

Canonical measures on the moduli spaces of compact Riemann surfaces

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Abstract. We study some explicit relations between the canonical line bundle and the Hodge bundle over moduli spaces for low genus. This leads to a natural measure on the moduli space of every genus which is related to the Siegel symplectic metric on Siegel upper half-space as well as to the Hodge metric on the Hodge bundle.

Keywords. Canonical measures; moduli spaces; compact Riemann surfaces; Hodge bundle; Siegel symplectic metric.

1. Introduction

As is very well-known today, Polyakov's (bosonic) string theory has produced a natural measure on the moduli spaces \mathcal{M}_g ($g \geq 1$). This measure has now been recognized as arising from pulling back the intrinsic Hodge metric of the Hodge line bundle E over \mathcal{M}_g to the canonical bundle K over \mathcal{M}_g via Mumford's (rather inexplicit) isomorphism $K \cong E^{13}$ for $g \geq 2$.^{*} (See, for example, the exposition in Nelson [6].) In this article we first show that there are explicit and canonical isomorphisms $K \cong E^{(g+1)}$ for low genus $g = 1, 2, 3$; in the case of genus 3 the isomorphism $K \cong E^4$ is to be interpreted as holding over the Zariski open subset $\mathcal{M}_3^0 = \mathcal{M}_3 - \mathcal{H}_3$ of nonhyperelliptic surfaces. As a result, we can pull back the Hodge metric via our isomorphisms for these low genera to obtain a hermitian metric on K – and therefore another intrinsic volume form on \mathcal{M}_1 , \mathcal{M}_2 and $\mathcal{M}_3 - \mathcal{H}_3$.

Because of the explicit nature of our isomorphisms we are able to actually exhibit this volume form in local (period matrix) coordinates on \mathcal{M}_g . To our surprise, it turned out that the volume form we are getting is nothing but the Riemannian volume form of Siegel's classical symplectic metric for $g = 1, 2, 3$. (For $g = 1$ this is the hyperbolic measure on \mathcal{M}_1 .) In particular, our volume form assigns finite (and computable!) total volume to \mathcal{M}_g for $g = 1, 2, 3$, whereas the Polyakov–Mumford measure of \mathcal{M}_g is infinite for any g .

The above identification of our measure on \mathcal{M}_g for low values of genus immediately allows us to extend our construction and define a corresponding measure on \mathcal{M}_g for every genus by utilizing Siegel's symplectic metric on Siegel upper half-space \mathcal{S}_g . This is explained in §4. The results here may thus be interpreted as asserting that the

* Powers of line bundles denote tensor product powers.

canonical line bundle K , (restricted to the Zariski open set of nonhyperelliptic points of \mathcal{M}_g for $g \geq 3$), carries a natural hermitian metric which is closely related to *both* the Hodge metric of the Hodge bundle and the symplectic metric of Siegel space.

Our method of proof involves description of the Hodge bundle E and associated bundles over \mathcal{M}_g for any g by explicitly finding the corresponding factors of automorphy for the action of the Torelli modular group on Torelli space \mathcal{T}_g . These explicit formulae (in §2 below) may hold some independent interest.

2. The moduli spaces and some vector bundles

The Teichmüller space, the Torelli space and the Riemann moduli space for compact Riemann surfaces of genus g (≥ 1) will be denoted respectively by T_g , \mathcal{T}_g and \mathcal{M}_g . For $g \geq 3$ we will denote the $(2g-1)$ dimensional analytic subvariety of hyperelliptic Riemann surfaces in these three spaces by H_g , \mathcal{H}_g and \mathcal{H}_g respectively. (For $g=1$ and 2 all Riemann surfaces are hyperelliptic – but this will not affect us.) The Zariski open subsets of nonhyperelliptic points in these spaces, for $g \geq 3$, will be written T_g^0 , \mathcal{T}_g^0 and \mathcal{M}_g^0 . For the basic theory of these spaces, and for the material below, one may consult the book [5]. We set $T_g^0 = T_g$, $\mathcal{T}_g^0 = \mathcal{T}_g$ and $\mathcal{M}_g^0 = \mathcal{M}_g$ for $g=1, 2$.

