AN EXACT SIMILARITY SOLUTION IN RADIATION-GAS-DYNAMICS

BY S. ASHRAF AND P. L. SACHDEV

(*Department of Applied Mathematics, Indian Institute of Science, Bangalore, India*)

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ABSTRACT

We have found an exact similarity solution of the point explosion problem in the case when the total energy of the shock wave that is produced is not constant but decreases with time and when the loss due to radiation escape is significant. We have compared the results of our exact solution with those of exact numerical solutions of Elliot and Wang and have explained the cause why our solution differs from theirs in certain aspects.

1. INTRODUCTION

SIMILARITY solutions in radiation-gas-dynamics have been given by Marshak,[1] Elliot[2] and Wang[3] in which the flow is headed by a shock wave. Elliot considered the explosion problem, solved earlier by Taylor[4] by introducing the radiation flux in its diffusion approximation. Wang[3] has discussed ' piston problem ' with radiation energy transfer in the thick limit, thin limit and also the general case with the idealised ' two direction ' approximation. In the present paper, we find an exact solution of the propagation of a strong spherical shock in a medium in which the density varies as R^{β} where R is the radial distance and β is a negative number and such that radiation flux is important. We assume a similarity form for radiation flux and make use of the " product solutions " of McVittie [5] to evaluate it. The radiation pressure and radiation energy are considered to be small in comparison to material pressure and energy respectively and therefore only radiation flux is taken into account. Our solution would give realistic results when the explosion, for example, takes place at a high altitude so that the loss of radiation energy from the shock becomes significant in the surrounding optically thin atmosphere. Thus, the total energy of the shock is not conserved, but decreases with time. We have not explicitly used radiative transfer equation, but have evaluated the radiative energy loss from the conservation equations. The solution, therefore, holds without any restriction on the optical properties of the medium.

2. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The equations of continuity, motion and energy are:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + 2 \frac{\rho u}{r} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \tag{2.2}$$

$$\frac{\partial E}{\partial t} + u \frac{\partial E}{\partial r} - \frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r}\right) + \frac{1}{\rho r^2} \frac{\partial}{\partial r}(Fr^2) = 0, \tag{2.3}$$

where ρ, u, p are density, particle velocity and pressure respectively at radial distance r and time t,

$$E = \frac{p}{(\gamma - 1)\rho}$$

where γ is the ratio of specific heats and F is the radiation flux.

The density in the undisturbed medium is assumed to be

$$\rho_a = \rho^* R^{\beta}, \tag{2.4}$$

where ρ^* and β are constants and R is the shock radius given by

$$\dot{R}^2 = A^2 R^{-\alpha}, \tag{2.5}$$

where A and α are constants.

We regard the shock to be a thin surface so that the radiation flux is continuous across it and we have the classical shock conditions for a strong shock:

$$p_2 = \frac{2}{\gamma + 1} \rho_1 \dot{R}^2, \tag{2.6}$$

$$\rho_2 = \frac{\gamma + 1}{\gamma - 1} \rho_1, \tag{2.7}$$

$$u_2 = \frac{2}{\gamma + 1} \dot{R}, \tag{2.8}$$

where the suffixes '2' and '1' give conditions just behind and just ahead of the shock respectively.

3. Similarity Solution

We assume the solution of the problem to be given in the similarity form:

$$u = \dot{R}\bar{u}(x),\ \ \rho = \rho_a\bar{\rho}(x),\ \ p = \rho_a\dot{R}^2\bar{p}(x),\ \ E = \dot{R}^2\bar{E}(x),$$

$$F = \rho_a\dot{R}^3\bar{F}(x), \tag{3.1}$$

where $x = r/R$ is the similarity variable. The equations (2.1)–(2.3) transform into

$$\beta + (\bar{u} - x)\frac{\bar{\rho}'}{\bar{\rho}} = -\left(\frac{2\bar{u}}{x} + \bar{u}'\right), \tag{3.2}$$

$$(\bar{u} - x)\bar{u}' - \frac{1}{2}\alpha\bar{u} = -\frac{\bar{p}'}{\bar{\rho}}, \tag{3.3}$$

$$(\bar{u} - x)\bar{E}' - \alpha\bar{E} + \frac{\bar{p}}{\bar{\rho}}\left(\frac{2\bar{u}}{x} + \bar{u}'\right) + \frac{1}{\bar{\rho}x^2}\frac{d}{dx}(\bar{F}x^2) = 0. \tag{3.4}$$

The shock conditions (2.6)–(2.8) change into

$$\bar{u}(1) = \frac{2}{\gamma+1},\ \ \bar{p}(1) = \frac{2}{\gamma+1},\ \ \bar{\rho}(1) = \frac{\gamma+1}{\gamma-1}. \tag{3.5}$$

We assume the product solution of the 'progressive wave' given by McVittie[5] in the form

$$u = \frac{a(t)}{t}r, \tag{3.6}$$

$$\rho = (\lambda + 1)ft^{2\alpha'}\eta^{\lambda-2}, \tag{3.7}$$

$$p = \alpha'^2 ft^{2}b(t)\eta^{\lambda}, \tag{3.8}$$

where

$$\eta = rt^{-\alpha'}, \tag{3.9}$$

a is some function of t and $f(t)$ and $b(t)$ are given by

$$a(t) = \frac{\alpha'\lambda - tf_t/f}{\lambda + 1} \tag{3.10}$$

$$b(t) = \frac{\lambda + 1}{\lambda \alpha'^2}(-a^2 + a - ta_t), \tag{3.11}$$

and α' and λ are some constants. These equations satisfy (2.1) and (2.2) identically.

