor is hypersurface orthogonal. (We are excluding the case of null Ricci tensor.)
(c) the space average of any of the scalars occurring in Raychaudhuri equation does not vanish.

Here the condition (b) allows a foliation of the space time into space sections and the averages referred to in (c) are defined as follows. Space average of any scalar χ is

$$< \chi >_s \equiv [\frac{\int \chi \sqrt{^3|g|} d^3x}{\int \sqrt{^3|g|} d^3x}]_{\text{limit over entire space}} \quad (3)$$

$< \chi >_s$ is thus invariant for all transformations involving the space coordinate x^i only.

We can orient the coordinates such that $\dot{v}^\alpha$ has only one nonvanishing component say along the coordinate x^1. As $\dot{v}^\alpha v_\alpha = 0, x^1$ is a spacelike coordinate. Again since with (2),

$$\dot{v}_i = \frac{1}{2}[ln(g_{00})]_{,i}, \dot{v}_0 = 0 \quad (4)$$

the three space vector $\dot{v}_i$ is a gradient vector and hence hypersurface orthogonal. Hence with the above stipulation

$$ds^2 = g_{00}dx^{0^2} + g_{11}dx^{1^2} + g_{ab}dx^a dx^b \quad (5)$$

where a, b run over the indices 2 and 3. One may wonder whether the metric forms (2) and (5) are globally valid with a single coordinate system. However, we note that all known regular solutions with non-vanishing $T_{\mu\nu}$ admit a single coordinate system if there be no discontinuity in $T_{\mu\nu}$. Such discontinuities, although not inconsistent with the condition of regularity, seems unappealing in a cosmological model and in our discussion we shall assume that the forms (2) and (5) are valid over entire space time with a single coordinate system. Obviously g_{00} is a function of x^1 and maybe x^0 and g_{11}, g_{ab} may be functions of all the four coordinates. As the tangent vector to x^1 coordinate line is a gradient, x^1 lines cannot form a closed loop. They may either run from $-\infty$ to $+\infty$ or in case they diverge from a point (as the radial lines in case of spherical or axial symmetry) they may run from zero to infinity. In any case if $\int^\infty \sqrt{|g_{11}|}dx^1$ converges to a finite value (i.e., $g_{11} \to 0$ as $x^1 \to +\infty$), then if there be no singularity at infinity one can see by a transformation that the x^1 lines are closed. (cf. the closed Friedmann universe in which $\int_0^\infty \frac{dr}{(1+r^2/4)}$ converges and one can transform r to an angular coordinate χ with domain 0 to 2π.) Again this would make the gradient vector vanish everywhere. We have thus a nontrivial $\dot{v}^\alpha$ only if $\int^\infty \sqrt{|g_{11}|}dx^1$ diverges.

Again, the scalar $\dot{v}^\alpha_{;\alpha}$ must vanish or oscillate about a mean vanishing value as $x^1 \to \pm\infty$ as otherwise the norm of $\dot{v}^\alpha$ would blow up - this is apparent when one recalls that in absence of singularity, the covariant divergence reduces to the ordinary divergence in a locally Lorentzian coordinate system. Now the space average of $\dot{v}^\alpha_{;\alpha}$ is

$$< \dot{v}^\alpha_{;\alpha} >_s \equiv [\frac{\int \dot{v}^\alpha_{;\alpha} \sqrt{^3|g|} d^3x}{\int \sqrt{^3|g|} d^3x}]_{\text{limit over entire space}} \quad (6)$$

If $\sqrt{^3|g|}$ diverges or remains finite at infinity, the denominator integral diverges and the vanishing of $\dot{v}^\alpha_{;\alpha}$ (or its mean value) at infinity will make the divergence of the numerator integral weaker. Consequently in the limit $< \dot{v}^\alpha_{;\alpha} >_s$ would vanish. In case $\sqrt{^3|g|}$ vanishes at infinity, this will be due to the vanishing of the two dimensional determinant $|g_{ab}|$ as we have seen that for nontrivial $\dot{v}^\alpha$, g_{11} cannot vanish. Thus, in this case, as this factor is common to both the denominator and the numerator integrals, the vanishing of $\dot{v}^\alpha_{;\alpha}$ at infinity again ensures $< \dot{v}^\alpha_{;\alpha} >_s = 0$

Note that in eq (1), all the terms on the left are positive definite as we are assuming the strong energy condition. Hence with $< \dot{v}^\alpha_{;\alpha} >_s = 0$, it follows,

$$- < \dot{\theta} >_s \geq \frac{1}{3} < \theta^2 >_s \quad (7)$$

It may happen that $\dot{\theta}$ and θ^2 both vanish at spatial infinity such that the relation (7) is an equality with both sides vanishing. That will lead to the result that space average of all the scalars in (1) vanish and thus prove our theorem. If that is not so, then either at every point

$$- \dot{\theta} \geq \frac{1}{3}\theta^2 \quad (8)$$

which will lead to a blow up of θ in the finite past or future or that in some regions of each space section

$$-\dot{\theta} > \frac{1}{3}\theta^2 \tag{9}$$

Integrating over the x^0 lines one finds a blowing up of θ in the finite past or future. As already noted, θ is a scalar formed from the Ricci tensor components, and so it cannot blow up in a nonsingular solution. Hence we conclude that (7) must be an equality with both sides vanishing and thus our theorem is proved.

In particular we note that if the space is closed so that the total spatial volume is finite, the theorem implies that the positive definite scalars in (1) will vanish everywhere or in other words there is no nontrivial singularity free solutions in case of closed space sections.

It may be worthwhile to make a comparison of the present theorem with that of Hawking and Penrose. As we have already remarked the Hawking-Penrose theorem is of little relevance so far as realistic models of the universe are concerned as closed trapped surfaces seem inevitable. On the other hand the present theorem depends on the consideration of infinite space integrals and hence it may overlook localized singularities which do not affect the infinite integrals. Such singularities are apparently taken care of by the trapped surface condition in Hawking Penrose theorem.

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