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# REPRESENTATION THEORETIC HARMONIC SPINORS FOR COHERENT FAMILIES

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**Abstract** Coherent continuation $\pi_2$ of a representation $\pi_1$ of a semisimple Lie algebra arises by tensoring $\pi_1$ with a finite dimensional representation F and projecting it to the eigenspace of a particular infinitesimal character. Some relations exist between the spaces of harmonic spinors (involving Kostant's cubic Dirac operator and the usual Dirac operator) with coefficients in the three modules. For the usual Dirac operator we illustrate with the example of cohomological representations by using their construction as generalized Enright-Varadarajan modules. In [9] we considered only discrete series, which arises as generalized Enright-Varadarajan modules in the particular case when the parabolic subalgebra is a Borel subalgebra.

**Key words** Semisimple lie group, Enright-Varadarajan module, Dirac cohomology, Zuckerman translation functor, Coherent family, harmonic spinor.

## 1. Introduction

These notes form an expanded version of the second author's talk in a Conference on *'Representations of Lie Groups and Applications'* held during December 15-18th, 2008 at the Institut Henri Poincaré, Paris. It is a report with a few additions of our joint work [9].

Let $G$ be a connected non-compact semisimple real Lie group with finite center and $K$ a maximal compact subgroup. We assume $G$ to be linear and also assume, although it is not essential, that both $G$ and $K$ have the same complex rank. Denote the Lie algebras by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively. We write

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \quad \mathfrak{p}_0 = \mathfrak{k}_0^{\perp}$$

for the corresponding orthogonal decomposition (Cartan decomposition) of $\mathfrak{g}_0$ with respect to the Killing form. Let $\theta$ denote the Cartan involution. Note that the killing form is positive definite on $\mathfrak{p}_0$. Complexifications of $\mathfrak{g}_0, \mathfrak{k}_0$ are denoted by $\mathfrak{g}, \mathfrak{k}$, respectively.We may define the Clifford algebra of $\mathfrak{p}$ and the corresponding spin representation $S$ of $\mathfrak{k}$.

Given a Harish Chandra module $(\pi, \mathcal{H})$ of $(\mathfrak{g}, K)$, there is a Dirac operator $\mathcal{D}_\pi : \mathcal{H} \otimes S \to \mathcal{H} \otimes S$ defined by

$$\mathcal{D}_\pi = \sum_j \pi(X_j) \otimes \gamma(X_j)$$

where $\{X_j\}$ is an orthonormal basis of $\mathfrak{p}_0$ and $\gamma$ is the Clifford multiplication on $S$.

J.-S. Huang and P. Pandžić [5] proved a remarkable result which marked a revival of interest in the role of the Dirac operator in representation theory. Focus on its role was a driving force starting from the 70's - *eg.* [1, 12, 15, 8, 16, 10, 11] to mention a few - not to mention numerous authors who used Dirac inequality to treat questions of unitarizability, notably, [17, 18, 19, 20, 21]. This result of Huang and Pandžić was a proof of a conjecture of Vogan, relating the infinitesimal character of an irreducible Harish Chandra module $\pi$ and an irreducible $\mathfrak{k}$-type occurring in the kernel of the formal Dirac operator $\mathcal{D}_\pi$, more precisely, not the kernel but $\mathrm{Ker}(\mathcal{D})/(\mathrm{Ker}(\mathcal{D}) \cap \mathrm{Im}(\mathcal{D}))$, the kernel modulo its intersection with the image of $\mathcal{D}$ which is called Dirac cohomology. Kostant who had introduced the cubic Dirac operator ([4, 6]) for a wide range of homogeneous spaces $G/H$ more general than the Riemannian symmetric space $G/K$, realized the potential of the Huang and Pandžić result and obtained an analogous result ([7]) for the cubic Dirac operator.

In [9] we were motivated to view in a coherent way how the spaces of (representation theoretic) harmonic spinors vary when the representation parameters change in a coherent way.

We illustrate in the following context: let $F(\nu)$ be the finite dimensional representation of $\mathfrak{g}$ with highest weight $\nu$, with respect to some positive system. Let $\{\pi(\mu)\}_{\mu\in\Lambda}$ be a coherent family of (virtual) representations of $G$ (see [2]). Typically in a positive cone contained in the parameter space $\Lambda$, this arises via the Zuckerman translation functor

$$F(\nu)^\star \otimes \pi(\mu+\nu) \longrightarrow \pi(\mu)$$

of tensoring with an irreducible finite-dimensional module and projecting to the central eigenspace corresponding to a shifted infinitesimal character. Here $F(\nu)^\star$ denotes the contragredient representation of $F(\nu)$.

Denote by $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_3$ the kernels of the Dirac operators associated with $\pi_{\mu+\nu}, F(\nu)^\star$ and $\pi_\mu$ respectively.

The problem is to understand how $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_3$ are related. For the case of discrete series representations we illustrated this in [9, (Theorem 4.2)]. More precisely, when $G$ has a compact Cartan subgroup, it is known that discrete series representations of $G$ arise as a particular case of Enright-Varadarajan $(\mathfrak{g}, K)$-

modules ([3, 14]), where $K$ still denotes a maximal compact subgroup of maximal rank of $G$. Now assume that $(\pi(\mu), \mathcal{H}(\mu))$ and $(\pi(\mu+\nu), \mathcal{H}(\mu+\nu))$ are two discrete series representations of $G$, regarded as $(\mathfrak{g}, K)$-modules, where $\mu$ is dominant integral regular and $\nu$ is dominant integral with respect to some positive system in $\mathfrak{g}$. Standard inclusions of Verma modules for $\mathfrak{g}$ give rise to an inclusion $\mathcal{H}(\mu) \hookrightarrow \mathcal{H}(\mu+\nu) \otimes F(\nu)$ which in turn gives rise to an includion $(\varphi : \mathcal{H}(\mu) \otimes S \hookrightarrow \mathcal{H}(\mu+\nu) \otimes F(\nu) \otimes S$. This commutes with the $K$-action on both sides. Moreover, we have a $K$-homomorphism $\beta : (\mathcal{H}(\mu+\nu) \otimes S) \otimes (F(\nu) \otimes S) \otimes S^{\star} \longrightarrow \mathcal{H}(\mu+\nu) \otimes F(\nu) \otimes S)$, by contracting the second factor $S$ and the fifth factor $S^{\star}$. The statement of [9, Theorem 4.2] is:

**Theorem.** $\varphi(\mathrm{Ker}(\mathcal{D}_{\pi(\mu)})) \subseteq \beta(\mathrm{Ker}(\mathcal{D}_{\pi(\mu+\nu)}) \otimes \mathrm{Ker}(\mathcal{D}_{F(\nu)}) \otimes S^{\star})$.