Recall that the Teichmüller modular group $\text{Mod}(g)$ and the Torelli modular group (which is identifiable with the symplectic group $\text{Sp}(2g, \mathbb{Z})$) acts by biholomorphic automorphisms on the complex manifolds T_g and \mathcal{T}_g respectively, producing \mathcal{M}_g as the quotient normal complex space. These group actions keep the subsets T_g^0 and \mathcal{T}_g^0 invariant – giving \mathcal{M}_g^0 as quotient.

On a Riemann surface X of genus $g \geq 1$ we fix a standard homology basis $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ for $H_1(X, \mathbb{Z})$, (characterized by having intersection matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$). The corresponding canonical dual basis $(\omega_1, \dots, \omega_g)$ of holomorphic 1-forms on X is uniquely determined by the normalization requirement: $\int_{\alpha_i} \omega_j = \delta_{ij}$. The $g \times g$ matrix $\pi(X)$ of β -periods [i.e. $\pi_{ij}(X) = \int_{\beta_j} \omega_i$] is then called the *canonical period matrix* for the marked Riemann surface X . $\pi(X)$ is a symmetric matrix with positive definite imaginary part, and this gives us the usual *period mapping* $\pi: \mathcal{T}_g \rightarrow \mathcal{S}_g$ of Torelli space into the Siegel upper half-space. Recall that \mathcal{S}_g is a hermitian symmetric domain of complex dimension $g(g+1)/2$, whereas \mathcal{T}_g is of dimension 1 if $g=1$ and dimension $(3g-3)$ for $g > 1$. It is known that π is a 2-to-1 holomorphic immersion of \mathcal{T}_g^0 into \mathcal{S}_g for $g \geq 3$. For $g=1$ and 2 the period map is a holomorphic embedding of \mathcal{T}_g onto its image in \mathcal{S}_g . Actually, for $g=1$, $\mathcal{S}_1 = \mathcal{T}_1 =$ upper half-plane U , and π can be regarded as the identity map. In genus 2, $\pi(\mathcal{T}_2)$ is “almost” all of \mathcal{S}_2 . See [5] for more details.

We are interested in the relationship between certain important vector bundles over the moduli spaces. It is important to recall that the Teichmüller space T_g is a contractible domain of holomorphy, so that every holomorphic vector bundle over it is holomorphically globally trivial. Consequently, any bundle over \mathcal{M}_g (or \mathcal{T}_g) can be described by a factor of automorphy for the corresponding modular group (or subgroup thereof) see Gunning [2]. The idea is to choose a global trivialization of the pull-back bundle over T_g and write down the action of the modular group on this pull-back.

One can construct the holomorphic vector bundles B_j , ($j \geq 1$), over T_g whose fiber

over $X \in T_g$ is the vector space of holomorphic j -forms on the Riemann surface X . One knows that for the bundle B_1 the canonical basis of 1-forms $(\omega_1(X), \dots, \omega_g(X))$ varies holomorphically with moduli (Bers [1]) – and so provides a *global holomorphic frame* for B_1 over T_g . Since, as we saw above, this global holomorphic frame is definable over the Torelli space \mathcal{T}_g , we see that the rank g bundle of 1-forms (we will still call it B_1) is holomorphically trivial over the Torelli space also. Let us describe the factor of automorphy for the action of the Torelli modular group $\Gamma = \text{Sp}(2g, \mathbb{Z})$ on B_1 over \mathcal{T}_g using this global trivialization.

Let $\gamma \in \text{Sp}(2g, \mathbb{Z})$ be the symplectic matrix

$$\gamma = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \quad (1)$$

partitioned into $g \times g$ blocks. γ acts by changing the chosen standard homology basis $(\alpha, \beta) = (\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ into another standard homology basis $(\tilde{\alpha}, \tilde{\beta}) = (P\alpha + Q\beta, R\alpha + S\beta)$. If $\omega = (\omega_1(\tau), \dots, \omega_g(\tau))$ was the canonical basis of 1-forms at a point $\tau \in \mathcal{T}_g$ dual to the original homology basis (α, β) , then we need to find the $GL(g, \mathbb{C})$ matrix that transforms ω to $\tilde{\omega} = (\tilde{\omega}_1, \dots, \tilde{\omega}_g)$, where the $\tilde{\omega}_j$ are the canonical dual 1-forms with respect to the new homology basis $(\tilde{\alpha}, \tilde{\beta})$. This requires that $\int_{\tilde{\alpha}_k} \tilde{\omega}_j = \delta_{jk}$; so a short calculation now shows that

