Proc. Indian Acad. Sci. (Math. Sci.), Vol. 91, Number 1, January 1982, pp. 39–51.

Long waves in inviscid compressible atmosphere II

P L SACHDEV and V S SESHADRI

Department of Applied Mathematics, Indian Institute of Science, Bangalore 560 012, India

MS received 4 June 1980 ; revised 16 December 1981

Abstract. Solitary waves have been found in an adiabatic compressible atmosphere which, in ambient state, has winds and temperature gradient, generalizing our earlier results for the isothermal atmosphere. Explicit results are obtained for the special case of linear temperature and linear wind distributions in the undisturbed conditions. An important result of the study is that the number of possible critical speeds of the flow depends crucially on whether the maximum Richardson number (which is variable in the present example) is greater or less than 1/4.

1. Introduction

In a previous paper, (Sachdev and Seshadri [3], I hereafter) we derived the equations governing solitary and other long waves in an isothermal atmosphere with wind shear. We discussed the existence and cardinality of solitary waves *via* an eigenvalue problem for a second order ordinary differential equation. In the present paper we generalize these results to more general atmospheres.

The plan of this paper is as follows. In § 2 we formulate our problem for some general atmospheres and discuss the zeroth and first order approximations. In § 3 we specialise the results of § 2 to the case of a linearly increasing temperature profile and derive a model equation. In § 4, the results of § 3 are applied to linearly increasing wind profile. Finally, we give the conclusions of our study in § 5.

2. Formulation and first order analysis

The more general atmosphere is again assumed to be adiabatic and inviscid, extending from the plane $y = 0$ to infinity in the vertical direction y. In its equilibrium state, it is assumed to have pressure and density distributions that are strictly decreasing functions of y, tending to zero as $y \to \infty$ in such a manner that the absolute temperature is a non-decreasing function of y. Further, an ambient shear flow is also assumed to exist, given by $u_0(y)$, the component of velocity in the horizontal direction.

The equations governing the propagation of atmospheric gravity waves are

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0, \tag{1a}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}, \tag{1b}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}-\rho g, \tag{1c}$$

$$\rho\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)-\gamma p\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right)=0, \tag{1d}$$

where p is the pressure, ρ is the density, (u, v) is the velocity vector, and g is the acceleration due to gravity.

A wave of permanent form moving with a velocity $(c, 0)$ is assumed to have been generated by some disturbance. This, for example, could be the result of an unsteady motion created much earlier. It is known from the theory of water waves that the propagation speed c of long water waves of permanent form is close to some critical value c^*. The purpose of the present analysis is to find the critical speeds of these waves for the model under study and also the equations governing their propagation.

As in *I*, equation (1) may be rendered non-dimensional by

$$X=x/H,\ Y=y/H,\ T=tc/H, R=\rho/\rho_g,$$
$$P=p/(\rho_g c^2), U=u/c,\ V=v/c\ ;$$

the small parameter for the present problem is provided by

$$\epsilon=L-\lambda, \tag{2}$$

where

$$L=(gH/c^2), \text{ and } \lambda=(gH/c^{*2}),$$

and the stretchings are defined by

$$\xi=\epsilon^{1/2}(X-T), \tag{3a}$$

$$\bar{V}=\epsilon^{1/2}V,\ \bar{U}=U-1. \tag{3b}$$

The notations here are the same as in *I*.

In terms of the new variables the undisturbed conditions, denoted by the suffix *e*, satisfy

$$\bar{U}_e(Y)=[u_0(YH)/c-1],\ \bar{V}_e(Y)=0, \tag{4}$$

$$\frac{dp_e}{dY}=-\lambda R_e(Y). \tag{5}$$

The governing equations for $\bar{U}, \bar{V}, R$ and P, the boundary conditions at the ground and the perturbation scheme for these functions are the same as 7 (a)-7 (d), (8), (9) and (10) in I.

To the zeroth order, we obtain

$$\bar{U}_0(Y)=\bar{U}_e(Y),\ R_0(Y)=R_e(Y), \tag{6}$$

while $P_0(Y)$ is obtained by solving

$$\frac{dP_0}{dY}=-LR_0(Y), \tag{7}$$

with the initial condition $P_0(0) = L$. By our assumption, $P_0(Y)$ and $R_0(Y)$ are strictly monotonic functions of Y, which decrease to zero as $Y \to \infty$, and the ratio P_0/R_0, which is proportional to the absolute temperature, is a nondecreasing function of Y. The equations for the first order approximations and the boundary conditions are again given by (12a)–(12d) and (13) in I.

Besides,

$$\bar{U}_1(\pm\infty, Y) = 0, \; \bar{V}_1(\pm\infty, Y) = 0,$$

$$P_1(\pm\infty, Y) = P_{e_1}(Y), \; R_1(\pm\infty, Y) = 0, \tag{8}$$

where $P_{e_1}(Y)$ is obtained by solving

$$\frac{dP_{e1}}{dY} = R_{e0}(Y), \tag{9}$$

with the initial condition $P_{e_1}(0) = -1$.

