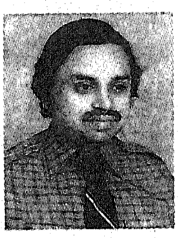
The Archimedes Principle and Gauss's Divergence Theorem

Subhashis Nag



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This article explores the connection between the Archimedes principle in physics and Gauss's divergence theorem in mathematics.

Introduction

Recently, while teaching a university course in vector-calculus, I remembered and revisited the business of deducing Archimedes' famous principle in hydrostatics from Gauss's divergence theorem (which relates certain surface integrals to volume integrals).(A well-known anecdote relates that Archimedes – having felt fluid pressure on his body (as I presume we all do) while immersed in a bathtub – understood it all in a flash, jumped out of the tub and emerged on the street in a state of nature, shouting *'Eureka!'*.) There is indeed a very elegant and deep connection between the Gauss divergence theorem and Archimedes' principle about the net effect due to pressure on any immersed solid. That study has generated the present article.

The main point for us will be to analyze the *resultant force* as well as the *resultant torque* (= turning moment) due to the entire system of forces arising from fluid pressure acting on the boundary surface of any three-dimensional body immersed in a homogeneous fluid at rest. (As we shall remark at the end of our calculations, the fluid need not even have been homogeneous for a rather general form of Archimedes' principle to be valid.)

I shall start by describing the resultant effect of an arbitrary system of forces acting at arbitrary points in $\mathbf{R}^3$. The mathematically minded reader may find the following discussion of three-dimensional statics independently useful.

Systems of Forces in $\mathbf{R}^3$

Let any finite (non-empty) set of force vectors $\mathbf{F}_i, 1 \leq i \leq$

N, be assigned as acting at arbitrary points of application $P_i \in \mathbf{R}^3$, in three-dimensional Euclidean space. For non-triviality, we may assume each $\mathbf{F}_i$ to be non-zero. The precise points of application are actually irrelevant as long as the *line of action*, say λ_i, for each force $\mathbf{F}_i$ is specified. (Namely, λ_i is the unique line passing through P_i in direction $\mathbf{F}_i$; indeed P_i may be chosen arbitrarily on λ_i in what follows.)

Associated to any such system of forces, we consider two fundamental vectors:

The resultant-force vector:

$$\vec{\mathbf{R}} = \sum_i \mathbf{F}_i \tag{1}$$

Torque-vector with respect to an arbitrary origin O:

$$\mathbf{Torq}(O) = \sum_i \vec{OP_i} \times \mathbf{F}_i \tag{2}$$

Clearly, $\mathbf{Torq}(O)$ is independent of the particular positions of the P_i along the λ_i. (Note: The $\times$ above denotes the usual vector-product ('cross-product') for vectors in $\mathbf{R}^3$.) (See any college-level mechanics text, e.g. French [2].)

Now, if we change the origin O to a different point O' we immediately deduce the following evident, but very basic, relationship:

$$\mathbf{Torq}(O) - \mathbf{Torq}(O') = \vec{OO'} \times \vec{\mathbf{R}} \tag{3}$$

Thus, given the resultant force $\vec{\mathbf{R}}$, the vector $\mathbf{Torq}(O)$ determines uniquely the torque-vector $\mathbf{Torq}(O')$, computed with respect to any other choice of origin.

The discussion above shows that it makes sense to make the following definition:

Mechanically Equivalent Systems: Two given systems of forces in $\mathbf{R}^3$: $\{System\ 1 : \mathbf{F}_i$ along lines $\lambda_i\}$, and $\{System\ 2 : \mathbf{G}_j$ along lines $\kappa_j\}$, are said to be *mechanically equivalent* if the corresponding resultant-force vectors are equal, and also the corresponding torque-vectors (computed with respect to any origin) are equal.

Remarks: We note for future use two evident yet important remarks:

(1) $\mathbf{Torq}(Q)$, as a vector-valued function of the point Q, is *constant on lines parallel to the resultant-force* $\vec{\mathbf{R}}$ – as is obvious from equation (3).