$$\begin{bmatrix} \tilde{\omega}_1 \\ \vdots \\ \tilde{\omega}_g \end{bmatrix} = [A_\gamma] \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_g \end{bmatrix}, \quad (2)$$

where the sought-for matrix A_γ is:

$$A_\gamma = (P^t + \pi(\tau)Q^t)^{-1}. \quad (3)$$

(Here M^t denotes the transpose of a matrix M ; $\pi(\tau) \in \mathcal{S}_g$, is the (canonical) period matrix of $\tau \in \mathcal{T}_g$, as defined before.) This A_γ is the factor of automorphy representing the 1-forms bundle over \mathcal{M}_g as a Γ -quotient of the B_1 bundle over \mathcal{T}_g .

Now, the Hodge line bundle E is defined as the determinant bundle of B_1 (i.e. $E = \Lambda^g B_1$). Therefore

$$\xi(\gamma, \tau) = [\det(P + Q\pi(\tau))]^{-1}, \quad \gamma \in \text{Sp}(2g, \mathbb{Z}), \quad \tau \in \mathcal{T}_g \quad (4)$$

is the factor of automorphy (with respect to the canonical trivialization) on Torelli space representing the Hodge bundle E over \mathcal{M}_g .

Remarks. (a) One can directly verify for formula (4) the cocycle condition

$$\xi(\gamma_1 \gamma_2, \tau) = \xi(\gamma_1, \gamma_2(\tau)) \cdot \xi(\gamma_2, \tau) \quad (5)$$

using the fact that the canonical period matrix at $\gamma(\tau)$ is

$$\pi(\gamma(\tau)) = (R + S\pi(\tau)) \cdot (P + Q\pi(\tau))^{-1}. \quad (6)$$

(b) To see whether a line bundle represented by a factor of automorphy $\xi(\gamma, \tau)$ is trivial, one has to find whether it has a nowhere vanishing global holomorphic section. Such a section exists if and only if there is a nowhere vanishing holomorphic function

φ on the covering space (\mathcal{T}_g here) such that

$$\xi(\gamma, \tau) = \varphi(\gamma(\tau))/\varphi(\tau), \quad \text{for } \gamma \in \Gamma, \quad \tau \in \mathcal{T}_g. \quad (7)$$

In particular, we can think of $\text{Pic}(\mathcal{M}_g)$ as equivalence classes of factors of automorphy for the Teichmüller modular group $\Gamma = \text{Mod}(g)$ on T_g , where ξ_1 and ξ_2 are equivalent if $\xi_1 \cdot \xi_2^{-1}$ is trivial in the above sense. This remark will be invoked in the next section.

3. The isomorphism $K \cong E^{g+1}$ in low genus

The canonical bundle $K(M)$ of a complex space M is the determinant line bundle of the holomorphic cotangent bundle of M . It is a basic fact that the holomorphic cotangent bundle of the moduli spaces (T_g , \mathcal{T}_g or \mathcal{M}_g) is identifiable, by Teichmüller's lemma (see [5]), as the bundle B_2 .

Since the symmetric tensor product of two 1-forms on a Riemann surface X is a 2-form, we have a natural vector bundle map v of the second symmetric tensor power of B_1 to B_2 (say over Torelli space):

$$S^2(B_1) \xrightarrow{v} B_2. \quad (8)$$

By Max Noether's well-known theorem one sees that the map v is a surjection over all of \mathcal{T}_1 and \mathcal{T}_2 , and over $\mathcal{T}_g^0 = \mathcal{T}_g - \mathcal{H}_g$ for $g \geq 3$. Note that, for $g \geq 2$, $S^2(B_1)$ is of rank $g(g+1)/2$ but B_2 is of rank $(3g-3)$. (These two numbers are equal precisely when $g=2$ or 3.) The fiber above X of the kernel of the above bundle map v corresponds to linear dependence relations amongst the $g(g+1)/2$ quadratic differentials

$$\theta_{ij} = \omega_i \otimes \omega_j \quad \text{on } X. \quad (9)$$

These "Noether relations" are actually nothing but differential versions of relations amongst the π_{ij} which describe the Schottky locus (i.e. the image of \mathcal{T}_g in \mathcal{S}_g).