Eliminating $\bar{U}_1$, $\bar{V}_1$, and R_1 from their perturbation equations we get a single differential equation for P_1:

$$\frac{\partial}{\partial Y}\left[\left(R_0\frac{\partial P_1}{\partial \xi} + \frac{\gamma P_0}{L}\frac{\partial^2 P_1}{\partial \xi\,\partial Y}\right)\Big/\left(\frac{dP_0}{dY} - \frac{\gamma P_0}{R_0}\frac{dR_0}{dY}\right)\right]$$

$$+ \frac{1}{L}\frac{\partial^2 P_1}{\partial \xi\,\partial Y} + \frac{1}{\bar{U}_0^2}\frac{\partial P_1}{\partial \xi} = 0. \tag{10}$$

We exclude critical levels occurring in the flow region, the function $\bar{U}_0(Y) = [u_0(YH)/c - 1]$ is therefore non-zero for $Y \in [0, \infty)$. The factor $[dP_0/dY - (\gamma P_0/R_0)\, dR_0/dY]$ is also not equal to zero for $Y \in [0, \infty)$ by our assumptions on $P_0(Y)$ and $R_0(Y)$.

The boundary condition $\bar{V}_1(\xi, 0) = 0$ can be shewn to be equivalent to

$$\frac{\gamma P_0}{LR_0}\frac{\partial^2 P_1}{\partial \xi\,\partial Y} + \frac{\partial P_1}{\partial \xi} = 0 \text{ at } Y = 0. \tag{11}$$

Further, we impose the condition that the first order terms do not grow larger in amplitude, compared to their zeroth order counterparts. This requires

$$\left|\frac{P_1}{P_0}\right| = 0(1) \text{ for all } Y. \tag{12}$$

Equation (10) can be transformed to

$$\frac{\partial^2 \bar{H}}{\partial \eta^2} + \left[LR_0\Big/\left(\gamma \bar{U}_0\frac{d\eta}{dY}\right)\right]\bar{H} = 0, \tag{13}$$

where

$$\bar{H} = \frac{1}{R_0}\frac{\partial P_1}{\partial \xi}, \tag{14}$$

and

$$\eta = \int_0^Y \left(\frac{1}{R_0 P_0}\frac{dP_0}{dY'} - \frac{\gamma}{R_0^2}\frac{dR_0}{dY'}\right) dY'. \tag{15}$$

If T denotes the absolute temperature normalized by $p_g/(\rho_g R_g)$ where R_g is the universal gas constant, we have

$$P/R = \lambda T. \tag{16}$$

If T_0 denotes the normalised temperature to the zeroth order, we have

$$\eta = \int_0^Y \frac{1}{R_0}\left[\frac{1}{T_0}\frac{dT_0}{dY'} - \frac{(\gamma-1)}{R_0}\frac{dR_0}{dY'}\right] dY'. \tag{17}$$

By our assumptions on $P_0(Y)$, $R_0(Y)$ and $T_0(Y)$, (17) gives

$$\eta(Y) > -(\gamma-1)\int_0^Y \frac{1}{R_0^2}\frac{dR_0}{dY'}\,dY' = (\gamma-1)\left[\frac{1}{R_0(Y)} - 1\right]. \tag{18}$$

This implies that $\eta \to \infty$ as $Y \to \infty$. Moreover, the integrand in (17) is strictly positive. We, therefore, conclude that the transformation (15) is a one-one mapping of $Y \in [0, \infty)$ onto $\eta \in [0, \infty)$.

Applying a theorem due to Hille and Wintner [2], we find that (13) has a solution $H_1(\eta)$, unique up to a multiplicative constant, such that

$$H_1(\eta) = 0(1),\ H_1'(\eta) = 0(1/\eta) \text{ as } \eta \to \infty, \tag{19}$$

if and only if

$$\int_0^\infty \left[R_0\eta\Big/\left(\bar{U}_0^2\frac{d\eta}{dY}\right)\right] d\eta < \infty,$$

that is,

$$\int_0^\infty [R_0\eta/\bar{U}_0^2(Y)]\,dY < \infty, \tag{20}$$

where η is given by (17). There is also another solution $H_2(\eta)$, non-unique, such that

$$H_2(\eta) = 0(\eta),\ H_2'(\eta) = 0(1) \text{ as } \eta \to \infty. \tag{21}$$

Assuming that (20) holds, we obtain

$$\frac{1}{R_0}\frac{\partial P_1}{\partial \xi} = \frac{\partial A_1}{\partial \xi} H_1(\eta) + \frac{\partial A_2}{\partial \xi} H_2(\eta), \tag{22}$$

where $A_1'(\xi)$ and $A_2'(\xi)$ are to be found from the boundary conditions. Integrating (22) with respect to ξ and using the conditions at $|\xi| = \infty$, we get.

$$\frac{P_1}{R_0} = [A_1(\xi) - A_1(\infty)]\,H_1(\eta) + [B_1(\xi) - B_1(\infty)]\,H_2(\eta) + \frac{P_{1_1}}{R_0}. \tag{23}$$

To satisfy condition (12), we put $[B_1(\xi) - B_1(\infty)] = 0$.