In other words, one can relate Dirac spinors for an irreducible representation, a finite dimensional irreducible representation and a third representation which is related to the first two via a Zuckerman translation. We have used the Enright-Varadarajan construction in the proof of this result for a description of the discrete series representations. In this article, we extend those results to the case of generalized Enright-Varadarajan modules [13] associated to arbitrary $\theta$-stable parabolic subalgebras. We have also described the Dirac cohomology of these modules including the non-unitary ones.

## 2. $\theta$-Stable Parabolic Subalgebra $\mathfrak{q}$ and Generalized Enright-Varadarajan Modules

Recall that we assume that $G$ has finite center and has a compact Cartan subgroup. As we have already mentioned, discrete series representations arise as a particular case of Enright-Varadarajan modules ([3, 14]). Recall that the Harish-Chandra parameter of a discrete class representation is non-singular and integral (when the group in question is the real form of a complex semi-simple Lie group). More generally, many of the cohomologically induced representations $A_{\mathfrak{q}}(\lambda)$ arise as generalized Enright-Varadarajan modules associated to a $\theta$-stable parabolic subalgebra [13]; the exact intersection of the two series of modules has not been written and it would be interesting to describe the intersections. Both the constructions describe the action of the centralizer of $\mathfrak{k}$ in the enveloping algebra of $\mathfrak{g}$ on a certain canonical $\mathfrak{k}$-type, which ought to facilitate this identification in view of a very useful powerful technical result of Harish-Chandra. Among the cohomologically induced representations $A_{\mathfrak{q}}(\lambda)$ the ones which have $(\mathfrak{g}, \mathfrak{k})$-cohomology when tensored with a finite dimensional $\mathfrak{g}$-module have non-singular integral infinitesimal character. We now describe briefly the construction of generalized Enright-Varadarajan modules which have a non-singular integral infinitesimal character. Let $\mathfrak{q}$ be a $\theta$-stable parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{r} \subset \mathfrak{q}$ be a $\theta$-stable Borel subalgebra of $\mathfrak{g}$. Fix a $\theta$-stable Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ such that $\mathfrak{c} \subset \mathfrak{r}$ and $\mathfrak{b} := \mathfrak{c} \cap \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$. (*Our equal rank assumption in fact implies that* $\mathfrak{b} = \mathfrak{c}$). Let $P$ be the corresponding ($\theta$-stable) system of positive roots. Define a Borel subalgebra $\mathfrak{r}_{\mathfrak{k}}$ of $\mathfrak{k}$ by $\mathfrak{r}_{\mathfrak{k}} = \mathfrak{r} \cap \mathfrak{k}$. The corresponding positive system of roots of $\mathfrak{k}$ with respect to $\mathfrak{b}$

is denoted $P_{\mathfrak{k}}$. We shall denote by $\delta$ and $\delta_{\mathfrak{k}}$ the half-sums of $P$ and $P_{\mathfrak{k}}$ respectively. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a Levi decomposition of $\mathfrak{q}$ so that $\mathfrak{c} \subset \mathfrak{l}$. We can write $P$ as a disjoint union $P = P_{\mathfrak{l}} \cup P_{\mathfrak{u}}$, so that $\mathfrak{u}$ is the sum of the root spaces corresponding to roots in $P_{\mathfrak{u}}$. Let $\mu$ be a non-singular integral weight which is dominant with respect to $P$. Let $\sigma$ be the unique element of the Weyl group $W_{\mathfrak{g}}$ of $\mathfrak{g}$ such that $\sigma(P) = P_{\mathfrak{l}} \cup -P_{\mathfrak{u}}$. Let $\sigma_{\mathfrak{k}}$ be the unique element of the Weyl group $W_{\mathfrak{k}}$ of $\mathfrak{k}$ such that $\sigma_{\mathfrak{k}}(P_{\mathfrak{k}}) = P_{\mathfrak{l}\cap\mathfrak{k}} \cup -P_{\mathfrak{u}\cap\mathfrak{k}}$. Consider the Verma module $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ for $\mathfrak{g}$ with highest weight $\sigma(\mu) - \delta$ with respect to $P$ and the Verma module $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma(\mu)-\delta}$ for $\mathfrak{k}$ with $P_{\mathfrak{k}}$-highest weight $\sigma(\mu) - \delta$. Evidently, $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma(\mu)-\delta}$ can be canonically identified with the $\mathcal{U}(\mathfrak{k})$-module generated by the highest weight vector of $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$. There is a unique $P_{\mathfrak{k}}$-dominant integral weight $\eta$ such that $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma(\mu)-\delta} \subseteq V_{\mathfrak{k},P_{\mathfrak{k}},\eta}$. One has $\sigma_{\mathfrak{k}}(\eta + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}} = \sigma(\mu) - \delta$. The $\mathfrak{k}$-module $V_{\mathfrak{k},P_{\mathfrak{k}},\eta}$ and the $\mathfrak{g}$-module $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ can both be simultaneously imbedded in a $\mathfrak{g}$-module $W_{\mathfrak{q},\mu}$, compatible with the imbedding $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma(\mu)-\delta} \subseteq V_{\mathfrak{k},P_{\mathfrak{k}},\eta}$ and having nice properties. Some of the important properties of the inclusions $V_{\mathfrak{g},P,\sigma(\mu)-\delta} \subseteq W_{\mathfrak{q},\mu}$ and $V_{\mathfrak{k},P_{\mathfrak{k}},\eta} \subseteq W_{\mathfrak{q},\mu}$ are the following (see [13]):

(i) $W_{\mathfrak{q},\mu}$ has a unique irreducible quotient $\mathfrak{g}$-module $D_{\mathfrak{q},\mu}$ which is $\mathfrak{k}$-finite,

(ii) the irreducible finite dimensional $\mathfrak{k}$-module $F_{\mathfrak{k},\eta}$ with $P_{\mathfrak{k}}$- highest weight $\eta$ occurs with multiplicity one in $D_{\mathfrak{q},\mu}$,

(iii) if $\chi_{P,-\mu}$ denotes the algebra homomorphism from $U(\mathfrak{g})^{\mathfrak{k}}$ into C defining the scalar by which $u \in \mathcal{U}(\mathfrak{g})^{\mathfrak{k}}$ acts on the highest weight vector of $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$, then the same homomorphism gives the action of $\mathcal{U}(\mathfrak{g})^{\mathfrak{k}}$ on $F_{\mathfrak{k},\eta} \subseteq D_{\mathfrak{q},\mu}$.