We can now state

Theorem 1. *There are canonical analytic isomorphisms of line bundles:*

- (i) $K \cong E^2$ on \mathcal{M}_1
- (ii) $K \cong E^3$ on \mathcal{M}_2
- (iii) $K \cong E^4$ on $\mathcal{M}_3 - \mathcal{H}_3 = \mathcal{M}_3^0$.

Proof. The global holomorphic frame $\omega = (\omega_1, \dots, \omega_g)$ for B_1 over \mathcal{T}_g gives rise (via v) to a global holomorphic frame $\theta_{ij} = \omega_i \otimes \omega_j$ ($(i, j) = 1, \dots, g(g+1)/2$) for B_2 over \mathcal{T}_g^0 for $g=1, 2, 3$. (For $g=1, 2$ we get $\mathcal{T}_g^0 \equiv \mathcal{T}_g$.) Any element $\gamma \in \text{Sp}(2g, \mathbb{Z})$ of the Torelli modular group acts on $(\omega_1, \dots, \omega_g)$ by the matrix A_γ of formula (3). Consequently, the corresponding automorphism on the θ_{ij} -frame for B_2 is the second symmetric power $S^2(A_\gamma)$ of A_γ . The question therefore reduces to relating the determinant of $S^2(A)$ to the determinant of a linear automorphism A . Indeed, the following general algebra lemma shows that $\det(S^2(A)) = (\det(A))^{g+1}$, where A operates on a g -dimensional vector space. We are through. \square

Multilinear algebra lemma. Let $A: V \rightarrow V$ be a linear automorphism of a g -dimensional vector space. Let $S^k(A): S^k(V) \rightarrow S^k(V)$ be the corresponding automorphism on the k th symmetric tensor powers of V . Then

$$\Lambda^{\binom{g+k-1}{k}} S^k(A) = [\Lambda^g(A)]^{\binom{g+k-1}{k} \cdot \frac{k}{g}}. \quad (10)$$

Remarks. $S^k(V)$ has dimension $\binom{g+k-1}{k}$; so the left side of (10) is $\det(S^k(A))$. Also note that the product of the exponents on the two sides of (10) match! [It is quite instructive to write down the rather pretty (functorially determined) matrix for $S^k(A)$ from the matrix for a general A . The relation between $\det(A)$ and $\det(S^k(A))$ then appears rather remarkable.]

Proof of lemma. We can choose a basis (e_1, \dots, e_g) of V such that A has upper triangular matrix in this basis (assume first that V is a complex vector space). Then, in the corresponding induced basis of $S^k(V)$ one recognizes that $S^k(A)$ has again upper triangular matrix! Thus now $\det(S^k(A))$ is the product of just the diagonal entries in $S^k(A)$ —and these depend *only* on the diagonal entries of A . One therefore sees that $\det(S^k(A))$ must be some (universal) power of $\det(A)$. It is now easy to verify that the exponents are as in (10). (For example, it is now enough to just check the result for $A = \text{diag}(d, d, \dots, d)$. Or one can do some combinatorial counting of exponents here.)

Finally note that, once having proved this over complex vector spaces, we are through in general because (10) is nothing but a polynomial identity with integer coefficients, (this is a “principle of permanence of analytic relationships”!) \square

COROLLARY 1.

Let E be the Hodge line bundle over \mathcal{M}_g .

- (a) The bundle E^{10} on \mathcal{M}_2 is globally analytically trivial.
- (b) The bundle E^9 on \mathcal{M}_3 is the line bundle of the divisor given by some integer times the hyperelliptic divisor \mathcal{H}_3 .

Proof. These facts follow by combining our Theorem 1 with Mumford’s [3] theorem that $K \cong E^{13}$ on \mathcal{M}_g for $g \geq 2$. It is to be remembered that \mathcal{H}_3 is a codimension 1 connected analytic subvariety in \mathcal{M}_3 . \square

Remarks. Mumford in [4] finds that $\text{Pic}(\mathcal{M}_2) = \mathbb{Z}/10\mathbb{Z}$ by looking at the geometry of the Teichmüller modular group. Our present corollary (a) confirms this from quite a different angle. Since we know explicitly from formula (4) the factor of automorphy for E , hence for E^{10} , it may be interesting to prove directly the triviality of E^{10} on \mathcal{M}_2 by finding the nowhere vanishing holomorphic φ on $\pi(\mathcal{T}_2)$ (from theta-nulls) which exhibit that $\xi^{10} = \varphi(\gamma(\tau))/\varphi(\tau)$. (See (7) of §2.)*.