Assuming $A_1(\infty) = 0$, we get

$$P_1 = R_0(Y)\,A_1(\xi)\,H_1(\eta) + P_{1\infty}(Y). \tag{24}$$

From the asymptotic behaviour of $H_1(\eta)$, given in (19), which arises as a result of imposing the condition (20) on the function $\bar{U}_0(Y)$, the right hand side of (24) is $0(R_0)$ as $Y \to \infty$. To satisfy the condition (12), however, it is sufficient that it is $0(P_0)$ as $Y \to \infty$. Since (P_0/R_0), which is proportional to the temperature, is assumed to be a non-decreasing function of Y we conclude that the condition (20) is only a sufficient condition for (12) to be true. For the isothermal case given in I, however, the condition (20) is also necessary for (12) to be satisfied. In this case it is easy to see that

$$\eta = (\gamma - 1)\,[\exp\,(Y) - 1]$$

and therefore (20) becomes

$$\int_0^\infty \frac{dY}{\bar{U}_0^2(Y)} < \infty,$$

which is the same as the condition (21) of I.

Once the expression for P_1 has been obtained, those for R_1, $\bar{V}_1$, $\bar{U}_1$ can be obtained from first order perturbation equations and the equilibrium conditions (4) and (5).

The vanishing of $\bar{V}_1(\xi, 0)$ gives

$$\frac{\partial P_1}{\partial \xi} + \frac{\gamma P_0}{L R_0}\frac{\partial^2 P_1}{\partial \xi\,\partial Y} = 0 \text{ at } Y = 0.$$

This, with the help of (24), becomes

$$H_1(0) - \gamma H_1'(0) = 0. \tag{25}$$

As was done for the isothermal case, the critical speeds in this case are obtained by solving (25).

The function $A_1(\xi)$, occurring in (24), is determined by considering the second order terms. However, if we continue the analysis for the general atmospheres it soon becomes intractable. We shall therefore restrict ourselves to the special case of a linearly increasing temperature distribution in the equilibrium state.

3. Linearly increasing temperature profile

Here we consider the special case of a linearly increasing temperature distribution in the equilibrium state, given by

$$T_e = 1 + \alpha Y, \tag{26}$$

where α is a positive constant. Then (26) together with (5) and the perfect gas relation gives

$$P_e = \lambda(1 + \alpha Y)^{-\mu}, \tag{27}$$

and

$$R_e = (1 + \alpha Y)^{-(1+\mu)} \tag{28}$$

where the definition $\mu = 1/\alpha$ has been introduced for convenience in writing. For Y fixed, the limit $\alpha \to 0$ gives the isothermal distributions of P_e and R_e of *I*.

The equation for P_1, corresponding to (10), may now be obtained as

$$\frac{\partial^3 P_1}{\partial \xi \partial Y^2} + \frac{(1+2\alpha)}{(1+\alpha Y)} \frac{\partial^2 P_1}{\partial \xi \partial Y} + \left[\frac{L(\alpha\gamma + \gamma - 1)}{\gamma \bar{U}_0^2 (1+\alpha Y)}\right] \frac{\partial P_1}{\partial \xi} = 0. \tag{29}$$

The boundary condition $\bar{V}_1(\xi, 0) = 0$ becomes

$$\gamma \frac{\partial^2 P_1}{\partial \xi \partial Y} + \frac{\partial P_1}{\partial \xi} = 0 \text{ at } Y = 0. \tag{30}$$

In this case, η is given by

$$\eta = \mu [(\gamma - 1) + \gamma\mu] [(1 + \alpha Y)^{1+\mu}], \tag{31}$$

and the condition (20) becomes

$$\int_0^\infty \frac{dY}{\bar{U}_0^2(Y)} < \infty. \tag{32}$$

Under the condition (32), we have two linearly independent solutions, $f(Y)$ and $g(Y)$, of the differential equation

$$\frac{d^2 \bar{F}}{dY^2} + \frac{(1+2\alpha)}{(1+\alpha Y)} \frac{d\bar{F}}{dY} \left[\frac{L(\alpha\gamma + \gamma - 1)}{\gamma \bar{U}_0^2 (1+\alpha Y)}\right] \bar{F} = 0, \tag{33}$$

such that

$$\begin{aligned} &f(Y) = 0\,[(1+\alpha Y)^{-(1+\mu)}],\ f'(Y) = 0\,[(1+\alpha Y)^{-(2+\mu)}], \\ &g(Y) = 0\,(1),\ g'(Y) = 0\,[(1+\alpha Y)^{-1}] \text{ as } Y \to \infty. \end{aligned} \tag{34}$$

The solution for $P_1(\xi, Y)$ becomes

$$P_1(\xi, Y) = A_1(\xi) f(Y) - (1+\alpha Y)^{-\mu}, \tag{35}$$

where we have used the equilibrium condition on P_1.

