THE ROTATION OF AN INFINITE PLANE LAMINA IN A VISCOUS COMPRESSIBLE FLUID

By S. D. Nigam

(*Department of Applied Mathematics, Indian Institute of Technology, Kharagpur, India*)

Received February 2, 1953

(Communicated by Dr. B. R. Seth, F.A.Sc.)

1. Introduction

The steady flow due to the rotation of an infinite plane lamina in an incompressible fluid has been discussed by Th. v. Karman.[1] The corresponding problem in a compressible viscous fluid presents enormous difficulties due to the non-linearity of the equations of motion, viscosity and heat conduction. It has been possible, however, to reduce the equations of motion to three non-linear equations by assuming heat conduction to be negligible and the equation of state to be of the form $p = p_0 + k\rho^\gamma$; p_0, k and γ are constants. The equations have been simplified further by assuming $\gamma = 5/3$ and then integrated after the manner of von Karman.

2. Equations of Motion

The equations of motion for a non-heat conducting viscous compressible fluid, in cylindrical co-ordinates are

$$\rho\left(u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}\right) = -\frac{\partial p}{\partial r} + \frac{\mu}{3}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z}\right] + \mu\left[\nabla^2 u - \frac{u}{r^2}\right] \quad (1)$$

$$\rho\left(u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right) = \mu\left[\nabla^2 v - \frac{v}{r^2}\right] \quad (2)$$

$$\rho\left(u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \frac{\mu}{3}\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z}\right] + \mu\nabla^2 w \quad (3)$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and the equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru\rho) + \frac{\partial}{\partial z}(w\rho) = 0, \quad (4)$$

where u, v, w are the radial, the azimuthal and the axial components of velocity respectively. If the lamina rotates with an angular velocity Ω about an axis perpendicular to its plane, we put

$$u = \Omega r f(z_1)\ ;\ \ v = \Omega r g(z_1)\ ;\ \ w = (\mu\Omega)^{\frac{1}{2}} h(z_1);\ \ p = p_0 - \mu\Omega p_1(z_1)$$

$$z_1 = (\Omega/\mu)^{\frac{1}{2}} z$$

and obtain the following equations:

$$\rho\,[f'^2 - g^2 + f'h] = f'' \tag{5}$$

$$\rho\,[2fg + g'h] = g'' \tag{6}$$

$$\rho h h' = p_1' + (2/3)(f' + 2h'') \tag{7}$$

$$2f\rho + \frac{d}{dz_1}(h\rho) = 0 \tag{8}$$

where a dash denotes differentiation with respect to z_1. We further assume, $\rho = -h^3(z_1)$. The equation for pressure using (8) becomes

$$p_1' = -h^4 h' \ \text{ or } \ p_1 = (1/5)\,\rho^{5/3}.$$

Therefore,* $\quad p = p_0 - (\mu\Omega/5)\,\rho^{5/3}. \qquad (9)$

Equations (5), (6) and (8) become

$$h^3\,[f^2 - g^2 + f'h] + f'' = 0 \tag{10}$$

$$h^3\,[2fg + g'h] + g'' = 0 \tag{11}$$

$$f + 2h' = 0 \tag{12}$$

The solutions of these equations have to satisfy the following boundary conditions:

$$f(0) = f(\infty) = 0;\ \ g(0) = 1,\ g(\infty) = 0;\ \ h(0) = 0 \tag{12 a}$$

For small values of z_1, we have

$$f = a_0 z_1 - \frac{a_0^3}{3584} z_1^8 - \frac{a_0^3 b_0}{2304} z_1^9 + \frac{a_0^3(3a_0^2 - 4b_0^2)}{23040} z_1^{10} + \ldots. \tag{13}$$

$$g = 1 + b_0 z_1 + \frac{a_0^4}{2304} z_1^9 + \frac{7a_0^4 b_0}{23040} z_1^{10} + \ldots. \tag{14}$$

$$h = -\frac{a_0}{4} z_1^2 + \frac{a_0^3}{64512} z_1^9 + \frac{a_0^3 b_0}{46080} z_1^{10} + \ldots. \tag{15}$$

For large values of z_1, we have

$$f = \mathrm{A}e^{-cz_1} - \frac{\mathrm{A}^2 + \mathrm{B}^2}{2cc_0} e^{-2cz_1} + \ldots. \tag{16}$$

$$g = \mathrm{B}e^{-cz_1} + \ldots. \tag{17}$$

---

* On account of the boundary conditions (12 *a*) the density becomes zero on the rotating lamina. This has been pointed out earlier by Lord Kelvin. [*Papers*, **1**, 83–87]. Also ref. C. Truesdell, *On the Equation of the Bounding Surface*, Publication, U. S. Naval Research Laboratory, 1951, 71–78.

$$h = -c_0 + \frac{A}{2c} e^{-cz_1} - \frac{A^2 + B^2}{8c^2 c_0} e^{-2cz_1} + \ldots ; \quad c_0^4 = c \tag{18}$$

The constants a_0, b_0, c, A and B have to be chosen so that f, g, h, f' and g' are continuous for some finite value of z_1 and it will then follow from the differential equations that all other derivatives are continuous.