Let $\ell(\cdot)$ denote the length function in $W_{\mathfrak{k}}$ as a minimal product of simple reflections. Choose a reduced expression $\sigma_{\mathfrak{k}} = s_1 \cdot s_2 \cdots s_m$. There is a chain of imbeddings of $\mathfrak{k}$-Verma modules $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{m+1}$ which can be fit inside the imbedding $V_1 = V_{\mathfrak{k},P_{\mathfrak{k}},\sigma_{\mathfrak{k}}(\eta+\delta_{\mathfrak{k}})-\delta_{\mathfrak{k}}} \subseteq V_{\mathfrak{k},P_{\mathfrak{k}},\eta} = V_{m+1}$.

(iv) The above mentioned $\mathfrak{g}$-module imbedding $V_{\mathfrak{g},P,\sigma(\mu)-\delta} \subseteq W_{\mathfrak{q},\mu}$ can be spread to a chain of $\mathfrak{g}$-module imbeddings $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ enlarging the chain of $\mathfrak{k}$-module imbeddings $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{m+1}$. In other words, $W_1 = V_{\mathfrak{g},P,\sigma(\mu)-\delta}, W_{m+1} = W_{\mathfrak{q},\mu}$ and we have $V_i \subseteq W_i, \forall i$.

We denote by $W_0$ the maximal proper $\mathfrak{g}$-submodule of $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$. Since $\sigma(\mu)$ is non-singular, integral and dominant with respect to $P_{\mathfrak{l}} \cup -P_{\mathfrak{u}}$ it is known that $W_0 = \sum_\gamma V_{\mathfrak{g},P,s_\gamma\sigma(\mu)-\delta}$. (see [24]). The sum is over the simple roots of $P_{\mathfrak{l}}$.

We can now state some additional properties of this chain of $\mathfrak{g}$-modules constructed in [13]. Let $\phi_i$ be the simple root of $P_{\mathfrak{k}}$ defined so that $s_i = s_{\phi_i}, i = 1, 2, \cdots, m$. Let $\mathfrak{m}_i$ be the 3-dimensional simple Lie algebra generated by the root vectors of $\mathfrak{k}$ corresponding to $\phi_i, i = 1, 2, \cdots, m$.

(v) Each $W_i$ is $\mathcal{U}(\mathfrak{n}_{\mathfrak{k}})$-free; here $\mathfrak{n}_{\mathfrak{k}}$ is the sum of all rootspaces of $\mathfrak{k}$ corresponding to roots in $P_{\mathfrak{k}}$.

(vi) $W_{i+1}/W_i$ is $\mathfrak{m}_i$-finite.

In [22, 23] Enright introduced a notion called the 'completion functor'; this takes place purely at the level of the three dimensional simple Lie algebra $sl_2$. Enright devised the completion functor as a natural extension of the idea of building $W_{i+1}$ from $W_i$ and to refine the construction of the chain in [3] to make it more transparent. He showed how the functorial properties of the completion functor imply that $W_{i+1}$ which is constructed disregarding the $\mathfrak{g}$-structure of $W_i$ but viewing it as nothing more than an $\mathfrak{m}_i$-module does indeed possess a canonical $\mathfrak{g}$-module structure extending the $\mathfrak{g}$-module $W_i$ and compatible wih the $\mathfrak{m}_i$-module structure of $W_{i+1}$. The following property of the inclusion $W_i \subseteq W_{i+1}$ is built in the definition of the completion functor:

**'Completion Property'**. Identify $\mathfrak{m}_i$ with $sl_2$. Consider the $sl_2$-Verma module inclusion $V_{-n-2} \subseteq V_n$. Then any $sl_2$-homomorphism $V_{-n-2} \longrightarrow W_i$ can be extended uniquely to $V_n \longrightarrow W_{i+1}$.

## 3. Statement of the Result

When $G$ has a compact Cartan subgroup, which we assume to hold, in the special case where $\mathfrak{q} = \mathfrak{r}$, (i.e., the parabolic subalgebra is a Borel subalgebra) it is a result due to Wallach ([14]) that the $(\mathfrak{g}, K)$-module $D_{\mathfrak{r},\mu}$ is isomorphic to the space of $\mathfrak{k}$-finite vectors in a discrete class representation of $G$ when $\mu$ is dominant regular with respect to the positive system whose root spaces together with the Cartan subalgebra span $\mathfrak{r}$. Using analogous arguments as in Wallach [14] and noting that a description of the action of the centralizer of $\mathfrak{k}$ in the enveloping algebra of $\mathfrak{g}$ on a canonical $\mathfrak{k}$-type is available for the construction of the modules $D_{\mathfrak{q},\mu}$ as well as $A_{\mathfrak{q}}(\lambda)$, a similar identification can be shown to exist between the more general $D_{\mathfrak{q},\mu}$ (for $\mu$ non-singular integral) associated to a $\theta$-stable parabolic subalgebra $\mathfrak{q}$ and the cohomologically induced representations $A_{\mathfrak{q}'}(\lambda)$ (having $(\mathfrak{g}, \mathfrak{k})$-cohomology when tensored with a suitable finite-dimensional module) associated to a related $\theta$-stable parabolic subalgebra $\mathfrak{q}'$. The sole requirement that we should get the same canonical $\mathfrak{k}$-type from the two constructions tells us how $\mathfrak{q}$ and $\mathfrak{q}'$ should be related and simultaneously also how $\mu$ and $\lambda$ are related.

**(3.1).** Let $\nu$ be $P$-dominant integral and $F(\nu)$ the finite dimensional irreducible representation for $\mathfrak{g}$ with highest weight $\nu$. So $\mu + \nu$ is $P$-dominant and regular and we have the irreducible $(\mathfrak{g}, K)$-modules $D_{\mathfrak{q},\mu}$ and $D_{\mathfrak{q},\mu+\nu}$ as above. We have a canonical $(\mathfrak{g}, K)$-module inclusion $\varphi : D_{\mathfrak{q},\mu} \hookrightarrow D_{\mathfrak{q},\mu+\nu} \otimes F(\nu)^\star$ which is a consequence of the inclusion of Verma modules for $\mathfrak{g}$: $V_{\mathfrak{g},P,\sigma(\mu)-\delta} \hookrightarrow V_{\mathfrak{g},P,\sigma(\mu+\nu)-\delta} \otimes F(\nu)^\star$. In turn this gives rise to a $K$-module inclusion $\varphi_S : D_{\mathfrak{q},\mu} \otimes S \hookrightarrow D_{\mathfrak{q},\mu+\nu} \otimes F(\nu)^\star \otimes S$. Moreover, we have a map $\beta : (D_{\mathfrak{q},\mu+\nu} \otimes S) \otimes (F(\nu)^\star \otimes S) \otimes S^\star \to D_{\mathfrak{q},\mu+\nu} \otimes F(\nu)^\star \otimes S$ by contracting the second factor $S$ and the fifth factor $S^\star$.