Substituting for $P_1(\xi, Y)$ from (35) in (30) we get the following equation for the critical speeds

$$f(0) + \gamma f'(0) = 0. \tag{36}$$

The first order quantities are found to be

$$P_1(\xi, Y) = A_1(\xi) f(Y) - (1+\alpha Y)^{-\mu}, \tag{37}$$

$$R_1(\xi, Y) = -A_1(\xi) f'(Y)/L, \tag{38}$$

$$\bar{V}_1(\xi, Y) = -\frac{A_1'(\xi)\,[f(Y) + \gamma(1+\alpha Y) f'(Y)]\,(1+\alpha Y)^{1+\mu}\,\bar{U}_0}{[(\gamma - 1)L + \gamma L\alpha]}, \tag{39}$$

and

$$\bar{U}_1(\xi, Y) = A_1(\xi)\left[\frac{\bar{U}_0\,[f(Y) + \gamma(1+\alpha Y) f'(Y)]\,(1+\alpha Y)^{1+\mu}}{(\gamma - 1)L + \gamma L\alpha} - \frac{(1+\alpha Y)^{1+\mu} f(Y)}{\bar{U}_0}\right]. \tag{40}$$

The equations governing the second order terms are the same as in *I* with $G_1 - G_4$ defined in the appendix.

The boundary condition $\bar{V}(\xi, 0) = 0$ gives

$$\bar{V}_2(\xi, 0) = 0. \tag{41}$$

The equation for P_2 may be derived by eliminating $\bar{U}_2$, $\bar{V}_2$ and R_2:

$$\frac{\partial^3 P_2}{\partial Y^2 \partial \xi} + \frac{(1 + 2a)}{(1 + aY)} \frac{\partial^2 P_2}{\partial Y \partial \xi} + \left[\frac{L(\gamma - 1 + \gamma a)}{\gamma(1 + aY)\bar{U}_0^2(Y)}\right] \frac{\partial P_2}{\partial \xi} = G_5, \tag{42}$$

here

$$G_5 = -\left[\frac{L(\gamma - 1 + \gamma a)}{\gamma(1 + aY)}\right]\left(\frac{U_0 G_1 - G_2}{\bar{U}_0^2}\right) + \left[\frac{\gamma a + \gamma - 1}{\gamma(1 + aY)}\right]\frac{\partial G_3}{\partial \xi}$$

$$+ \frac{1}{\gamma(1 + aY)}\frac{\partial}{\partial Y}\left[\gamma(1 + aY)\frac{\partial G_3}{\partial \xi} + \frac{(1 + aY)^{1+\mu} G_4}{\bar{U}_0(Y)}\right]. \tag{43}$$

The general solution of (42) is given by

$$\frac{\partial P_2}{\partial \xi} = B_1(\xi) f(Y) + B_2(\xi) g(Y) + g(Y) \int_0^Y \frac{G_5(\xi, Y') f(Y')}{W(f, g)(Y')} dY'$$

$$- f(Y) \int_0^Y \frac{G_5(Y') g(Y')}{W(f, g)(Y')} dY', \tag{44}$$

where $B_1(\xi)$ and $B_2(\xi)$ are to be found from the boundary conditions. We shall now analyse the various terms in (44) for large Y. The expression for G_5 given in (43) involves G_1, G_2, G_3 and G_4. As Y becomes large, one may check that the term which contributes most to G_5 is proportional to $\bar{U}_0^2 f(Y)/(1 + aY)$ and this, from (34), is of the order of $\bar{U}_0^2(1 + aY)^{-(2+\mu)}$ as $Y \to \infty$. On the other hand $W(f, g)(Y)$, the Wronskian of the two linearly independent solutions $f(Y)$ and $g(Y)$ of (33), is easily seen to be proportional to $(1 + aY)^{-(2+\mu)}$. Thus, as $Y \to \infty$, the leading terms in $G_5(\xi, Y) f(Y)/[W(f, g)(Y)]$ and $G_5(\xi, Y) g(Y)/[W(f,g)(Y)]$ would be proportional to $\bar{U}_0^2(Y)(1 + aY)^{-(1+\mu)}$ and $\bar{U}_0^2(Y)$ respectively. Assuming now that $\bar{U}_0(Y)$ is such that

$$\int_0^\infty \bar{U}_0^2(Y')(1 + aY')^{-(1+\mu)} dY' < \infty, \tag{45}$$

we find $\partial P_2/\partial \xi = 0(1)$ as $Y \to \infty$ unless we choose

$$B_2(\xi) = -\int_0^\infty \frac{G_5(\xi, Y') f(Y')}{W(f, g)(Y')} dY'. \tag{46}$$

