Hyperviscosity, Galerkin Truncation, and Bottlenecks in Turbulence

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It is shown that the use of a high power α of the Laplacian in the dissipative term of hydrodynamical equations leads asymptotically to truncated inviscid conservative dynamics with a finite range of spatial Fourier modes. Those at large wave numbers thermalize, whereas modes at small wave numbers obey ordinary viscous dynamics [C. Cichowlas *et al.*, Phys. Rev. Lett. **95**, 264502 (2005)]. The energy bottleneck observed for finite α may be interpreted as incomplete thermalization. Artifacts arising from models with $\alpha > 1$ are discussed.

A single Maxwell daemon embedded in a turbulent flow would hardly notice that the fluid is not exactly in thermal equilibrium because incompressible turbulence, even at very high Reynolds numbers, constitutes a tiny perturbation on thermal molecular motion. Dissipation in real fluids is just the transfer of macroscopically organized (hydrodynamic) energy to molecular thermal energy. Artificial microscopic systems can act just like the real ones as far as the emergence of hydrodynamics is concerned; for instance, in lattice gases the "molecules" are discrete Boolean entities [1] and thermalization is easily observed at high wave numbers [2]. Another example has been found recently by Cichowlas *et al.* [3] wherein the Euler equations of ideal nondissipative flow are (Galerkin) truncated by keeping only a finite—but large—number of spatial Fourier harmonics. The modes with the highest wave numbers k then rapidly thermalize through a mechanism discovered by Lee [4] and studied further by Kraichnan [5], leading in three dimensions (3D) to an equipartition energy spectrum $\propto k^2$. The thermalized modes act as a fictitious microworld on modes with smaller wave numbers in such a way that the usual dissipative Navier–Stokes dynamics is recovered at large scales [6].

All the known systems presenting thermalization are conservative. As we shall show thermalization may be present in dissipative hydrodynamic systems when the dissipation rate increases so fast with the wave number that it mimics ideal hydrodynamics with a Galerkin truncation. This is best understood by considering hydrodynamics with hyperviscosity: the usual momentum diffusion operator (a Laplacian) is replaced by the αth power of the Laplacian, where $\alpha > 1$ is the dissipativity. Hyperviscosity is frequently used in turbulence modeling to avoid wasting numerical resolution by reducing the range of scales over which dissipation is effective [7].

The unforced hyperviscous 1D Burgers and multidimensional incompressible Navier–Stokes (NS) equations are

$$\partial_t v + v \partial_x v = -\mu k_G^{-2\alpha}(-\partial_x^2)^\alpha v; \qquad (1)$$

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p - \mu k_G^{-2\alpha}(-\nabla^2)^\alpha \boldsymbol{v}; \qquad \nabla \cdot \boldsymbol{v} = 0. \qquad (2)$$

The equations must be supplemented with suitable initial and boundary conditions. We employ 2π-periodic boundary conditions in space, so that we can use Fourier decompositions such as $\boldsymbol{v}(\boldsymbol{x}) = \sum_k \hat{\boldsymbol{v}}_k e^{i\boldsymbol{k} \cdot \boldsymbol{x}}$. Note that minus the Laplacian is a positive operator, with Fourier transform k^2, which can be raised to an arbitrary power α. The coefficient μ is taken positive to make the hyperviscous operator dissipative. The Galerkin wave number $k_G > 0$ is chosen off-lattice so that no wave number is exactly equal to k_G. In Fourier space the hyperdissipation rate is $\mu(k/k_G)^{2\alpha}$, where $k \equiv |\boldsymbol{k}|$.

If we now hold μ and k_G fixed and let $\alpha \to \infty$ we see that the hyperdissipation rate tends to zero, for $k < k_G$, and to infinity, for $k > k_G$. This implies that in the limit of infinite dissipativity, the solution of a hyperviscous hydrodynamical equation converges to that of the corresponding inviscid equations Galerkin-truncated at wave number k_G.

To define inviscid Galerkin truncation precisely, we rewrite Eqs. (1) and (2) in the abstract form $\partial_t v = B(v, v) + L_\alpha v$, where B is a quadratic form representing the nonlinear term (including the pressure p in the NS case). The truncation projector P_{k_G} is the linear, low-pass filtering operator that, when applied to v, sets all Fourier harmonics with $k > k_G$ to zero. The inviscid, Galerkin-truncated equation, with initial condition v_0, is

$$\partial_t u = P_{k_G} B(u, u), \qquad u_0 = P_{k_G} v_0. \qquad (3)$$

Since u can be written in terms of a finite number of modes with $k < k_G$, Eq. (3) is a dynamical system of finite dimension. In addition to momentum, it conserves the energy and other quadratic invariants for the inviscid equations [5]. There is good numerical evidence—but no rigorous

proof—that the solutions of the Galerkin-truncated invis-
cid Burgers and 3D Euler equations tend, at large times, to
statistical equilibria defined by their respective invariants.