*Note added in proof

(G Gonzales-Dies has answered this (thesis, King’s College, London) by showing that $\varphi(\tau) = \prod_{\text{even}} \theta^2[\varepsilon](0, \pi(\tau))$ —i.e., the product of the squares of the 10 even theta-nulls—gives a nowhere vanishing section of E^{10} over \mathcal{M}_2 (Private communication.)

A question. To describe K on \mathcal{M}_g , for $g \geq 4$, by a factor of automorphy for $\mathrm{Sp}(2g, \mathbb{Z})$ we need to know whether the canonical bundle (or better still B_2) over Torelli space is trivial. We have seen above that K (and B_2) over \mathcal{T}_g^0 is trivial for $g = 1, 2, 3$. Does this phenomenon persist in higher genus?

4. A canonical measure on \mathcal{M}_g for all g

The Hodge bundle E , and more so B_1 itself, carries the completely intrinsic *Hodge hermitian metric* on its fibers. This is given by pairing any two holomorphic 1-forms φ and ψ on a Riemann surface X as follows:

$$(\varphi, \psi) = i \iint_X \varphi \wedge \bar{\psi}. \quad (11)$$

Theorem 2. *The Hodge hermitian metric on E^{g+1} ($g = 1, 2, 3$) pulled back by the isomorphisms of Theorem 1 produces a hermitian metric on K (over \mathcal{M}_1 , \mathcal{M}_2 and \mathcal{M}_3^0). This gives a volume form on \mathcal{M}_g ($g = 1, 2, 3$) which can be explicitly written down as*

$$d(\mathrm{vol}) = \frac{\prod_{(i,j)=1}^{g(g+1)/2} (d\pi_{ij} \wedge d\bar{\pi}_{ij})}{[\det(\mathrm{Im} \pi_{ij})]^{g+1}} \quad (12)$$

in period-matrix coordinates π_{ij} for $\mathcal{T}_g \subset \mathcal{S}_g$. (Thus, $d(\mathrm{vol})$ is a volume form on Siegel upper half-space \mathcal{S}_g which is invariant under the Torelli modular group.)

Remarks. (i) For $g = 1$, $d(\mathrm{vol})$ is just the hyperbolic (Poincare volume) measure: $(d\tau \wedge d\bar{\tau})/(\mathrm{Im} \tau)^2$ on upper half τ -plane.

(ii) For any g the formula (12) as a volume form on \mathcal{S}_g is nothing but (apart from a constant factor) the Riemannian volume form obtained from Siegel's symplectic hermitian metric on \mathcal{S}_g :

$$ds^2 = \mathrm{trace}[(\mathrm{Im} \pi_{ij})^{-1} (d\pi_{ij})(\mathrm{Im} \pi_{ij})^{-1} (d\bar{\pi}_{ij})]. \quad (13)$$

This metric, and hence its volume element (12), is invariant under the *full* real symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ (and not just $\mathrm{Sp}(2g, \mathbb{Z})$). The volume element for (13) can be seen in Siegel [8], p. 130. It is $1/2^g$ times the volume element from (12).

Proof of Theorem. The canonical frame $(\omega_1, \dots, \omega_g)$ for B_1 , and the induced frame $\theta_{ij} = \omega_i \otimes \omega_j$ for B_2 , give holomorphic trivializations of B_1 and B_2 over \mathcal{T}_g^0 for these low genera. Pulling back the Hodge norm from E^{g+1} to K we see that

$$\left\| \prod_1^{g(g+1)/2} \theta_{ij} \right\|^2 = [\det(\omega_i, \omega_j)]^{g+1}.$$

The induced modular-invariant volume form on \mathcal{T}_g^0 is therefore

$$d(\mathrm{vol}) = \frac{\prod_1^{g(g+1)/2} (\theta_{ij} \wedge \bar{\theta}_{ij})}{[\det(\omega_i, \omega_j)]^{g+1}}. \quad (14)$$

By Riemann's bilinear relations one sees that the denominator is the same as the denominator in (12). Now, the quadratic differentials θ_{ij} have to be interpreted as cotangents to Torelli space, using Teichmüller's lemma. A famous variational formula of Rauch does precisely that (for $g \geq 2$) and says

$$\omega_i \otimes \omega_j = d\pi_{ij}. \quad (15)$$

See [5], pp. 260–263 for a proof. Therefore, (15) and (14) prove (12) (for $g \geq 2$ at least).