3. DETERMINATION OF CONSTANTS

We follow a method of approximation to determine the coefficients. It is highly probable that the expressions (13), (14) and (15) are convergent for $z_1 \leqslant 1$. The coefficients of various terms are decreasing rather rapidly, so that, even for $z_1 = 1$ we neglect terms $\sim O(z_1^8)$. Again, in (16), (17) and (18) we neglect terms $\sim O(e^{-3cz_1}.)$ The equations for constants are

$$a_0 = Ae^{-c} - \frac{A^2 + B^2}{2\,cc_0} e^{-2c}$$

$$1 + b_0 = Be^{-c}$$

$$-\frac{a_0}{4} = -c_0 + \frac{A}{2c} e^{-c} - \frac{A^2 + B^2}{8\,c^2 c_0} e^{-2c}$$

$$a_0 = -c\left[Ae^{-c} - \frac{A^2 + B^2}{cc_0} e^{-2c}\right]$$

$$b_0 = -c\,Be^{-c}.$$

The constants are

$$a_0 = 3\cdot634,\ b_0 = -\cdot833,\ c = 5,\ c_0 = 1\cdot49,\ A = 1184\cdot232,\ B = 24\cdot735.$$

The functions f, g, h, are given in Fig. 1.

f and g tend to zero exponentially and become indistinguishable from zero for $z_1 = 2$. Hence, if, μ/Ω is small, $\mu/\Omega r$ and $v/\Omega r$ are appreciable only in a thin layer $[\delta = (\mu/\Omega)^{\frac{1}{2}}\cdot 2]$ near the disk. This points to the existence of a boundary layer adjacent to the rotating lamina.

The solution applies strictly only to an infinite disc; but if we neglect the edge effect we can find the frictional moment on a rotating disk of radius a. The shearing stress is given by

$p_{z\phi} = \mu \frac{\partial v}{\partial z} = (\mu\Omega^3)^{\frac{1}{2}}\, rg'(0) \ldots\ldots$ at the disk, so that the moment is

$$M = -\int_0^a 2\pi r^2 p_{z\phi}\, dr = \pi a^4 (\mu\Omega^3)^{\frac{1}{2}} \times \cdot 4165.$$

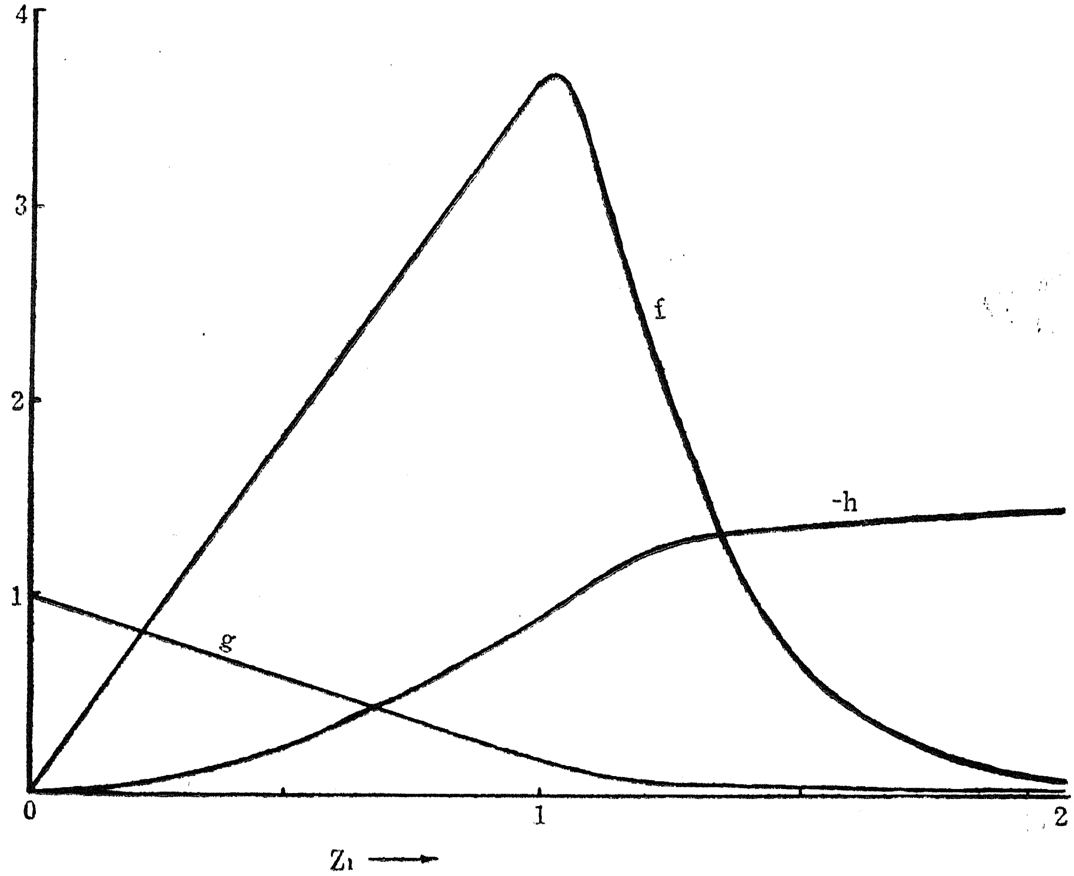

FIG. 1 The flow functions

4. Summary and Discussion of Results

In the corresponding problem for the incompressible viscous fluid[2] the functions mod. f and mod. h are everywhere less than the maximum value of mod. g. In our case the functions mod. f and mod. h are at some places greater than the maximum value of mod. g. The radial and the axial components of velocity at some places exceed the velocity of the disc. This is quite in accord with the phenomena associated with the compressibility of the fluids. But for these compressibility effects, which are confined to the neighbourhood of the disc, the general features of flow are same as in incompressible fluids. There is a steady axial flow towards the rotating lamina; this is necessary to preserve continuity. The fluid moves radially outwards near the lamina.

In conclusion, I thank Mr. K. S. Rangasami and Mr. R. S. Gupta for the help they have given me in checking up the numerical results and in drawing the sketches.

References

1. Karman, Th. v. .. *Zeit. f. angew. Math. u. Mech.*, 1921, **1**, 244–47.
2. Goldstein, S. .. *Modern dev. in Fluid Dynamics*, **1**, The Clarendom Press, Oxford, 1938, 111.