A rigorous proof of the convergence, as $\alpha \to \infty$, of
solutions of the hyperviscous Burgers equation (1) and of
the hyperviscous NS equation (2) in any dimension to
those of the associated Galerkin-truncated, inviscid equa-
tion will be given elsewhere. It uses standard tools of
functional analysis; note that the formidable mathematical
difficulties that beset the ordinary ($\alpha = 1$) 3D NS equation
disappear for $\alpha \geq 5/4$ [8].

From a physicist's point of view the convergence result
looks rather obvious, though it has hardly been noted
before (see, however, Refs. [9–11]): as $\alpha \to \infty$ all the
modes with $k > k_G$ are immediately suppressed by an
infinite dissipation, whereas those with $k < k_G$ obey invis-
cid truncated dynamics. Not surprisingly, the fate of cou-
plings between triads of modes whose wave numbers
straddle k_G is a delicate point. In a Galerkin truncation
any such triad should be left out. It may be shown that for
$\alpha \to \infty$ such straddling couplings are suppressed, not only
for the Burgers and NS equation but also for the hyper-
viscous magnetohydrodynamical equations and for some
turbulence closures, specifically, the direct interaction ap-
proximation (DIA) [9] and the eddy-damped-quasi-nor-
mal-markovian (EDQNM) approximation [12]. Hence
the convergence to the corresponding Galerkin-truncated
equations holds for all the aforementioned equations in any
dimension of space.

There are, however, interesting exceptions among hy-
drodynamical equations for which the result does not hold.
They include the kinetic theory of resonant wave interac-
tions [13] and the Markovian random coupling model [14].

Indeed, the resonant wave interaction theory arises in the
limit when the period of the waves goes to zero and this
limit does not commute with the limit of a vanishing
damping time for modes having $k > k_G$; a similar remark
can be made about the MRCM equation.

Let us stress that systems with a finite dissipativity—
however large—are quite different from Galerkin-
truncated systems. For example, consider the 3D NS equa-
tion with a random force, delta-correlated in time, for
which we know the mean energy input ε per unit volume.
It is still true that, for $\alpha \to \infty$, the solution of this equation
converges to that of the Galerkin-truncated equation, but
this time with a random force. If E_0 is the initial energy,
this solution has a mean energy $E(t) = E_0 + \varepsilon t$, which
grows indefinitely in time. But, as soon as α is given a
finite value, however large, a statistical steady state, in
which energy input and hyperviscous energy dissipation
balance, is achieved at large times. Such a steady state
presents an interesting interplay of thermalization and dis-
sipation, when α is large, as we show below.

The direct numerical simulation (DNS) of the Galerkin-
truncated 3D Euler equations in Ref. [3] used 1600^3
Fourier modes. Large-α simulations of Eq. (2) would
require significantly higher resolution to identify the vari-
ous spectral ranges that we can expect, namely, inertial,
thermalized, and far-dissipation ranges and transition re-
gimes between these. Fortunately, Bos and Bertoglio [15]
have shown that key features of the Galerkin-truncated
Euler equations, such as the presence of inertial and ther-
malized ranges, can be reproduced by the two-point
EDQNM closure [12] for the energy spectrum. For
Eq. (2), with stochastic, white-in-time, homogeneous,
and isotropic forcing with spectrum $F(k)$, the hyperviscous
EDQNM equations are

$$\left[\partial_t + 2\mu\left(\frac{k}{k_G}\right)^{2\alpha}\right]E(k,t) = \iint_{\Delta_k} dp\, dq\, \theta_{kpq} b(k,p,q)\frac{k}{pq}E(q,t)[k^2 E(p,t) - p^2 E(k,t)] + F(k),$$

$$\theta_{kpq} = \frac{1}{\mu_k + \mu_p + \mu_q}, \quad b(k,p,q) = \frac{p}{k}(xy + z^3), \quad \mu_k = \mu\left(\frac{k}{k_G}\right)^{2\alpha} + \lambda\left[\int_0^k p^2 E(p,t)dp\right]^{1/2}.$$

(4)