Now, we can state our result. Denote by $\mathcal{W}_1^o, \mathcal{W}_2$ and $\mathcal{W}_3^o$ the kernels of the Dirac operators associated with $D_{\mathfrak{q},\mu+\nu}, F(\nu)^\star$ and $D_{\mathfrak{q},\mu}$ respectively. Denote by $\mathcal{W}_1$, (resp.$\mathcal{W}_3$) the subspace of $\mathcal{W}_1^o$, (resp.$\mathcal{W}_3^o$), spanned by $\mathfrak{k}$-types of $P_{\mathfrak{k}}$-highest $\xi$ such that $\xi + \delta_{\mathfrak{k}}$ is $W_{\mathfrak{g}}$-conjugate to $\mu + \nu$, (resp.$\mu$).

**(¶) Remark** : For unitary representations $\mathcal{W}^o = \mathcal{W}$. Folklore has it that probably even for the non-unitary ones in the coherent continuation considered here, this equality may have been noted somewhere in the literature; we have not come across this explictly.

Our method gives a complete description (see Prop. 3.12 and remarks preceding it) of $\mathcal{W}_3$ (hence, also of $\mathcal{W}_1$ by replacing $\mu$ by $\mu+\nu$) without restricting to the unitary case, which is used in the proof of the following result.

**Theorem 1.** *We have:* $\varphi_S(\mathcal{W}_3) \subseteq \beta(\mathcal{W}_1 \otimes \mathcal{W}_2 \otimes S^\star)$.

*Proof* : The method involves relating $\mathcal{W}_1$ and $\mathcal{W}_3$ to the kernel of the Dirac operator acting on $V_{\mathfrak{g},P,\sigma(\mu+\nu)-\delta} \otimes S$ and $V_{\mathfrak{g},P,\sigma(\mu)-\delta} \otimes S$. (See 3.4, 3.5 and 3.7 below).

It is easy to describe the kernel of the Dirac operator acting on $F(\nu)^\star \otimes S$. Using the formula for the square of the Dirac operator [12, Proposition 3.1], [17, Lemma 2.5] and [12, Lemma 8.1] this kernel is the $\mathcal{U}(\mathfrak{k})$-span of $x \otimes s$ where $x$ is a $P_\mathfrak{k}$-highest weight vector of $F(\nu)^\star$ with $P_\mathfrak{k}$-dominant weight $w(-\nu)$ in the Weyl group orbit of $-\nu$ and $s$ is a $P_\mathfrak{k}$-highest weight vector of an irreducible component of $S$ of weight $-\delta_\mathfrak{k} - w\delta$.

Next, we describe (see 3.2, 3.3 and 3.6) some elements in the kernel of the Dirac operator $\mathcal{D}_{\mathfrak{q},\mu+\nu}$ acting on $D_{\mathfrak{q},\mu+\nu} \otimes S$ as in [9]. (Similar remarks for $D_{\mathfrak{q},\mu} \otimes S$ will hold.) Later we use "Vogan's Conjecture" (now a 'Theorem' due to Huang and Pandžić) which reads off the infinitesimal character of an irreducible $(\mathfrak{g}, K)$-module by looking at the $P_\mathfrak{k}$-highest weight of an irreducible $\mathfrak{k}$-type in the kernel of the Dirac operator for that module to complement this description and conclude that there is nothing besides the elements in the kernel described this way.

**(3.2)**. Let $y'$ be a $P_\mathfrak{k}$-highest weight vector of weight $\gamma'$ in $V_{\mathfrak{g},P,\sigma(\mu+\nu)-\delta} \otimes S$ annihilated by the Dirac operator. Assume that $\gamma' + \delta_\mathfrak{k}$ is $\sigma_\mathfrak{k}(P_\mathfrak{k})$-dominant and non-singular. Let $\gamma$ be a $P_\mathfrak{k}$-dominant integral weight such that $\gamma + \delta_\mathfrak{k} \in W_\mathfrak{k}(\gamma' + \delta_\mathfrak{k})$. Then there is a $P_\mathfrak{k}$-highest weight vector $\tilde{y}$ of $W_{\mathfrak{q},\mu+\nu} \otimes S$ of weight $\gamma$ such that $\mathcal{U}(\mathfrak{k}) \cdot y' \subseteq \mathcal{U}(\mathfrak{k}) \cdot \tilde{y}$. The existence of such a vector results by successively using the completion property (section 2) following through the chain of modules $W_i, i = 1, \dots, m+1$. Suppose that $y' = y_1, \dots, y_i, \dots, y_{m+1} = \tilde{y}$ are the successive vectors obtained this way. The $\mathcal{U}(\mathfrak{k})$-modules generated by them gives a chain of $\mathfrak{k}$-Verma modules. Since the Dirac operator annihilates $\mathcal{U}(\mathfrak{k})(y')$, property (vi) (section 2) implies that $\mathcal{U}(\mathfrak{m}_1)(y_2)$ has finite image under the Dirac operator. Property (v) ('no torsion') implies that this image has to be zero. Thus the Dirac operator annihilates $\mathcal{U}(\mathfrak{k})(y_2)$. By induction, one concludes that $\tilde{y}$ goes to zero under the Dirac operator. Hence, also the image $y$ of $\tilde{y}$ in $D_{\mathfrak{q},\mu+\nu} \otimes S$ is in the kernel of the Dirac operator. We apply these observations by making the simplest choice for $y'$. Namely, $y' = x \otimes s$, where $x$ is the $P$-highest weight vector of $V_{\mathfrak{g},P,\sigma(\mu+\nu)-\delta}$ of weight $\sigma(\mu+\nu) - \delta$ and $s$ is the $P_\mathfrak{k}$-highest weight vector of $S$ of weight $\delta - \delta_\mathfrak{k}$.

**(3.3)**. Let $\overline{y}'$ be a $P_\mathfrak{k}$-highest weight vector of weight $\overline{\gamma}'$ in $V_{\mathfrak{g},P,\sigma(\mu)-\delta} \otimes S$ annihilated by the Dirac operator. Assume that $\overline{\gamma}' + \delta_\mathfrak{k}$ is $\sigma_\mathfrak{k}(P_\mathfrak{k})$-dominant and non-singular. Let $\overline{\gamma}$ be a $P_\mathfrak{k}$-dominant integral weight such that $\overline{\gamma} + \delta_\mathfrak{k} \in W_\mathfrak{k} \cdot (\overline{\gamma}' + \delta_\mathfrak{k})$. Then there is a $P_\mathfrak{k}$-highest weight vector $\tilde{\overline{y}}$ of $W_{\mathfrak{q},\mu} \otimes S$ of weight $\overline{\gamma}$ such that

$\mathcal{U}(\mathfrak{k}) \cdot \overline{y}' \subseteq \mathcal{U}(\mathfrak{k}) \cdot \tilde{\overline{y}}$. One can show that $\tilde{\overline{y}}$ (hence, also its image $\overline{y}$ in $D_{\mathfrak{q},\mu} \otimes S$) is in the kernel of the Dirac operator. Apply these observations by choosing $\overline{y}' = \overline{x} \otimes s$, where $\overline{x}$ is the $P$-highest weight vector of weight $\sigma(\mu) - \delta$ of $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ and $s$ is the $P_{\mathfrak{k}}$-highest weight vector of $S$ of weight $\delta - \delta_{\mathfrak{k}}$. Note that if $\xi$ is the $P_{\mathfrak{k}}$-highest weight of this choice of $\overline{y}'$, then $\xi + \delta_{\mathfrak{k}} = \sigma(\mu) \in W_{\mathfrak{g}}(\mu)$.