We take this to be the case and obtain

$$\frac{\partial P_2}{\partial \xi} = B_1(\xi) f(Y) + g(Y) \int_\infty^Y \frac{G_5(\xi, Y') g(Y')}{W(f, g)(Y')} dY'$$

$$- f(Y) \int_0^Y \frac{G_5(\xi, Y') g(Y')}{W(f, g)(Y')} dY'. \tag{47}$$

It is easy to see from the asymptotic behaviour of the integrands in (47), given earlier, that $\partial P_2/\partial\xi$ is still not of the same order as $\partial P_1/\partial\xi$ as $Y \to \infty$. For, if we take a linear ambient velocity profile $u_0(y) = \beta y$, so that $\bar{U}_0(Y) = (\beta HY/c - 1)$, and $0 < \alpha < \frac{1}{2}$, we see that the conditions (32) and (45) are satisfied, and

$$\left| \frac{\partial P_2}{\partial \xi} \Big/ \frac{\partial P_1}{\partial \xi} \right| = 0\,(Y^3) \text{ as } Y \to \infty. \tag{48}$$

Consequently, if we take P_2 as given by the integral of (47) with respect to ξ, the perturbation expansion for P will be valid only for a limited range of Y; the range will depend on the ambient velocity profile. In the above example of a linearly increasing ambient velocity profile the perturbation expansion for P will be valid, to first order, for all Y which satisfy $Y^3 = 0\,(1/\epsilon)$. In our subsequent analysis we shall restrict our discussion to distances Y, determined by the ambient wind, for which the perturbation expansions remain uniformly valid.

The second order term, $\bar{V}_2$, is then obtained as

$$\begin{aligned}\frac{\bar{V}_2}{\bar{U}_0} = &-\frac{(1+\alpha Y)^{1+\mu}}{[(\gamma-1)+\gamma\alpha]\,L}\Bigg[\{f(Y)+\gamma(1+\alpha Y)f'(Y)\}\Bigg\{B_1(\xi) \\ &- \int_0^Y \frac{G_5\,g(Y')}{W(f,g)(Y')}\,dY'\Bigg\} + \{g(Y)+\gamma(1+\alpha Y)\,g'(Y)\} \\ &\times \int_\infty^Y \frac{G_5 f(Y')}{W(f,g)(Y')}\,dY'\Bigg] + \frac{(1+\alpha Y)^{2+2\mu}\,G_4(\xi, Y)}{L\,[(\gamma-1)+\gamma\alpha]\bar{U}_0} \\ &+ \frac{\gamma(1+\alpha Y)^{2+\mu}}{L\,[\gamma-1)+\gamma\alpha]}\,\frac{\partial G_3}{\partial \xi}.\end{aligned} \tag{49}$$

Applying the condition (41) we get

$$\begin{aligned}&\left[\frac{g(0)+\gamma g'(0)}{d}\right]\int_0^\infty [(1+\alpha Y)^{2+\mu} f(Y)\,G_5]\,dY \\ &+ \frac{G_4(\xi,0)}{\bar{U}_0(0)} + \gamma\frac{\partial G_3}{\partial \xi}(\xi,0) = 0,\end{aligned} \tag{50}$$

where we have made use of the condition (36) and where $d = W(f, g)(0)$. We now specify the functions $f(Y)$ and $g(Y)$ by requiring $d = 1$ and $g(0) + \gamma g'(0) = 1$. Equation (50), after some lengthy calculations, reduces to

$$m_2\frac{\partial A_1}{\partial \xi} + m_1 A_1\frac{\partial A_1}{\partial \xi} + m_0\frac{\partial^3 A_1}{\partial \xi^3} = 0, \tag{51a}$$

where

$$m_0 = -\int_0^\infty \frac{(1+\alpha Y)^{1+\mu}\,\bar{U}_0^2(Y)\,[f+\gamma\alpha(1+\alpha Y)f']^2}{\gamma L\,[(\gamma-1)+\gamma\alpha]}\,dY, \tag{51b}$$

$$m_1 = \frac{(\gamma - 1)}{\gamma L} f^2(0) + \int_0^\infty (1 + \alpha Y)^{2+\mu} f(Y) \left[B_1 f^2 + B_2 f f' + B_3 (f')^2\right] dY, \tag{51c}$$

$$m_2 = \frac{[(\gamma - 1) + \gamma \alpha]}{\gamma} \int_0^\infty \frac{(1 + \alpha Y)^{1+\mu} f^2(Y)}{\bar{U}_0^2(Y)} dY, \tag{51d}$$

$$B_1 = -\frac{(3 + \alpha)(1 + \alpha Y)^{\mu - 1}}{\gamma \bar{U}_0^2} + \frac{4 \bar{U}_0' (1 + \alpha Y)^\mu}{\gamma \bar{U}_0^3} - \frac{L[\gamma \alpha + (\gamma - 1)](1 + \alpha Y)^\mu}{\gamma \bar{U}_0^4} + \frac{(1 + \alpha)(1 + \alpha Y)^{\mu - 2}}{[(\gamma - 1) + \alpha \gamma] L},$$