(2) There exists $Q \in \mathbf{R}^3$ such that the torque-vector computed with respect to Q is zero, i.e, $\mathbf{Torq}(Q) = \mathbf{0}$, if and only if the force system is equivalent to a *single* force acting along some line in $\mathbf{R}^3$. The point Q can be chosen arbitrarily on this line.

What Force Systems can be Reduced to a Single Force Only? We inquire: under what circumstances can we get a mechanically equivalent system consisting of just *one* force $\vec{\mathbf{R}}$ (assumed non-zero) acting along some specific line, say Λ, in $\mathbf{R}^3$? By the remark number (2) above, we see that we must be able to choose $Q \in \mathbf{R}^3$ so that $\mathbf{Torq}(Q) = \mathbf{0}$. The equation (3) then implies that we must be able to solve for the vector $\vec{OQ}$ from the equation:

$$\mathbf{Torq}(O) = \vec{OQ} \times \vec{\mathbf{R}} \tag{4}$$

The equation shows, of course, (recall the defining properties of the cross-product), that $\mathbf{Torq}(O)$ *must be orthogonal to* $\vec{\mathbf{R}}$. Consequently the *necessary condition* for a solution Q to exist is that:

$$\vec{\mathbf{R}} \cdot \mathbf{Torq}(O) = 0 \tag{5}$$

A simple calculation now shows that if (5) is valid then the solutions Q to (4) lie along the following line:

$$\vec{OQ} = \frac{1}{|\vec{\mathbf{R}}|^2}(\vec{\mathbf{R}} \times \mathbf{Torq}(O)) + t\vec{\mathbf{R}}, \text{ any real } t \tag{6}$$

The torque vanishes when computed with respect to any point on this particular line. Consequently, subject to condition (5), the equation (6) represents the line Λ along which

$\vec{\mathbf{R}}$ acts, and there is then no residual torque or any other forces.

Algorithm for Reducing an Arbitrary Force System: Our analysis demonstrates that, given an arbitrary system of forces acting along specified lines in $\mathbf{R}^3$, we obtain a mechanically equivalent reduced system by following the simple step-by-step algorithm below:

First step: Calculate $\vec{\mathbf{R}}$ and $\mathbf{Torq}(O)$ (the latter w.r.t. an arbitrary origin O)

Case 1: If $\vec{\mathbf{R}} = \mathbf{0}$, then equation (3) shows that the torque-vector is *independent* of the origin. Thus the system reduces to a pure 'couple', (representable (in infinitely many ways) as a pair of equal and opposite forces acting along some pair of parallel lines.)

Case 2: If $\vec{\mathbf{R}}$ is non-zero then check the dot-product:

$\vec{\mathbf{R}} \cdot \mathbf{Torq}(O)$

Case 2a: If $\vec{\mathbf{R}} \cdot \mathbf{Torq}(O) = 0$: then get a *single force* $\vec{\mathbf{R}}$ as the reduced equivalent system, acting *along the line whose equation was exhibited in (6) above.*

Case 2b: $\vec{\mathbf{R}} \cdot \mathbf{Torq}(O)$ $non-zero$: This is the *generic* case. A reduced equivalent system must consist of the force $\vec{\mathbf{R}}$ (say applied at origin O), together with a couple representing the torque $\mathbf{Torq}(O)$.

Interestingly enough, we shall demonstrate in a later section that the system of forces arising from fluid pressure acting on any immersed surface S in a homogeneous fluid, *always reduces to Case 2a* above! Namely, there is no turning moment left over. That is part of the beauty of Archimedes' principle – (although this matter is seldom explained in a school or college course).