The statement analogous to Theorem 1 relating $(V_{\mathfrak{g},P,\sigma(\mu+\nu)-\delta} \otimes S) \otimes (F^\star(\nu) \otimes S) \otimes S^\star$ and $(V_{\mathfrak{g},P,\sigma(\mu)-\delta} \otimes S)$, namely the fact that

**(3.4)**. $\varphi_S(\overline{y}') \in \beta(y' \otimes \mathcal{W}_2 \otimes S^\star)$ is evident.

**(3.5)**. Properties (v) and (vi) of the chain $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ and the fact that the maps $\varphi_S$ and $\beta$ are restrictions of corresponding maps obtained by changing $W_1$ to $W_{m+1}$ imply (using 3.4) statements analogous to 3.4 relating $(W_{\mathfrak{q},\mu+\nu} \otimes S) \otimes (F^\star(\nu) \otimes S) \otimes S^\star$ and $(W_{\mathfrak{q},\mu} \otimes S)$ and these in turn can finally be related to $(D_{\mathfrak{q},\mu+\nu} \otimes S) \otimes (F^\star(\nu) \otimes S) \otimes S^\star$ and $(D_{\mathfrak{q},\mu} \otimes S)$.

We recall the last two properties of the chain of $\mathfrak{g}$-modules $W_i, i = 1, 2, \cdots, m+1$ before the completion property -

(v) Each $W_i$ is $\mathcal{U}(\mathfrak{n}_{\mathfrak{k}})$-free; here $\mathfrak{n}_{\mathfrak{k}}$ is the sum of all rootspaces of $\mathfrak{k}$ corresponding to roots in $P_{\mathfrak{k}}$.

(vi) $W_{i+1}/W_i$ is $\mathfrak{m}_i$-finite.

Having fixed a reduced expression $\sigma_{\mathfrak{k}} = s_1 \cdot s_2 \cdots s_m$, define submodules $\overline{W}_1, \overline{W}_2, \cdots, \overline{W}_m, \overline{W}$ and $W_X$ as in [13, 4.3] and [13, 4.5]. Let $\overline{V}_1 \subseteq \overline{V}_2 \subseteq \cdots \subseteq \overline{V}_m \subseteq \overline{V}_{m+1}$ be a chain of Verma modules for $\mathfrak{k}$. Assume that $\overline{V}_{i+1}/\overline{V}_i$ is $\mathfrak{m}_i$-finite. Let $v_{m+1}$ be a highest weight vector of $\overline{V}_{m+1}$ of weight $\mu_{m+1}$ which is $P_{\mathfrak{k}}$-dominant integral. Now assume that $\overline{V}_{m+1} \subseteq W_{m+1}$ so that for each $i = 1, \cdots, m+1$, $\overline{V}_i \subseteq W_i \cap \overline{V}_{m+1}$. We have the following important property which tells us how to detect whether $\overline{V}_{m+1}$ is nonzero mod $W_X$ by inspecting whether $\overline{V}_1$ is nonzero mod $W_0$.

(vii) If $\overline{V}_{m+1} \neq 0$ mod $W_X$, then $\overline{V}_1 \neq 0$ mod $W_0$.

For many of the properties of the chain of modules listed so far, only the $\mathfrak{k}$-module structure is relevant and the properties continue to hold when the chain is replaced with a new one obtained by tensoring the original chain by a finite dimensional $\mathfrak{k}$-module. The completion property and properties (v),(vi) and (vii) remain true even when the chain of $\mathfrak{g}$-modules $W_1 \subseteq \cdots \subseteq W_{m+1}$ is replaced by a new chain of $\mathfrak{k}$-modules $W_1 \otimes F \subseteq \cdots \subseteq W_{m+1} \otimes F$ where $F$ is any finite dimensional $\mathfrak{k}$-module. We will be interested in applying these properties to the chain $W_1 \otimes S \subseteq \cdots \subseteq W_{m+1} \otimes S$, where $S$ is the spin module.

**(3.6).** Using (vii) we wish to construct more non-zero elements in the Dirac kernel essentially via constructions similar to 3.2 and 3.3. Recall that $\sigma(P_{\mathfrak{l}} \cup P_{\mathfrak{u}}) = \sigma(P) = P_{\mathfrak{l}} \cup -P_{\mathfrak{u}}$. Write $P' = P_{\mathfrak{l}} \cup -P_{\mathfrak{u}}$. Let $\overline{P}_{\mathfrak{l}}$ be a positive system in the root system $\Delta_{\mathfrak{l}} = P_{\mathfrak{l}} \cup -P_{\mathfrak{l}}$ such that $\overline{P}_{\mathfrak{l}\cap\mathfrak{k}} = P_{\mathfrak{l}\cap\mathfrak{k}}$. Put $\overline{P} = \overline{P}_{\mathfrak{l}} \cup P_{\mathfrak{u}}$. Note that $\overline{P}_{\mathfrak{k}} = P_{\mathfrak{k}}$. Put $\overline{P}' = \overline{P}_{\mathfrak{l}} \cup -P_{\mathfrak{u}}$. Note that $\overline{P}'_{\mathfrak{k}} = P'_{\mathfrak{k}} = \sigma_{\mathfrak{k}}(P_{\mathfrak{k}}) = P_{\mathfrak{l}\cap\mathfrak{k}} \cup -P_{\mathfrak{u}\cap\mathfrak{k}}$. Choose $\overline{\sigma} \in W_{\mathfrak{g}}$ such that $\overline{\sigma}(P) = \overline{P}'$. The crucial observation is that the irreducible quotients (as