In genus 1 it is directly possible to identify $\omega_1^2 = \theta_{11}$ as the cotangent vector $d\tau$, where the upper-half τ -plane U is the Teichmüller and the Torelli space $T_1 = \mathcal{T}_1 = U$.

The normalized ω_1 on a torus $X_\tau = \mathbb{C}/L(1, \tau)$ is clearly dz (here the α -loop is the projection of the segment $[0, 1]$ and the β -loop is the projection of the segment $[0, \tau]$). Now one writes down the affine (Teichmüller mapping) f of X_τ onto X_σ , and calculates its Beltrami coefficient $\mu = \bar{\partial}f/\partial f$ on X_τ . The formula for μ is

$$\mu = \left(\frac{\tau - \sigma}{\sigma - \bar{\tau}} \right) \frac{d\bar{z}}{dz} \quad \text{on } X_\tau, \text{ for any } \tau, \sigma \in U. \quad (16)$$

We compute the Teichmüller pairing $\iint \mu \theta_{11}$ of μ and $\theta_{11} = dz^2$ to get

$$\begin{aligned} (\mu, \theta_{11}) &= \int_{X_\tau} \int \left(\frac{\tau - \sigma}{\sigma - \bar{\tau}} \right) dz \wedge d\bar{z} \\ &= -2i \left(\frac{\tau - \sigma}{\sigma - \bar{\tau}} \right) \cdot (\text{area of period parallelogram for } X_\tau). \end{aligned} \quad (17)$$

Setting now $\sigma = \tau + \varepsilon d\tau$, and letting $\varepsilon \rightarrow 0$, we get

$$(\mu, \theta_{11}) = \varepsilon d\tau + O(\varepsilon^2). \quad (18)$$

This shows that θ_{11} corresponds to $d\tau$, as required. \square

From Theorem 2 and Remark (ii) we now see the correct generalization of the measure we are getting in low genera. Take the Riemannian metric induced on $\mathcal{T}_g^0 = \mathcal{T}_g - \mathcal{H}_g$ from the symplectic metric (13) on \mathcal{S}_g , (\mathcal{T}_g^0 is immersed in \mathcal{S}_g by the period mapping π), and then take the corresponding volume form on \mathcal{T}_g^0 (which is certainly modular-invariant). We thus get a volume form on \mathcal{M}_g^0 for every g .

Siegel [9], p. 6, computed the (finite) volume of $\mathcal{S}_g/\text{Sp}(2g, \mathbb{Z})$ with the volume element of (13). Thus our measures give finite total mass to \mathcal{M}_g at least for $g = 1, 2, 3$. We do not know at present whether the volume of \mathcal{M}_g will be finite for $g \geq 4$ because the Schottky problem ($\mathcal{T}_g \subset \mathcal{S}_g$) complicates matters thoroughly.

Remark. Note that no simple expression like (12) was possible for the Polyakov measure on \mathcal{M}_g because the Mumford isomorphism is inexplicit, so one does not know which wedge of holomorphic quadratic differentials should correspond to $(\omega_1 \wedge \cdots \wedge \omega_g)^{13}$.

5. Another related canonical metric on T_g

The Riemannian metric on T_g^0 induced from Siegel's metric (13) on Siegel's upper half-space came up, as we saw, very naturally in the calculations of this paper. This

metric had previously been given consideration by Royden [7]. This metric on moduli has a close connection to another completely natural hermitian metric on T_g induced via the metric on the Riemann surface X obtained from the embedding of X in its flat canonically polarized Jacobi variety.