Some Remarks and a Question: Here is a query that the mathematically minded among you may enjoy investigating. In the generic **Case 2b**, we saw that we can never choose Q so that $\mathbf{Torq}(Q)$ vanishes. What then is the set of points

$Q \in \mathbf{R}^3$ such that the vector $\mathbf{Torq}(Q)$ achieves its *minimum possible magnitude*? This is an instructive calculus problem of optimization. The reader will discover that this 'best' set of torque-centers Q comprises either a line, λ , or a plane, Π. In either case it is clear that the line or plane must be parallel to $\vec{\mathbf{R}}$ (since, by Remark (1) after equation (3), we know that $\mathbf{Torq}(Q)$ stays constant along lines parallel to $\vec{\mathbf{R}}$). *Prove that one gets a line or a plane consisting of 'minimizing' centers Q according as the rank of the following symmetric matrix, Quad($\vec{\mathbf{R}}$), is 2 or 1, (respectively).*

$$\mathrm{Quad}(\vec{\mathbf{R}}) = \begin{vmatrix} {r_2}^2 + {r_3}^2 & -r_1 r_2 & -r_1 r_3 \\ -r_2 r_1 & {r_3}^2 + {r_1}^2 & -r_2 r_3 \\ -r_3 r_1 & -r_3 r_2 & {r_1}^2 + {r_2}^2 \end{vmatrix}$$

where $\vec{\mathbf{R}} = (r_1, r_2, r_3)$. Note that the above matrix, whose entries are quadratic expressions in the components of the resultant $\vec{\mathbf{R}}$, always has determinant zero (check this identity!). Its rank is therefore always strictly less than three. Rank 0 is being ignored because that entails $\vec{\mathbf{R}} = \mathbf{0}$, (which we disallowed for Case 2b).

Gauss's Divergence Theorem

Let S be a smooth closed (= compact) surface in $\mathbf{R}^3$ enclosing (i.e., bounding) a three-dimensional region V. We shall write $\partial V = S$.

We let $\hat{\mathbf{n}}(x, y, z)$ denote the unit normal pointing outward at the general point $(x, y, z) \in S$. Let $\mathbf{G}$ be any vector field on $V \cup S$, possessing continuous partial derivatives on an open set containing $V \cup S$. Then the well-known *Gauss divergence theorem* states:

$$\int\int_S \mathbf{G} \cdot \hat{\mathbf{n}} \; \mathrm{d}S = \int\int\int_V \mathrm{div}(\mathbf{G}) \; \mathrm{d}V \qquad (7)$$

Notation: Recall that for a vector field $\mathbf{G}$ we set $\mathrm{div}(\mathbf{G}) = \nabla \cdot \mathbf{G}$ and $\mathrm{curl}(\mathbf{G}) = \nabla \times \mathbf{G}$ where $\nabla = \hat{\imath}\partial/\partial x + \hat{\jmath}\partial/\partial y + \hat{k}\partial/\partial z$. Therefore, $\mathrm{div}(\mathbf{G}) = \partial P/\partial x + \partial Q/\partial y + \partial R/\partial z$, if $\mathbf{G} = \hat{\imath}P + \hat{\jmath}Q + \hat{k}R$. We shall also denote partial derivatives by subscripts, where convenient. Finally note that $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$

denote the usual orthonormal set of unit basis vectors in the coordinate directions.

It is important to note that the divergence theorem relates the integral of a derivative of **G** (namely div(**G**)) over a three-dimensional region V, to the surface integral (the 'flux through the surface S') of the vector field **G** itself over the boundary surface. The double integral on the left side is with respect to the element of surface area on S, whereas the triple integral on the right is with respect to the volume element in $\mathbf{R}^3$.

The proof can, in fact, be reduced to several judicious applications of the standard fundamental theorem of calculus (which asserts that integration is the inverse process to differentiation). Karl Friedrich Gauss (1777–1855) discovered the above theorem while engaged in his research on electrostatics. The books by R Courant (a classic calculus text), and that by M Spivak, listed in the Suggested Reading, are good places to look for the divergence theorem. In Chapter V, section 5 of his text Courant indeed explains – very briefly – the idea that is behind what we are presently working out; also Spivak's last exercise (no. 5–36) on the last page of the book talks about the idea being presented in this article, (but, in both cases without mention of the possible turning effects arising from pressure).