$$B_2 = \frac{2(1 + \alpha Y)^{\mu - 1}}{L[(\gamma - 1) + \gamma \alpha]} + \frac{4 \bar{U}_0' (1 + \alpha Y)^{1+\mu}}{\bar{U}_0^3} + \frac{(2\alpha - 4/\gamma)(1 + \alpha Y)^\mu}{\bar{U}_0^2},$$

and

$$B_3 = \left\{\frac{1 + 2\alpha^2 \gamma + \alpha \gamma - 2\alpha}{L[(\gamma - 1) + \gamma \alpha])}\right\} (1 + \alpha Y)^\mu - \frac{(1 + \alpha Y)^{1+\mu}}{\bar{U}_0^2}.$$

It is easy to see that, as $\alpha \to 0$, (51a) reduces to (43a) with its co-efficients given by (43b)-(43d) of I. The latter corresponds to the equation for solitary and cnodial waves for an isothermal atmosphere. As in the isothermal case, the solitary and cnoidal wave solutions of (51a) may be written out. We may also show that the speed of the solitary waves, c, which is close to a critical speed c^*, is such that $|c| > |c^*|$.

Equations (37)-(40) give the first order solution with $A_1(\xi)$ satisfying (51a). The quantities, P_1, R_1, $\bar{U}_1$ and $\bar{V}_1$, are of the same order, for all Y, as their zeroth order counterparts. However, the second order solutions grow in relation to the first order quantities as $Y \to \infty$, the growth depending on the ambient velocity profile. Consequently, the solutions given by (37)-(40) would remain uniformly valid only in a limited range of Y provided by the ambient shear flow.

4. An example

We now apply the results of § 3 to the special case

$$u_0(y) = \beta y, \; \beta > 0. \tag{52}$$

The critical speeds can be obtained by solving the differential equation

$$\frac{d^2 \bar{F}}{dY^2} + \frac{(1 + 2\alpha)}{(1 + \alpha Y)} \frac{d\bar{F}}{dY} + \frac{Ri_m}{(1 + \alpha Y)[Y - c^*/(\beta H)]^2} \bar{F} = 0, \tag{53}$$

together with the conditions

$$\bar{F} = 0 \, [1/(1 + \alpha Y)^{1+\mu}] \text{ as } Y \to \infty, \tag{54a}$$

$$\bar{F}(0) + \gamma \bar{F}'(0) = 0, \tag{54b}$$

where

$$Ri_m = (\alpha\gamma + \gamma - 1)\, g/\gamma H\beta^2, \tag{55}$$

is the maximum Richardson number. Equation (53) follows from (33) and (52). We preclude critical levels, thus requiring that the two singularities of (53) do not lie in the flow field.

Equation (53) can be transformed to the hypergeometric equation

$$Z(1 - Z)\frac{d^2 G}{dZ^2} + [2k - \{1 + (k + 1 + \mu) + k\}\, Z]\frac{dG}{dZ}$$
$$- k(k + 1 + \mu)\, G = 0, \tag{56}$$

by introducing the new variables

$$G(Z) = (Y - c^*/(\beta H))^{-k}\, \bar{F}(Y), \tag{57}$$

$$Z = -\alpha (Y - c^*/(\beta H))/\delta \tag{58}$$

where

$$k = 1/2 \pm (1/4 - Ri_m/\delta)^{1/2}, \tag{59}$$

and $\quad \delta = 1 + \alpha c^*/(\beta H)$.

The boundary conditions (54a, b) become

$$G = 0\,(1/[(1 + \alpha Y)^{1+\mu}\, (Y - c^*/(\beta H))^k]) \text{ as } Y \to \infty, \tag{60}$$

and

$$\left[\left(\frac{1 - \delta}{\delta}\right) + \gamma k\right] G\left(\frac{\delta - 1}{\delta}\right) + \gamma\left(\frac{\delta - 1}{\delta}\right) G'\left(\frac{\delta - 1}{\delta}\right) = 0. \tag{61}$$

The solution of (56) which has the asymptotic behaviour given by (60) is

$$G(Z) = (Y - c^*/(\beta H))^{-(1+\mu+k)}\, F(a', b';\ c';\ Z^{-1}), \tag{62}$$

where

$$a' = 3/2 + \mu + ik_2,\ b' = 3/2 + \mu - ik_2,\ c' = 2 + \mu,$$

and $\quad k_2 = (Ri_m/\delta - 1/4)^{1/2}$.

