ON SLOW MOTION OF A SPHERE IN A VISCOUS LIQUID

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Abstract

Stokes' and Seth's solutions for the slow motion of a sphere in a viscous, incompressible liquid have been discussed from the viewpoint of the structure of the velocity field and its relation to the drag of the sphere. The problem is analysed from a different angle in this paper. It is believed that it throws more light on the *physics* of the problem.

1. Introduction

The steady motion of a sphere in a viscous incompressible liquid when the Reynolds number is small has been the subject of a number of investigations, Stokes,[1] Oseen,[2] Lamb[3] and Seth.[4, 5] The equations of motion are

$$\mu \nabla^2 \mathbf{V} = \nabla p \tag{1}$$

$$\nabla \cdot \mathbf{V} = 0. \tag{2}$$

The boundary conditions are

$$\begin{aligned} \mathbf{V} &= \mathrm{U}\boldsymbol{i} \quad \text{on } r = a \\ \mathbf{V} &= 0 \quad \text{at } \infty. \end{aligned} \tag{3}$$

A simple and instructive way of solving the problem is to put

$$\mathbf{V} = \nabla\left(\frac{\partial \phi}{\partial x}\right) + \nabla \times \mathbf{D} \tag{4}$$

where

$$\mathbf{D} = \boldsymbol{j}\left(-\frac{\partial \mathrm{X}}{\partial \mathrm{Z}}\right) + \boldsymbol{k}\left(\frac{\partial \mathrm{X}}{\partial \mathrm{Y}}\right). \tag{5}$$

This gives the pressure

$$p = p_0 + \mu \frac{\partial}{\partial x} \nabla^2 X \tag{6}$$

and the Equations (1) and (2) reduce to

$$\nabla^2 \phi = 0, \quad \nabla^4 X = 0. \tag{7}$$

The appropriate expressions for ϕ and X are

$$\phi = -\frac{Ua^3}{4} \cdot \frac{1}{r}, \quad X = \frac{3Uar}{4}. \tag{8}$$

The liquid exerts on the sphere a resultant force $6\pi\mu Ua$ along XO and therefore to maintain a uniform motion of the sphere with a velocity U along OX a force of magnitude $6\pi\mu Ua$ along OX is required.

We make the following observations:

(1) The velocity (4) can be expressed as the sum of two types of terms

$$\mathbf{V} = \mathbf{V}_{\text{rot.}} + \mathbf{V}_{\text{irrot.}} \tag{9}$$

such that

$$\nabla \times \mathbf{V} = \nabla \times \mathbf{V}_{\text{rot.}}$$

(2) The drag on the sphere comes entirely from $\mathbf{V}_{\text{rot.}}$

There is hardly anything to be desired in the solution of the above boundary value problem from the mathematical point of view. But it will surely lead to a better understanding of the physics of the problem if we could incorporate the external force $6\pi\mu Ua$ in the equations of motion (1) and then generate the flow field outside the moving sphere from this force. An interesting innovation in this direction was made by Seth[4,5] who enquired what field will be created by a force $6\pi\mu Ua$ acting at a point (say origin) and moving with a velocity U along OX. Using the analogy between the equations of linear elasticity and the Equations (1) and (2) he was able to show that this velocity field was $\mathbf{V}_{\text{rot.}}$ of (9). This velocity does not satisfy the boundary conditions (3). In order to obtain the complete solution it was necessary to superpose on $\mathbf{V}_{\text{rot.}}$ the velocity field due to a doublet of strength $-Ua^3/4\ \boldsymbol{i}$ situated at the origin and this gave the complete velocity field outside a moving sphere. For the steady motion of any solid we have the following theorem:

Seth's Theorem.—The slow viscous motion of a solid can be obtained by superposing, on an irrotational motion due to a 'generalized doublet' a solution due to a 'concentrated force' in the direction of motion.

On re-examining the problem we find that the moving sphere does not exert a concentrated force at the origin, on the other hand, it exerts on the liquid a system of stresses distributed over the surface whose resultant should be $6\pi\mu Ua$. The stress system has been incorporated in the equations of motion and it has been shown that the entire velocity field (4) can be derived from the stress system.

2. Mathematical Formulation and the General Solution

The equations of motion appropriate to the physical problem are

$$\mu\nabla^2 \mathbf{V} - \nabla p + \rho\mathbf{P} = 0 \tag{10}$$

and

$$\nabla . \mathbf{V} = 0 \tag{10 a}$$

where $\mathbf{P} = (X, Y, Z)$ is the body force per unit mass within a volume T and vanishes outside T. If we take the velocity field as

$$\mathbf{V} = \nabla \times \mathbf{A} \tag{11}$$

where $\mathbf{A} = (F, G, H)$ is a vector potential then the Equation (10 *a*) is identically satisfied. We express the body force $\mathbf{P}$ in the form

$$\mathbf{P} = \nabla\Phi + \nabla \times \mathbf{B}, \quad \mathbf{B} = (L, M, N) \tag{12}$$