The Resultant of Pressure Forces: A Generalized Archimedes' Principle

Let a solid region V, (a compact region of $\mathbf{R}^3$ with non-empty interior), having a smooth boundary surface $\partial V = S$, be immersed in a tank of liquid. We fix the xy plane to be the horizontal surface of the liquid, with the positive z-axis pointing upwards. Thus the fluid region is the lower half-space: $\{(x, y, z) \in \mathbf{R}^3 : z \leq 0\}$. (Note *Figure 1*)

From basic physics we recall that the pressure at any point $P(x, y, z)$, which is force per unit area, appears due to the weight of the column of liquid above that point. This force always acts in the direction that is *normal* to any element of surface that sits there at $P(x,y,z)$ (precisely because a liquid in equilibrium cannot support any shearing stresses) and its magnitude is independent of the direction of the normal to the area element.

Note on units: We shall identify the units of weight with the units of mass in what follows, i.e., we measure unit magnitude of force to be the gravitational force acting on one unit of mass. This eliminates the g factor in our equations.

Let ρ denote the density of the fluid. (Note:The perceptive reader will notice that in what follows, Theorem Eureka.R will go through for even a general inhomogeneous fluid (i.e. the density ρ could as well be a function of (x,y,z) varying from point to point), while Theorem Eureka.Torq. will go through for certain inhomogeneous fluids – for instance, if the density is a function of z alone.)

The fluid pressure, call it $\mu(x,y,z)$, at any point (x,y,z), is obtained, as explained, from the weight of (an infinitesimally thin) vertical column of liquid standing above that point. Consequently:

$$\mu(x,y,z) = \int_{(x,y,z)}^{(x,y,0)} \rho \mathrm{d}z \qquad (8)$$

the line integral being along the vertical line segment. In a homogeneous liquid, one has simply $\mu(x,y,z) = -\rho z$.

The General Mathematical Problem: Forgetting all the physics now, the chief mathematical problem we propose is to analyze the system of forces arising from a pressure of magnitude μ per unit area, acting in the inward normal direction all over any immersed surface S. This system of forces is therefore obtained by partitioning the entire surface S into tiny area elements, typically denoted ΔS, with ΔS containing the point $P(x,y,z) \in S$; the force acting at P on the area element ΔS is of magnitude equal to $\mu(P)\Delta S$ in the direction of the inward normal to the surface element ΔS. Namely,

$$\text{Force at } P \text{ on surface-element} \Delta S = -\mu(x,y,z)\Delta S\, \hat{\mathbf{n}}(x,y,z) \qquad (9)$$

(The minus sign arises because we defined $\hat{\mathbf{n}}(P)$ as denoting the *outward* unit normal to S at P.)

The mathematical 'Archimedes problem' is to now find the resultant force and the resultant torque due to these forces acting over the entire closed surface S. (Of course, we must in our calculations, proceed to the limit – as the partitioning of S becomes finer and finer – in the time-tested tradition of integral calculus.)

The Resultant Force $\vec{\mathbf{R}}$: $\vec{\mathbf{R}}$ is the vector sum of the forces (9) over all the area elements in the partitioning of S. Passing from Riemann sum to integral (replacing ΔS by $\mathrm{d}S$), we deduce that the three components of $\vec{\mathbf{R}}$ are:

$$\vec{\mathbf{R}} \cdot \hat{\imath} = -\int\int_S \mu(x,y,z)\hat{\imath} \cdot \hat{\mathbf{n}}\mathrm{d}S \qquad (10\mathrm{i})$$

$$\vec{\mathbf{R}} \cdot \hat{\jmath} = -\int\int_S \mu(x,y,z)\hat{\jmath} \cdot \hat{\mathbf{n}}\mathrm{d}S \qquad (10\mathrm{j})$$

$$\vec{\mathbf{R}} \cdot \hat{k} = -\int\int_S \mu(x,y,z)\hat{k} \cdot \hat{\mathbf{n}}\mathrm{d}S \qquad (10\mathrm{k})$$