After changing this solution to similarity form which requires a to be a constant, evaluating $\bar{F}$ from (3.4) and applying the boundary conditions (3.5). we finally obtain the following solution:

$$\bar{u}(x) = \frac{2}{\gamma + 1} x, \tag{3.12}$$

$$\bar{p}(x) = \frac{2}{\gamma + 1} x^{\lambda}, \tag{3.13}$$

$$\bar{\rho}(x) = \frac{\gamma + 1}{\gamma - 1} x^{\lambda - 2} \tag{3.14}$$

$$\bar{F}(x) = \frac{4(7 - \gamma + \beta + \beta\gamma)}{(\gamma - 1)^2(\beta + 5\gamma + \beta\gamma + 1)} x^{(\gamma+1)(\beta+3)/\gamma-1}, \tag{3.15}$$

where

$$\lambda = \frac{2 + (\beta + 2)(\gamma + 1)}{\gamma - 1}.$$

The temperature is obtained from

$$T = \frac{\bar{p}}{\bar{\rho}} \frac{\dot{R}^2}{\bar{R}}$$

where $\bar{R}$ = gas constant,

so that if

$$T = \frac{\dot{R}^2}{\bar{R}} \bar{T},$$

we have

$$\bar{T} = \frac{2(\gamma - 1)}{\gamma + 1} x^2. \tag{3.16}$$

This solution is an example of exact solutions in radiation-gas-dynamics corresponding to the exact solutions obtained in ordinary gas dynamics by McVittie,[5] Sedov,[6] etc.

Results and Discussion

For density to remain finite at the centre and for the flux not to be negative anywhere we have from (3.14) and 3.15)

$$\beta\gamma + \beta + 6 > 0, \tag{3.17}$$

$$7 - \gamma + \beta + \beta\gamma > 0. \tag{3.18}$$

For the permissible values of β corresponding to values of $\gamma = 1{\cdot}2$, $1{\cdot}5$ and $5/3$, we have tabulated the powers of x for $\bar{p}$, $\bar{\rho}$, and $\bar{F}$ in Table I. We have also tabulated the values of

$$\delta = \frac{2(3 + \beta - \alpha)}{\alpha + 2}$$

which gives the variation of total energy in the shock

$$\left(\alpha = \frac{2[4 + (\gamma + 1)(\beta + 1)]}{\gamma + 1}\right),$$

$$E_{tot.} = 4\pi \int_0^R (E + \tfrac{1}{2} u^2)\, \rho r^2\, dr = const\ t^\delta \tag{3.19}$$

and the ratio of mean radiative flux to the rate of change of total energy given by

$$\mu = \frac{\dfrac{1}{R}\displaystyle\int_0^R 4\pi r^2 F\, dr}{\dfrac{dE_{tot.}}{dt}} = -\frac{\gamma - 1}{\beta + \beta\gamma + 6\gamma} \tag{3.20}$$

corresponding to different values of γ and β.

(i) This solution predicts velocity, density, pressure and radiation flux to be zero at the centre. The values of all these physical quantities monotonically decrease from the highest at the shock to zero at the centre. The effect of change of β and γ and consequently the change in the rate of energy loss by radiation is much smaller for velocity and pressure than for density. This is in agreement with the results of Wang.[3] These parameters (except velocity which is a linear function of x) show a steep rise near the shock. This rise becomes much larger for small values of β. The velocity and temperature behind the shock depend only on γ and are independent of β.

TABLE I

	γ	β	λ	$\lambda-2$	$\frac{(1+\gamma)(\beta+3)}{\gamma-1}$	$\delta=\frac{2(3+\beta-\alpha)}{\alpha+2}$	μ	$-100\times\mu$
1	1·2	0	32	30	33	−0·6905	−0·0277	2·77
2	1·2	−1	21	19	22	−0·5807	−0·04	4
3	1·2	−2	10	8	11	−0·35	−0·0714	7·14
4	1·2	−2·5	4·5	2·5	5·5	−0·1035	−0·1176	11·76
5	1·5	0	14	12	15	−0·6111	−0·0555	5·55
6	1·5	−1	9	7	10	−0·4615	−0·0769	7·69
7	1·5	−2	4	2	5	−0·125	−0·125	12·5
8	5/3	0	11	9	12	−0·5714	−0·0666	6·66
9	5/3	−1	7	5	8	−0·4	−0·0909	9·09
10	5/3	−2	3	1	4	0	0	0

(ii) The temperature and radiation flux are particularly large near the shock, showing that the very high temperature region is near the shock boundary. The radiation flux is everywhere positive unlike the results obtained by Wang[3] for the piston problem wherein the radiation flux becomes negative away from the shock towards the piston. Comparing our results with those of Elliot[2] for his $\bar{K}=100$ which corresponds to large radiation effects we find that velocity, pressure and radiation flux behind the shock have a very similar distribution to that we have obtained and in fact all these parameters show a monotonic decrease to zero towards the centre but in his case the temperature is almost uniform throughout the shock region. This is possibly due to the diffusion approximation used by Elliot[2] which renders the temperature more or less uniform in the entire shock region. In his other results which correspond to milder radiation effects, the radiation flux assumes its maximum somewhere within the shock region and becomes zero both at the centre and very small at the shock.

(iii) μ which is a measure of mean of the loss of energy due to radiation increases when β decreases from 0 to $-2 \cdot 5$, say, that is, when the undisturbed density falls off rapidly and the atmosphere is more rarefied.

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