$\mathfrak{g}$-modules) of $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ and $V_{\mathfrak{g},\overline{P},\overline{\sigma}(\mu)-\overline{\delta}}$ are isomorphic. Both are isomorphic to $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathcal{W}_{\mathfrak{l},\sigma(\mu)-\delta}$, where $W_{\mathfrak{l},\sigma(\mu)-\delta}$ is the finite-dimensional irreducible $\mathfrak{l}$-module whose highest weight with respect to $P_{\mathfrak{l}}$ is $\sigma(\mu) - \delta = \sigma(\mu - \delta) - 2\delta_{\mathfrak{u}}$. (see [24]). Then the highest weight of $W_{\mathfrak{l},\sigma(\mu)-\delta}$ with respect to $\overline{P}_{\mathfrak{l}}$ is $\overline{\sigma}(\mu) - \overline{\delta}$ which equals $\overline{\sigma}(\mu - \delta) - 2\delta_{\mathfrak{u}}$. We have a surjection $V_{\mathfrak{l},P_{\mathfrak{l}},\sigma(\mu)-\delta} \longrightarrow W_{\mathfrak{l},\sigma(\mu)-\delta}$. By lemma 9 in [3] the $P_{\mathfrak{l}\cap\mathfrak{k}}$ highest weight vector $\hat{v}_{\mathfrak{l}}$ of weight $\overline{\sigma}(\mu - \delta) - 2\delta_{\mathfrak{u}}$ in $W_{\mathfrak{l},\sigma(\mu)-\delta}$ can be pulled back to a $P_{\mathfrak{l}\cap\mathfrak{k}}$ highest weight vector $\overline{v}_{\mathfrak{l}}$ of $V_{\mathfrak{l},P_{\mathfrak{l}},\sigma(\mu)-\delta}$. We identify $V_{\mathfrak{l},P_{\mathfrak{l}},\sigma(\mu)-\delta} = \{v \in V_{\mathfrak{g},P,\sigma(\mu)-\delta} \mid X_\alpha \cdot v = 0, \forall \alpha \in P_{\mathfrak{u}}\}$. Then, $X_\alpha \cdot \overline{v}_{\mathfrak{l}} = 0, \forall \alpha \in P_{\mathfrak{k}}$.

Note that $\overline{\delta} - \delta_{\mathfrak{k}}$ is the highest weight of an irreducible $\mathfrak{k}$-type in the spin module $S$. Let $s_{\overline{\delta}-\delta_{\mathfrak{k}}}$ be a corresponding highest weight vector. By following the construction outlined in 3.3 taking $\overline{y}' = \overline{v}_{\mathfrak{l}} \otimes s_{\overline{\delta}-\delta_{\mathfrak{k}}}$ we get $\tilde{\overline{y}} \in W_{\mathfrak{q},\mu} \otimes S$ and its image in $D_{\mathfrak{q},\mu} \otimes S$ in the kernel of the Dirac operator $\mathcal{D}_{\mathfrak{q},\mu}$ on $D_{\mathfrak{q},\mu} \otimes S$, one for each choice of $\overline{P}_{\mathfrak{l}}$ as above. Note that if $\xi$ is the $P_{\mathfrak{k}}$-highest weight of $\overline{y}'$, then $\xi + \delta_{\mathfrak{k}} = \overline{\sigma}(\mu) \in W_{\mathfrak{g}}(\mu)$. Thus all the elements constructed in this way in $\mathcal{W}_3^o$ in fact belong to the subspace $\mathcal{W}_3$. Further replacing $\mu$ by $\mu + \nu$ we get similar elements in the kernel of the Dirac operator (which in fact lie in $\mathcal{W}_1$) $\mathcal{D}_{\mathfrak{q},\mu+\nu}$ on $D_{\mathfrak{q},\mu+\nu} \otimes S$ as in 3.2, one for each choice of $\overline{P}_{\mathfrak{l}}$.

Since we are dealing with both $D_{\mathfrak{q},\mu+\nu} \otimes S$ and $D_{\mathfrak{q},\mu} \otimes S$ we denote the vectors $\overline{y}', \tilde{\overline{y}}$ corresponding to the two cases with a subscript, in other words, by $\overline{y}'_{\mu+\nu}, \tilde{\overline{y}}_{\mu+\nu}$ and $\overline{y}'_{\mu}, \tilde{\overline{y}}_{\mu}$ respectively. Similar to 3.4 and 3.5, it is evident that

**(3.7).** $\varphi_S(\overline{y}'_\mu) \in \beta(\overline{y}'_{\mu+\nu} \otimes \mathcal{W}_2 \otimes S^\star)\text{and}\varphi_S(\tilde{\overline{y}}_\mu) \in \beta(\tilde{\overline{y}}_{\mu+\nu} \otimes \mathcal{W}_2 \otimes S^\star)$

**Remark** : For notational convenience, we denote the Dirac operator on $W_{\mathfrak{q},\mu} \otimes S$ also by $\mathcal{D}_{\mathfrak{q},\mu}$. We do not need a separate notation for the Dirac operator on $W_1 \otimes S$ as this is the restriction of $\mathcal{D}_{\mathfrak{q},\mu}$.

**(3.8).** Next, we will show that there are no elements in $\mathcal{W}_3$ other than what is in the linear span of the ones indicated above and furthermore that $\mathcal{W}_3$ maps bijectively onto the Dirac cohomology of $D_{\mathfrak{q},\mu}$, using Vogan's conjecture (see [5 pp.185,186], [25, Chap. 3]) relating Dirac cohomology and infinitesimal character.

Let $z \in D_{\mathfrak{q},\mu} \otimes S$ and suppose that $z$ is a non-zero $P_{\mathfrak{k}}$-highest weight vector of weight $\xi$ in $\mathcal{W}_3$, i.e., that $\xi + \delta_{\mathfrak{k}} \in W_{\mathfrak{g}} \cdot \mu$. By lemma 9 in [3] the $P_{\mathfrak{k}}$-highest weight vector $z \in D_{\mathfrak{q},\mu} \otimes S$ of weight $\xi$ can be pulled back to a $P_{\mathfrak{k}}$-highest weight vector $\overline{z} \in W_{\mathfrak{q},\mu} \otimes S$ of weight $\xi$. We observe that $\overline{z} \neq 0$ mod $W_X$, while, $\mathcal{D}_{\mathfrak{q},\mu}\overline{z} = 0$ mod $W_X \otimes S$. Consider the chain of $\mathfrak{k}$-modules $W_1 \otimes S \subseteq W_2 \otimes S \cdots \subseteq W_i \otimes S \subseteq \cdots \subseteq W_{m+1} \otimes S$. The intersection of $W_1 \otimes S (= V_{\mathfrak{g},P,\sigma(\mu)-\delta} \otimes S)$ and the $\mathfrak{k}$-Verma module generated by $\overline{z}$ contains the $\mathfrak{k}$-Verma module $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma_{\mathfrak{k}}(\xi+\delta_{\mathfrak{k}})-\delta_{\mathfrak{k}}}$ which is non-zero mod $W_0 \otimes S$. (See remark following property (vii)). We fix a non-zero highest weight vector $\tilde{z}$ of $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma_{\mathfrak{k}}(\xi+\delta_{\mathfrak{k}})-\delta_{\mathfrak{k}}}$ of weight $\sigma_{\mathfrak{k}}(\xi + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}}$.