Here $E(k,t)$ is the energy spectrum, Δ_k defines the set of
$p \geq 0$ and $q \geq 0$ such that k, p, q can form a triangle, $x, y,$
z are the cosines of its angles and the eddy-damping
parameter λ is expressed in terms of the Kolmogorov
constant. The EDQNM equations have been studied nu-
merically for more than three decades [16], but their hyper-
viscous versions Eq. (4) present new difficulties that we
overcome as follows. Since we are interested in the steady
state we use an iterative method: the emission term,
$E(p,t)E(q,t)$ in Eq. (4), is considered as a renormalization
of the force $F(k)$; the absorption term $E(q,t)E(k,t)$ is
treated as a renormalization of the hyperviscous damping
[17]. We then construct a sequence of energy spectra that,
at stage $n + 1$, is just the renormalized force divided by the
renormalized damping, both based on stage n. This gives
rapid convergence to the steady state at low wave numbers,

but, beyond a certain (α-dependent) wave number, con-
vergence slows down dramatically and it is better to use
time marching to obtain the steady state. At large values of
k and α the problem becomes very stiff, so we use a slaved
fourth-order Runge–Kutta scheme [18]. We discretize k
logarithmically, with N_c collocation points per octave.
Triad interactions involving wave number ratios signifi-
cantly larger than N_c are poorly represented [19]; so, since
wave number ratios of up to 50 play an important role for
large α, we have used $N_c \lesssim 90$; this is computationally
demanding because the complexity of the code is $O(N_c^3)$.
We force at the lowest wave number ($k = 1$) in our nu-
merical study of Eq. (4) with $k_G = 10^5$, $\lambda = 0.36$, and $1 \leq$
$\alpha \leq 729$. The resulting compensated, steady-state energy
spectra $k^{+5/3}E(k)$ are shown in Fig. 1; flat regions, extend-
ing over two to five decades of k (depending on α), are

close to the Kolmogorov inertial range; for large α there is a distinct thermalized range with $E(k) \sim k^2$ (also found in the transition between classical and quantum superfluid turbulence [20]), as we expect from our discussion of the Galerkin-truncated Euler equations. In the far-dissipation range $k > k_G$ the spectra fall off very rapidly. For all values of α the far-dissipation range is preceded by a bump or bottleneck. It is also observed, in some experiments [21] and DNS of Navier-Stokes, with a shape that is quite independent of the Reynolds number [22]. The bottleneck for $\alpha = 1$ has previously been explained as the inhibition of the energy cascade from low to high wave numbers because of viscous suppression of the cascade in the dissipation range [23]. Our work provides an alternative explanation: the usual bottleneck may be viewed as incomplete thermalization.

At large values of α the thermalized range gives rise to an eddy viscosity ν_{eddy}. This acts on modes with wave numbers lower than those in the thermalized range; the corresponding damping rate is $\nu_{\mathrm{eddy}} k^2$. The eddy viscosity can be expressed as an integral over the thermalized range [15,19]. As α grows, so does ν_{eddy} and, eventually, the renormalized viscous damping overwhelms the hyperviscous damping for modes at low wave numbers (below those in the thermalized range). The dynamics of these modes is then governed by the usual $\alpha = 1$ equation. Not surprisingly, then, we find a pseudodissipation range around $k \simeq 10^4$ that is shown in an expanded scale in the inset of Fig. 1; a similar range for the Galerkin-truncated case is discussed in Ref. [15] and is already visible in the DNS of Ref. [3]. For large α the inset of Fig. 1 also shows a secondary bottleneck range for $10^3 < k < 10^4$; this may be viewed as the usual ($\alpha = 1$) EDQNM bottleneck stemming from ν_{eddy}.

Our results apply to compressible flows also. We have studied the simplest instance, that is the unforced hyperviscous 1D Burgers equation (1). Its solution converges to the entropy solution, i.e., the standard solution with shocks, obtained when $k_G \rightarrow \infty$ for any $\alpha \geq 1$ [24]. Here we are interested in the large-α behavior at fixed k_G. We do not