The nature of the eigenvalues is obtained by substituting $G(Z)$ from (62) into (61) to get

$$\bar{G}(\delta) = [(1 - \delta)^2 - \gamma\alpha (1 + 1/\alpha)(1 - \delta)]\, F(a', b';\ c';\ \delta/(\delta - 1))$$
$$+ \gamma\alpha\delta F'(a', b';\ c';\ \delta/(\delta - 1)) = 0, \tag{63}$$

where the prime over F, the hypergeometric function, denotes the derivative with respect to the last argument. Since we require c^* to be negative, to avoid critical levels, we seek the roots δ of (63) which lie between 1 and $-\infty$. We also observe that the function $\bar{G}(\delta)$, given in (63), is analytic in δ for δ belonging to the range $(-\infty, 1)$ and, consequently, the limit points of the zeros of $\bar{G}(\delta)$, if any, could only be $\delta = -\infty$ or $\delta = 1$. Three distinct cases arise depending on whether the maximum Richardson number, Ri_m, is less than, equal to, or greater than 1/4. The nature of the eigenvalues can be predicted by knowing the asymptotic behaviour of the hypergeometric function for the above three cases and the analysis is similar to the one presented in paper I, the details may be found in [4]. The number of roots of $\bar{G}(\delta)$ in (63) is finite when Ri_m is less than or equal to 1/4, while there is an infinity of them in the neighbourhood of $\delta = 1$ for Ri_m greater than 1/4.

While these results are obtained from the eigenvalue problem, our theory, as in the isothermal case, does not include solitary waves travelling near critical speeds c_n^* for which n is arbitrarily large. This is because for large n, c_n^*, which is negative, is very close to zero and the origin $Y = 0$ comes close to the critical level $Y = c^*/(\beta H)$, which lies below the ground. It is easily verified that, for large n, R_1 and $\bar{U}_1$ at the origin, given by (38) and (40) become large. Hence the spectrum of possible critical speeds, as obtained from the eigenvalue problem, has to be restricted so that the critical level is not in the close vicinity of the flow field.

5. Conclusions

In the present paper we have extended the results of I to more general atmospheres. We have considered an atmosphere in which, initially, the pressure and density were strictly decreasing with Y and the temperature was a non-decreasing function of Y. With the help of a simple transformation, a growth condition on the ambient wind profile was obtained which ensured that the first order terms are of the same order as the zeroth order ones. The problem of finding the critical speeds in this case also reduced to solving an eigenvalue problem for a second order ordinary differential equation. Further analysis in this general case, however, was found to become too involved and therefore, for simplicity, a linearly increasing temperature profile was assumed. This corresponds to assuming the atmosphere to be a pure thermosphere. Results similar to those stated in the isothermal case were obtained. Also, for a linear wind profile, the first order equations posed an eigenvalue problem for the Gauss's hypergeometric equation whose eigenvalues give the possible critical speeds of the flow. In this case the Richardson number varied with height and decreased to zero as Y increased to infinity. We have shown that if the maximum Richardson number, Ri_m, is greater than 1/4, then the number of possible critical speeds is infinite. On the other hand, if $Ri_m \leqslant 1/4$, only a finite number of them were found to exist. This change in the behaviour of the number of critical speeds with the maximum Richardson number can be explained on the basis of the behaviour of the solutions near the critical level, as for the isothermal case. Also, our comments made in the isothermal case, on the extent of the validity of our solution and the finiteness of the spectrum of the critical speeds, if we wish to stay away from the critical levels, remain valid in this case too. Therefore, generally, the limited range of validity of the solution as well as the need to keep the critical level out of the domain of the solution both inhibit the cardinality of the solitary waves so that even for $Ri_m > 1/4$ we shall have only a small number of possible solitary waves in the atmosphere.

Acknowledgement

This work was in part carried out under the University Grants Commission project 'Wave Phenomena' No. 23-884/78 CSRIF-Code No. 001/Maths/78.

Appendix

In this appendix we give the G_i's for the case of linear temperature profile. Writing $\eta = (1 + a Y)$, and using the notations as in *I*, we have,

$$G_1 = \left[\frac{\eta^{\mu}}{\gamma L \bar{U}_0} f^2 + \frac{\eta^{\mu}}{L^2(\gamma - 1 + \gamma a)} (a \bar{U}_0 + \eta \bar{U}_0') ff' + \frac{\eta^{1+\mu}}{L^2(\gamma - 1 + \gamma a)} (\gamma \eta \bar{U}_0' + (\gamma - 1 + 2\gamma a) \bar{U}_0) f'^2 \right] A_1 \frac{\partial A_1}{\partial \xi},$$