where Φ, L, M, N are known functions of X, Y, Z. Taking divergence and curl of (12) respectively, we obtain

$$\nabla^2 \Phi = \nabla \cdot \mathbf{P}$$

$$\nabla^2 \mathbf{B} = \nabla \times \mathbf{P}.$$

They are Poisson's equations and their solutions are

$$\Phi = -\frac{1}{4\pi}\iiint (\mathbf{P}' \cdot \nabla R^{-1})\, dx'\, dy'\, dz' \tag{13}$$

and

$$\mathbf{B} = -\frac{1}{4\pi}\iiint (\mathbf{P}' \times \nabla R^{-1})\, dx'\, dy'\, dz' \tag{14}$$

where $\mathbf{P}'$ denotes the value of $\mathbf{P}$ at any point (x', y', z') within T, R is the distance of this point from (x, y, z) and the integration extends through T.

Substituting (11) and (12) in the equations of motion (10) and taking the divergence and curl separately we get

$$p = \rho\Phi = -\frac{\rho}{4\pi}\iiint (\mathbf{P}' \cdot \nabla R^{-1})\, dx'\, dy'\, dz' \tag{15}$$

and the equation

$$\mu\nabla^2 \mathbf{A} + \rho\mathbf{B} = 0. \tag{16}$$

Using the identity

$$\nabla^2\left(\frac{\partial R}{\partial x}\right) = 2\,\frac{\partial R^{-1}}{\partial x}$$

and the Equation (14), the solution of (16) can be written as

$$\mathbf{A} = \frac{\rho}{8\pi\mu}\iiint (\mathbf{P}' \times \nabla R)\, dx'\, dy'\, dz'. \tag{17}$$

From (11) and (17) we obtain the expressions for the components of velocity in terms of the system of stresses acting in the region T,

$$\mathbf{V} = \frac{\rho}{8\pi\mu}\iiint \left\{\frac{\mathbf{P}'}{R} + \frac{\mathbf{P}'\,.\,(\boldsymbol{r}-\boldsymbol{r}')}{R^3}\,[(\boldsymbol{r}-\boldsymbol{r}')\,.\,(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})]\right\} dx'\, dy'\, dz' \tag{18}$$

where

$$\boldsymbol{r} = (x, y, z) \qquad \text{and} \quad \boldsymbol{r}' = (x', y', z').$$

3. THE MOTION OF A SPHERE

When a sphere of radius 'a' moves along OX with a velocity U in a viscous, incompressible liquid, it exerts on the liquid a system of stresses on the surface $r = a$ whose resultant along OX is $6\,\pi\mu Ua$. In (19), putting

$$\rho X = \frac{6\pi\mu Ua}{4\pi a^2}\cdot\frac{\delta\,(r' - a)}{r'} \tag{19}$$

and

$$Y = Z = 0$$

where $r' = |\boldsymbol{r}'|$ and δ is the Dirac-delta function, we obtain

$$u = \frac{3U}{4a}\int_0^{\pi}\int_a^{\infty}\frac{\delta\,(r' - a)}{R}\sin\theta'\, r'\, dr'\, d\theta'$$

$$-\frac{3U}{8a}\frac{\partial^2}{\partial x^2}\int_0^{\pi}\int_a^{\infty} R\,\delta\,(r' - a)\, r'\sin\theta'\, dr'\, d\theta'$$

where

$$R^2 = a^2 + r^2 - 2ar\cos(\theta' - \theta)$$

and $r = |\boldsymbol{r}|$. Therefore, after integration

$$u = \frac{3Ua}{4}\left[\frac{1}{r} + \frac{1}{r^3}\left(x^2 + \frac{a^2}{3}\right) - \frac{x^2 a^2}{r^5}\right].$$

Similarly

$$v = \frac{3Ua}{4}\, xy\left(\frac{a^2}{r^5} - \frac{1}{r^3}\right),$$

and

$$w = \frac{3Ua}{4}\, xz\left(\frac{a^2}{r^5} - \frac{1}{r^3}\right).$$

The pressure is obtained from the Equation (15)

$$p = p_0 + \frac{3\mu Ua}{2} \cdot \frac{x}{r^3}.$$

This agrees with the expression (4) found by solving the boundary value problem.

4. Conclusion

The motion of a sphere can be interpreted in any of the following three ways:

(i) Solution of the boundary value problem satisfying the equations of motion (1) and continuity (2) and *two* boundary conditions for velocity on $r = a$ and the condition at infinity (3).

(ii) Solution of the equations of motion with a concentrated force $6\pi\mu Ua$ acting at the origin satisfying the equations of continuity *plus* the solution due to a doublet. The strength of the doublet is determined by satisfying only *one* boundary condition (say, the normal component of velocity on $r = a$).

(iii) The *particular* solution of the equations of motion (10) with a stress distribution (19) on $r = a$ satisfying the equation of continuity. This automatically satisfies the boundary condition (3).

5. References

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