have to resort to closure now since we can solve the primitive equation (1) directly by a pseudospectral method. If we choose a single initial condition the resulting spectrum is noisy because, unlike the ordinary Burgers equation, its Galerkin-truncated version and thus also the high-α versions are believed to be chaotic dynamical systems [25]. So we solve (1) with the two-mode random initial condition $v_0(x) = \sin x + \sin(2x + \phi)$, where ϕ is distributed uniformly in the interval $[-\pi, \pi]$. We use 2^{14} collocation points and set $\mu = 1$, $k_G = 342.1$, and $\alpha = 1000$. In Fig. 2 we show the Burgers energy spectrum $E(k) = |\tilde{v}(k)|^2$, averaged over 20 realizations of the phase ϕ at various times. At the latest output times the spectrum is almost completely flat, i.e., thermalized, with equipartition of the energy between all the Fourier modes. At earlier times $E(k)$ behaves approximately as k^{-2} in an inertial range that corresponds to shocks in physical space; there is a thermalized range at higher wave numbers up to k_G; for $k > k_G$ the spectrum falls very rapidly. No pseudodissipation range is observed here between the inertial and thermalized ranges as seen in the 3D NS case (Fig. 1). Perhaps the data are too noisy, but a careful examination of $v(x)$ in physical space indicates that this phenomenon might arise from the compressible nature of the Burgers dynamics: thermalization begins over the whole physical range (as high-frequency noise with wave number $\approx k_G$); noise generated close to shocks is absorbed by them and not enough is left to produce any appreciable eddy viscosity that could broaden the shocks.

We now summarize our main findings from the study of hyperviscous hydrodynamical equations with powers α of the Laplacian ranging from unity to very large values.

The simplest results are obtained for very large α. The solutions of the 1D Burgers equation or the Navier–Stokes equations in any space dimension d are then very close to the solutions of the corresponding Galerkin-truncated equations, displaying thermalization at wave numbers below k_G. The detailed scenario will of course be affected by the dimension of space. In 3D, with enough resolution, we may be able to observe up to five ranges: an inertial range,
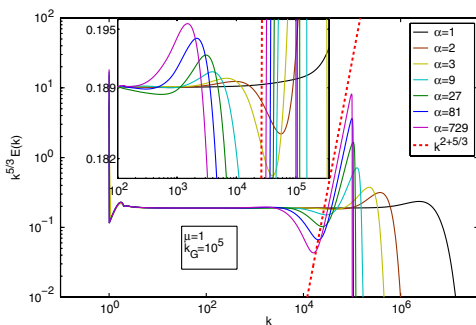

FIG. 1 (color online). Log-log plots of the compensated spectrum $k^{5/3} E(k)$ versus k from a numerical integration of the hyperviscous EDQNM Eq. (4) for different values of α; inset: enlarged spectra showing a secondary bottleneck (see text).
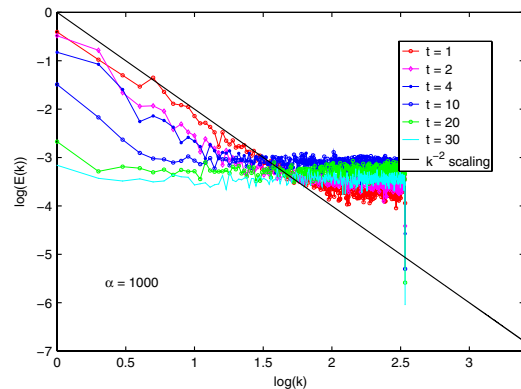

FIG. 2 (color online). Log-log plots of averaged energy spectra (see text) versus wave number k for the hyperviscous Burgers equation (1) at various times.

a secondary bottleneck, a pseudodissipation range, a thermalized range, and a far-dissipation range. Because of enstrophy conservation and of the predominance of Fourier-space nonlocal interactions, the 2D case is rather special and deserves a separate study.

The most relevant case is of course that of ordinary dissipation ($\alpha = 1$). The energy-spectrum bottleneck generally observed at high Reynolds numbers in 3D incompressible turbulence may be viewed as an incomplete thermalization: as we increase α larger and larger bottlenecks are present, eventually displaying thermalization on their rising side.

We finally deal with the case of moderately large α of the sort used in many simulations [7]. How safe is this procedure and what kind of artifacts can we expect?

Using large values of α in simulations to "avoid wasting resolution" is hardly advocated by anybody, but we now understand what goes wrong: a huge thermalized bottleneck will develop at high wave numbers, whose action on smaller wave numbers is an ordinary $\alpha = 1$ dissipation with an eddy viscosity much larger than what would be permissible in a normal $\alpha = 1$ simulation.

When α is chosen just a bit larger than unity (e.g., $\alpha = 2$ which is standard in oceanography [7]) the advantage of widening the inertial range may be offset by artifacts at bottleneck scales; indeed, even an incomplete thermalization will bring the statistical properties of such scales closer to Gaussian, thereby reducing the rather strong intermittency which would otherwise be expected [26]. For similar reasons spurious isotropization can be expected for problems with an anisotropic constraint, such as rapidly rotating or stratified flow or MHD with a strong uniform magnetic field.

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