$$\begin{aligned} G_2 = \Bigg[\Bigg\{ &- \left(\frac{\bar{U}_0'}{L(\gamma - 1 + \gamma a)} - \frac{1}{\bar{U}_0}\right)^2 \eta^{1+\mu} \\ &+ \frac{\bar{U}_0}{L(\gamma - 1 + \gamma a)} \left(\frac{(a+1)\eta^{\mu} \bar{U}_0'}{L(\gamma - 1 + \gamma a)} + \frac{\eta^{1+\mu} \bar{U}_0''}{L(\gamma - 1 + \gamma a)} - \frac{(a+1)\eta^{\mu}}{\bar{U}_0}\right)\Bigg\} f^2 + \Bigg\{\frac{\bar{U}_0}{L(\gamma - 1 + \gamma a)} \left(\frac{2\gamma \eta^{2+\mu} \bar{U}_0''}{L(\gamma - 1 + \gamma a)}\right. \\ &\left. + \frac{(\gamma + 1 + \gamma a)\eta^{1+\mu} \bar{U}_0'}{L(\gamma - 1 + \gamma a)} - \frac{2\gamma(a+1)\eta^{1+\mu}}{\bar{U}_0}\right) \\ &- 2\left(\frac{\bar{U}_0'}{L(\gamma - 1 + \gamma a)} - \frac{1}{\bar{U}_0}\right)\left(\frac{\gamma \eta^{2+\mu} \bar{U}_0'}{L(\gamma - 1 + \gamma a)}\right)\Bigg\} ff' \\ &+ \Bigg\{\frac{\gamma \bar{U}_0}{L(\gamma - 1 + \gamma a)} \left(\frac{\eta^{2+\mu} \bar{U}_0'}{L(\gamma - 1 + \gamma a)} + \frac{\gamma \eta^{3+\mu} \bar{U}_0''}{L(\gamma - 1 + \gamma a)} \right. \\ &\left. - \frac{\eta^{2+\mu}}{\bar{U}_0} - \frac{\gamma^2 \eta^{3+\mu}}{L^2(\gamma - 1 + \gamma a)^2}\right)\Bigg\} f'^2 \Bigg] A_1 \frac{\partial A_1}{\partial \xi}, \end{aligned}$$

$$G_3 = -\frac{A_1 f'}{L} + \frac{\bar{U}_0^2 (f + \gamma \eta f')}{L(\gamma - 1 + \gamma a)} \frac{\partial^2 A_1}{\partial \xi^2},$$

$$\begin{aligned} G_4 = -\left(\frac{f \bar{U}_0}{L \eta^{1+\mu}}\right) \frac{\partial A_1}{\partial \xi} &+ \frac{1}{L(\gamma - 1 + \gamma a)} \Bigg[\left\{\frac{\gamma(a+1)\bar{U}_0}{\eta} - \bar{U}_0'\right\} f^2 \\ &+ \{(1 + \gamma - \gamma a)\bar{U}_0 - 2\gamma\eta \bar{U}_0'\} ff' + \{2\gamma \eta \bar{U}_0 - \gamma^2 \eta^2 \bar{U}_0' \\ &- \gamma^2 (1 + 2a) \eta \bar{U}_0\} f'^2 \Bigg] A_1 \frac{\partial A_1}{\partial \xi}, \end{aligned}$$

and

$$\begin{aligned} G_5 = \left[\frac{(\gamma - 1 + \gamma a)}{\gamma \eta \bar{U}_0^2} f\right] \frac{\partial A_1}{\partial \xi} &+ (c_1 f^2 + c_2 f f' + c_3 f'^2) A_1 \frac{\partial A_1}{\partial \xi} \\ &+ \frac{1}{L(\gamma - 1 + \gamma a)} \Bigg[\left\{\frac{(\gamma - 1 + 2\gamma a)\bar{U}_0^2}{\gamma \eta} + 2a \bar{U}_0 \bar{U}_0' - 1\right\} f \\ &+ \{\gamma a + 2\gamma \eta \bar{U}_0 \bar{U}_0'\} f' \Bigg] \frac{\partial^3 A_1}{\partial \xi^3}, \end{aligned}$$

where c_1, c_2 and c_3 are given by

$$c_1 = \frac{(a+1)\,\eta^{\mu-2}}{L(\gamma-1+\gamma a)} - \frac{(a+3)\,\eta^{\mu-1}}{\gamma \bar{U}_0^2} + \frac{4\eta^{\mu}\,\bar{U}_0'}{\gamma \bar{U}_0^3} - \frac{L(\gamma-1+\gamma a)}{\gamma \bar{U}_0^4}\,\eta^{\mu},$$

$$c_2 = \frac{2\eta^{\mu-1}}{L(\gamma-1+\gamma a)} + \frac{(2a-4/\gamma)\,\eta^{\mu}}{\bar{U}_0^2} + \frac{4\eta^{1+\mu}\,\bar{U}_0'}{\bar{U}_0^3},$$

and

$$c_3 = \frac{(1+2a^2\gamma + a\gamma - 2a)\eta^{\mu}}{((\gamma-1)+\gamma a)\,L} - \frac{\eta^{1+\mu}}{\bar{U}_0^2}.$$

References

[1] Abramowitz M and Stegun I A (eds) 1965 *Handbook of mathematical functions* (New York : Dover Publications)

[2] Hille E 1969 *Lectures on ordinary differential equations* (Massachusetts : Addison-Wesley) pp. 428–432

[3] Sachdev P L and Seshadri V S 1979 *Phys. Fluids* 22 60

[4] Seshadri V S 1977 Some analytic approach to nonlinear wave propagation in non-uniform media Ph.D. Thesis, Indian Institute of Science, Bangalore