We know that the weights of the $\mathfrak{g}$-Verma module $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ are of the form $\sigma(\mu) - \delta - \sum_{\alpha \in P} m_\alpha \alpha$, where $m_\alpha$ are non-negative integers. Let $\overline{P}_{\mathfrak{l}}, \overline{P}, \overline{\sigma}$ have the same meaning as in 3.6. For any such $\overline{P}$ the weights of the $\mathfrak{g}$-Verma module $V_{\mathfrak{g},\overline{P},\overline{\sigma}(\mu)-\overline{\delta}}$ are of the form $\overline{\sigma}(\mu) - \overline{\delta} - \sum_{\alpha \in \overline{P}} m_\alpha \alpha$, where $m_\alpha$ are non-negative integers. The weights of the spin module $S$ (a $\mathfrak{k}$-module) are of the form $\delta - \delta_{\mathfrak{k}} - < A >$ for certain subsets $A \subseteq P$ disjoint from $P_{\mathfrak{k}}$ and $< A >= \sum_{\alpha \in A} \alpha$. For the same reason, for the positive systems $\overline{P}$ considered above (in particular, $\overline{P}_{\mathfrak{k}} = P_{\mathfrak{k}}$) the weights of the spin module $S$ are also of the form $\overline{\delta} - \overline{\delta}_{\mathfrak{k}} - < A >$ for certain subsets $A \subseteq \overline{P}$ disjoint from $P_{\mathfrak{k}}$. Both $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ and $V_{\mathfrak{g},\overline{P},\overline{\sigma}(\mu)-\overline{\delta}}$ have the same irreducible quotient, namely, $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathcal{W}_{\mathfrak{l},\sigma(\mu)-\delta}$ and $\tilde{z}$ has a non-zero image in this quotient.

**(†) Remark** : The following fact is clear from the above observations. For each $\overline{P}_{\mathfrak{l}}$ as above recall the vector $\overline{y}'$ described in the second paragraph of (3.6). Then, $(W_1/W_0) \otimes S$ = the line spanned by the image of $\overline{y}'$ $\oplus$ subspace spanned by weight vectors corresponding to weights different from that of $\overline{y}'$. Moreover, the Dirac operator $\mathcal{D}_{W_1/W_0}$ annhilates the first summand and the image of $\mathcal{D}_{W_1/W_0}$ is contained in the second summand.

**(3.9).** Let $\beta$ be the weight of $\tilde{z}$. Let $\overline{P}_{\mathfrak{l}}, \overline{P}, \overline{\sigma}$ have the same meaning as in 3.6. Then the obsevations in the paragraph preceding above remark imply that $\beta + \delta_{\mathfrak{k}}$ is of the form $\overline{\sigma}(\mu) - \sum_{\alpha \in \overline{P}} m_\alpha \alpha - < A >$, where $m_\alpha (\alpha \in \overline{P})$ are non-negative integers and $A \subseteq \overline{P}$ is a subset disjoint from $P_{\mathfrak{k}}$. Furthermore, $\beta + \delta_{\mathfrak{k}} = \sigma_{\mathfrak{k}}(\xi + \delta_{\mathfrak{k}})$.

**(3.10).** Since $\xi + \delta_{\mathfrak{k}} \in W_{\mathfrak{g}} \cdot \mu$ it follows that $\beta + \delta_{\mathfrak{k}} \in W_{\mathfrak{g}} \cdot \mu$. Since $\xi + \delta_{\mathfrak{k}}$ is dominant regular with respect to $P_{\mathfrak{k}}$, $\sigma_{\mathfrak{k}}(\xi + \delta_{\mathfrak{k}})$ (=$\beta + \delta_{\mathfrak{k}}$) is dominant with respect to $\sigma_{\mathfrak{k}}(P_{\mathfrak{k}})$. Since $\sigma_{\mathfrak{k}}(P_{\mathfrak{k}}) = P_{\mathfrak{l}\cap\mathfrak{k}} \cup -P_{\mathfrak{u}\cap\mathfrak{k}}$, we note that $\beta + \delta_{\mathfrak{k}}$ is dominant regular with respect to $P_{\mathfrak{l}\cap\mathfrak{k}} \cup -P_{\mathfrak{u}\cap\mathfrak{k}}$.

**(3.11).** Let us now make a particular choice of the positive system $\overline{P}_{\mathfrak{l}}$ in (3.6). Choose $\overline{P}_{\mathfrak{l}} \subseteq P_{\mathfrak{l}} \cup -P_{\mathfrak{l}}$ such that $\beta + \delta_{\mathfrak{k}}$ is dominant regular with respect to $\overline{P}_{\mathfrak{l}}$. Define two positive systems $\overline{P}$ and $Q$ as follows: as earlier, $\overline{P} = \overline{P}_{\mathfrak{l}} \cup P_{\mathfrak{u}}$, $Q = \{\alpha \in (P \cup -P) \mid \beta + \delta_{\mathfrak{k}}(\alpha) > 0\}$. Again, following earlier notation, let $\overline{\sigma} \in W_{\mathfrak{g}}$ be the unique element such that $\overline{\sigma}(P) = \overline{P}_{\mathfrak{l}} \cup -P_{\mathfrak{u}}$. From what we have said above, we note

$$P_{\mathfrak{l}\cap\mathfrak{k}} \cup -P_{\mathfrak{u}\cap\mathfrak{k}} \cup \overline{P}_{\mathfrak{l}} \subseteq \overline{\sigma}(P) \cap Q. \qquad (**)$$