We conclude with some explanations and open questions regarding this last metric mentioned above. Notice that given any canonical choice of a (conformal) Riemannian metric on a (varying) Riemann surface X , there is a naturally induced metric on the Teichmüller space $T(X)$. Indeed, to define a hermitian (co-)metric on $T(X)$ one needs to assign a hermitian inner product on the space of holomorphic quadratic differentials on X (since these comprise the cotangent vectors to $T(X)$). Such a pairing is always definable by the formula

$$\langle \varphi, \psi \rangle = \int_X \int \varphi \bar{\psi} H^{-1} \quad (19)$$

where $ds^2 = H(z)|dz|^2$ is the chosen conformal metric on X . For example, when ds^2 is chosen as the hyperbolic metric, we obtain the Weil–Peterson metric on $T(X)$ by this construction (see, for example, [5], p. 404.).

Our idea is to use the metric on the compact Riemann surface obtained from the Abel–Jacobi embedding $X \hookrightarrow J(X)$, where $J(X)$ is equipped with its canonical flat metric. In fact,

$$J(X) = A(X)^*/H_1(X, \mathbb{Z}), \quad (20)$$

where $A(X)$ is the g -dimensional vector space of holomorphic 1-forms on X . The Hodge inner product (11) on $A(X)$ gives a dual inner product to the dual vector space, and this makes $J(X)$ a flat torus in the canonical way. The pull-back metric on X is a conformal (Kähler) metric which assigns total area g to X , and has non-positive curvature everywhere on X . These assertions are very easy to verify. Indeed, note that the curvature inequality follows from general principles because X is embedded as a minimal surface in $J(X)$; (any Kähler submanifold of a Kähler manifold is a minimal variety). So the two principal curvatures are equal and opposite (mean curvature must vanish) – hence the Gaussian curvature is non-positive on X , as stated.

A computation shows that the local expression for this Jacobian-induced metric on X is $ds^2 = H(z)|dz|^2$ where

$$H(z) = \sum_{j,k=1}^g \omega_j(z) \lambda_{jk} \overline{\omega_k(z)}. \quad (21)$$

Here $(\omega_1, \dots, \omega_g)$ is the standard Riemann normalized basis for $A(X)$ (see §2), and the λ -matrix is the inverse of the positive-definite $(\operatorname{Im} \pi_{ij})$ matrix. $H(z)$ can also be expressed as $k(z, z)$ where

$$k(z, \zeta) = \sum_{j=1}^g u_j(z) \overline{u_j(\zeta)} \quad (22)$$

is the reproducing kernel for $A(X)$ with respect to the Hodge pairing (11). Here (u_1, \dots, u_g) is any orthonormal basis for $A(X)$ with respect to (11). (Note: $k(z, \zeta)$ is a “bi-Abelian differential” on X and $H(z) = k(z, z)$ is an area form.)

Using this H in formula (19) we thus obtain a natural "Jacobian-induced" hermitian metric on T_g . One checks that this metric is modular invariant on T_g because of the naturality of the construction.

The Jacobian-induced metric on T_g and the Siegel-space-induced metric are comparable via the expressions shown for each. Therefore, in order to study questions like finite volume for \mathcal{M}_g in the Siegel metric, and also for many other reasons as well as for its own sake, it would be interesting to find out whether this Jacobian-induced metric on T_g is (i) complete or not; (ii) has negative curvature or not; (iii) gives finite volume for \mathcal{M}_g or not. We hope to report on these matters in the future.

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References

- [1] Bers L, Holomorphic differentials as functions of moduli, *Bull. Am. Math. Soc.* **67** (1961) 206–210
- [2] Gunning R, *Riemann surfaces and generalized theta functions* (New York: Springer-Verlag) (1976)
- [3] Mumford D, Stability of projective varieties, *L. Ens. Math.* **24** (1977) 39–110
- [4] Mumford D, Abelian quotients of the Teichmüller modular group, *J. d'Anal. Math.* **18** (1967) 227–244
- [5] Nag S, *The complex analytic theory of Teichmüller spaces* (New York: John Wiley Interscience) (1988)
- [6] Nelson P, Lectures on strings and moduli space, *Physics Reports* (1987)
- [7] Royden H, Invariant metrics on Teichmüller space, in *Contributions to analysis* (eds) L Ahlfors *et al* (New York: Academic Press) (1974)
- [8] Siegel C L, *Topics in complex function theory* (New York: John Wiley Interscience) Vol. 33 (1973)
- [9] Siegel C L, *Symplectic geometry* (New York: Academic Press) (1964)