The basic point is: $\overline{\sigma}(\mu)$ is dominant regular with respect to $\overline{\sigma}(P)$, $\beta + \delta_{\mathfrak{k}}$ is dominant regular with respect to $Q$ and they are in the same $W_{\mathfrak{g}}$ orbit. This situation forces $\beta + \delta_{\mathfrak{k}} = \overline{\sigma}(\mu)$ + a negative integral combination of $\overline{\sigma}(P) \cap -Q$. From $(**)$ it is clear that $\overline{\sigma}(P) \cap -Q$ is disjoint from $P_{\mathfrak{k}} \cup -P_{\mathfrak{k}} \cup P_{\mathfrak{l}} \cup -P_{\mathfrak{l}}$; this implies $\overline{\sigma}(P) \cap -Q \subseteq -P_{\mathfrak{u}}$. So, $\beta + \delta_{\mathfrak{k}} = \overline{\sigma}(\mu)$ + a non-negative integral combination of roots in $P_{\mathfrak{u}}$. Comparing this with 3.9, the non-negative integral combination in the last sentence is null and the quantity $-\sum_{\alpha \in \overline{P}} m_\alpha \alpha - < A >$ in 3.9 is also null (for the particular $\overline{P}_{\mathfrak{l}}$ as in (3.11). In particular upto a non-zero scalar multiple the vector $\tilde{z}$ in the second paragraph of (3.8) and the vector denoted by $\overline{y}'$ in the second paragraph of (3.6) (for the particular choice of $\overline{P}_{\mathfrak{l}}$ in (3.11)) coincide mod $W_0 \otimes S$. Hence the element $z \in D_{\mathfrak{q},\mu} \otimes S$ in the Dirac kernel we started with in the beginning of the second paragraph of (3.8) coincides upto a non-zero scalar multiple with the element in the kernel of the Dirac operator $\mathcal{D}_{\mathfrak{q},\mu}$ on $D_{\mathfrak{q},\mu} \otimes S$ constructed as indicated in (3.6) starting with $\overline{y}'$.

**(††) Remark** : It should be noted that the same arguments prove (using remark (†) above (3.9) and property (vii)) that $z \notin \mathcal{D}_{\mathfrak{q},\mu}(D_{\mathfrak{q},\mu} \otimes S)$. Thus $\mathcal{W}_3$ maps injectively into the Dirac cohomology of $D_{\mathfrak{q},\mu}$. But by Vogan's conjecture ([5], [25, Chap. 3]) every element of the Dirac cohomology of $D_{\mathfrak{q},\mu}$ has a representative in $\mathcal{W}_3$.

**(‡) Remark** : Let $\mathrm{V}_{\mathfrak{k},\epsilon}$ be a finite-dimensional $\mathfrak{k}$-type with $P_{\mathfrak{k}}$-highest weight $\epsilon$ occurring in $D_{\mathfrak{q},\mu} \otimes S$ such that $\epsilon + \delta_{\mathfrak{k}}$ is in the $W_{\mathfrak{g}}$-orbit of $\mu$. From the formula for the square of the Dirac operator [12, Proposition 3.1] and [17, Lemma 2.5] it is seen that $\mathcal{D}_{\mathfrak{q},\mu}(\mathcal{D}_{\mathfrak{q},\mu}(\mathrm{V}_{\mathfrak{k},\epsilon})) = 0$. If $\mathcal{D}_{\mathfrak{q},\mu}(\mathrm{V}_{\mathfrak{k},\epsilon}) \neq 0$ it would contradict (††). Hence we conclude that $\mathcal{D}_{\mathfrak{q},\mu}(\mathrm{V}_{\mathfrak{k},\epsilon}) = 0$. Bearing in mind that our $D_{\mathfrak{q},\mu}$ are not necessarily unitary this phenomenon is noticeable.

This ends the discussion begun in 3.8 and culminates in the following result.

**Proposition 3.12.** *Let $\mathfrak{r}$ be a $\theta$-stable Borel subalgebra of $\mathfrak{g}$ corresponding to a positive system $P$. Let $\mathfrak{q}$ be a $\theta$-stable parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{r}$. Fix a Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Let $\mu$ be a non-singular integral weight dominant with respect to $P$. Let $D_{\mathfrak{q},\mu}$ be the generalized Enright-Varadarajan module as in* [13]. *Let $P_{\mathfrak{k}}$ be the positive system of roots in $\mathfrak{k}$ corresponding to the $\mathfrak{k}$-Borel subalgebra $\mathfrak{r} \cap \mathfrak{k}$. Let $\sigma$ be the unique element of the Weyl group $W_{\mathfrak{g}}$ of $\mathfrak{g}$ such that $\sigma(P) = P_{\mathfrak{l}} \cup -P_{\mathfrak{u}}$. Let $\sigma_{\mathfrak{k}}$ be the unique element of the Weyl group $W_{\mathfrak{k}}$ of $\mathfrak{k}$ such that $\sigma_{\mathfrak{k}}(P_{\mathfrak{k}}) = P_{\mathfrak{l}\cap\mathfrak{k}} \cup -P_{\mathfrak{u}\cap\mathfrak{k}}$.*

*(i) The Dirac cohomology*

$$\mathrm{Ker}(\mathcal{D}_{\mathfrak{q},\mu}) / \left(\mathrm{Im}(\mathcal{D}_{\mathfrak{q},\mu}) \cap \mathrm{Ker}(\mathcal{D}_{\mathfrak{q},\mu})\right)$$

*of $D_{\mathfrak{q},\mu}$ as a $\mathfrak{k}$-module has a decomposition*

$$\bigoplus_{\kappa \in (W_{\mathfrak{l}}/W_{\mathfrak{l}\cap\mathfrak{k}})} (\tau_{\sigma_{\mathfrak{k}}^{-1}\kappa\sigma(\mu)-\delta_{\mathfrak{k}}}).$$

*Here the sum is over $\kappa$ in the Weyl group of $\mathfrak{l}$ such that $\kappa(P_{\mathfrak{l}}) \supseteq P_{\mathfrak{l}\cap\mathfrak{k}}$. The representation $\tau_{\sigma_{\mathfrak{k}}^{-1}\kappa\sigma(\mu)-\delta_{\mathfrak{k}}}$ is the irreducible finite-dimensional representation of $\mathfrak{k}$ with $P_{\mathfrak{k}}$-highest weight $\sigma_{\mathfrak{k}}^{-1}\kappa\sigma(\mu) - \delta_{\mathfrak{k}}$. Each summand occurs with multiplicity one.*

*(ii) Let $\mathcal{W}_{\mathfrak{q},\mu}$ be the subspace of vectors in $D_{\mathfrak{q},\mu} \otimes S$ spanned by irreducible finite-dimensional $\mathfrak{k}$-submodules $V_{\mathfrak{k},\epsilon}$ with $P_{\mathfrak{k}}$-highest weight $\epsilon$ such that $\epsilon+\delta_{\mathfrak{k}}$ is $W_{\mathfrak{g}}$-conjugate to $\mu$. On this space the casimir operator $\Omega_{\mathfrak{k}}$ of $\mathfrak{k}$ acts as multiplication by the scalar $\|\mu\|^2 - \|\delta_{\mathfrak{k}}\|^2$. Moreover,*

$$\mathcal{W}_{\mathfrak{q},\mu} \cap \mathrm{Ker}(\mathcal{D}_{\mathfrak{q},\mu}) = \mathcal{W}_{\mathfrak{q},\mu} \cap \mathrm{Ker}(\mathcal{D}_{\mathfrak{q},\mu}^2) = \mathcal{W}_{\mathfrak{q},\mu}$$

*and this space maps bijectively onto the Dirac cohomology.*

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