But the above three are precisely the surface 'flux' integrals for the following three vector fields, respectively:

$$\mathbf{G}_1 = -\mu\hat{\imath}, \quad \mathbf{G}_2 = -\mu\hat{\jmath}, \quad \mathbf{G}_3 = -\mu\hat{k} \qquad (11)$$

where the $\mathbf{G}_i$ are vector fields defined in the entire fluid region $\{z \leq 0\}$. Therefore, the *Gauss divergence theorem (7) implies* that the resultant force is:

$$\vec{\mathbf{R}} = \hat{\imath}\int\int\int_V \mathrm{div}(\mathbf{G}_1)\mathrm{d}V +$$

$$\hat{\jmath}\int\int\int_V \mathrm{div}(\mathbf{G}_2)\mathrm{d}V + \hat{k}\int\int\int_V \mathrm{div}(\mathbf{G}_3)\mathrm{d}V \qquad (12)$$

But clearly we have: $\mathrm{div}(\mathbf{G}_3) = -\partial\mu/\partial z = \rho$ (see the definition (8) of μ). Consequently we have proved a basic result:

Theorem Eureka.R. *The resultant-force $\vec{\mathbf{R}}$, which is the vector sum of all the pressure forces acting on the compact immersed body V, is:*

$$\vec{\mathbf{R}} = -\hat{\imath}\int\int\int_V \frac{\partial\mu}{\partial x}\mathrm{d}V - \hat{\jmath}\int\int\int_V \frac{\partial\mu}{\partial y}\mathrm{d}V + \hat{k}\int\int\int_V \rho\mathrm{d}V$$

The $\hat{k}$ component is thus exactly equal to the weight of the fluid displaced by the immersed body, acting in the upward direction. The $\hat{\imath}$ and $\hat{\jmath}$ components vanish since the pressure depends only on depth (i.e., $\mu_x = \mu_y = 0$).

I surely need not remind the reader that the assertion above regarding the $\hat{k}$ component is exactly what they teach you in school physics as Archimedes' principle.

Analysis of the Resultant Torque: As we have learnt earlier, the resultant-force is *not* by any means the entire story about any given force system in 3-space. We need to understand also the torque-vector $\mathbf{Torq}(O)$, and thence obtain a reduced mechanically equivalent system. That is the goal we now pursue.

Let $\vec{\mathbf{r}}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ denote the position vector $\vec{OP}$ for $P(x,y,z) \in S$. Since the force on the element ΔS at P is $-\mu(x,y,z)\Delta S\hat{\mathbf{n}}(x,y,z)$ (see (9)), we observe that:

$$\mathbf{Torq}(O) = -\int\int_S \vec{\mathbf{r}} \times (\mu\hat{\mathbf{n}})\mathrm{d}S \qquad (13)$$

When decomposed into components we get:

$$\mathbf{Torq}(O)\cdot\hat{\imath} = \int\int_S \vec{\mathbf{r}} \times (\mu\hat{\imath})\cdot\hat{\mathbf{n}}\mathrm{d}S, \qquad (14)$$

and, of course, two other similar expressions for the $\hat{\jmath}$ and $\hat{k}$ components. (Each integrand is a 'scalar triple product'.)

The Gauss divergence theorem again applies immediately to each of these three integrals, and we obtain:

$$\mathbf{Torq}(O)\cdot\hat{\imath} = \int\int\int_V \mathrm{div}(\vec{\mathbf{r}} \times (\mu\hat{\imath}))\mathrm{d}V \qquad (15\mathrm{i})$$

$$\text{Torq}(O)\cdot\hat{\jmath} = \int\int\int_V \text{div}(\vec{\mathbf{r}}\times(\mu\hat{\jmath}))\text{d}V \qquad (15\text{j})$$

$$\text{Torq}(O)\cdot\hat{k} = \int\int\int_V \text{div}(\vec{\mathbf{r}}\times(\mu\hat{k}))\text{d}V \qquad (15\text{k})$$

We are now admirably positioned to state and prove the basic and final theorem:

Theorem Eureka.Torq. *The resultant torque-vector about the origin, owing to all the pressure forces on the surface of the immersed body V, is*

$$\text{Torq}(O) = \hat{\imath}\int\int\int_V [y\rho + z\mu_y]\text{d}V - \hat{\jmath}\int\int\int_V [x\rho + z\mu_x]\text{d}V + \hat{k}\int\int\int_V [y\mu_x - x\mu_y]\text{d}V$$

Since $\mu_x = \mu_y = 0$ throughout the fluid region, the torque simplifies to:

$$\text{Torq}(O) = \hat{\imath}\int\int\int_V y\rho\text{d}V - \hat{\jmath}\int\int\int_V x\rho\text{d}V \qquad (16)$$

But the resultant force $\vec{\mathbf{R}}$ (vide Theorem Eureka.**R***) acts vertically upward, whereas the torque vector lies in the horizontal plane – namely $\vec{\mathbf{R}}$ is orthogonal to* Torq(O)*. The system of forces arising from pressure therefore falls under Case 2a of the first section. Consequently, the entire system of forces is equivalent to a single upward force equal to the weight of the fluid displaced – acting (as we will prove below) in a vertical line passing through the center of mass of the region V of fluid displaced.*

Proof: We need simply calculate the divergences of each of the vector fields appearing on the right sides of equations (15i,j,k). One obtains:

$$\text{div}(\vec{\mathbf{r}}\times(\mu\hat{\imath})) = y\rho + z\mu_y$$

$$\text{div}(\vec{\mathbf{r}}\times(\mu\hat{\jmath})) = -x\rho - z\mu_x$$

$$\text{div}(\vec{\mathbf{r}}\times(\mu\hat{k})) = -x\mu_y + y\mu_x$$

[Note: The computation is greatly facilitated by the identity: $\mathrm{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \mathrm{curl}(\mathbf{F}) - \mathbf{F} \cdot \mathrm{curl}(\mathbf{G})$, and the fact that $\mathrm{curl}(\vec{\mathbf{r}}) = \mathbf{0}$.] That proves the main equation (actually for any general choice of the density function ρ – see remarks below). Thus (16) immediately follows, as desired.

The rest is easy. Clearly, $\vec{\mathbf{R}} \cdot \mathbf{Torq}(O) = 0$ – so we are indeed under the purview of Case 2a of the first section, as asserted. Therefore, the system reduces to a single force: $\vec{\mathbf{R}} = \hat{k}(\textit{weight of fluid displaced})$, acting along a vertical line. The position of that vertical line is immediately verified to pass through the 'center of mass' (of the *fluid displaced* – not of the immersed body, of course (!)). This point (CM) is the point whose coordinates are:

$$\text{Center of mass(CM)} = \left(\frac{\int\int\int_V x\rho \mathrm{d}V}{\int\int\int_V \rho \mathrm{d}V}, \frac{\int\int\int_V y\rho \mathrm{d}V}{\int\int\int_V \rho \mathrm{d}V}, \frac{\int\int\int_V z\rho \mathrm{d}V}{\int\int\int_V \rho \mathrm{d}V}\right)$$

That the final line of action passes through CM follows by applying either equation (3) (which checks that $\mathbf{Torq}(CM) = 0$), or, alternatively, by working out equation (6). That computation is simple, and we are through.

The Fish with the Turning Stomach: We have arrived at a rather satisfying and general statement of Archimedes' principle – with the added insight that the pressure forces always constitute a system under the special Case 2a. Let us note that if the poor fish (see *Figure 1*) experiencing **all** this pressure happens to have its own center of mass not in a vertical line with the center of mass of the fluid that it has displaced, then, (though fishy it may sound), it may well feel a certain turning effect in its stomach (and wonder what it had had for lunch). The fish is then advised to turn to either Dr. Archimedes or Dr. Gauss for help.

An Important Remark: A most interesting point in the mathematical derivation above is that we can just as easily carry out our analysis in a possibly *inhomogeneous* liquid of varying density, namely we may take the density at $P(x, y, z)$

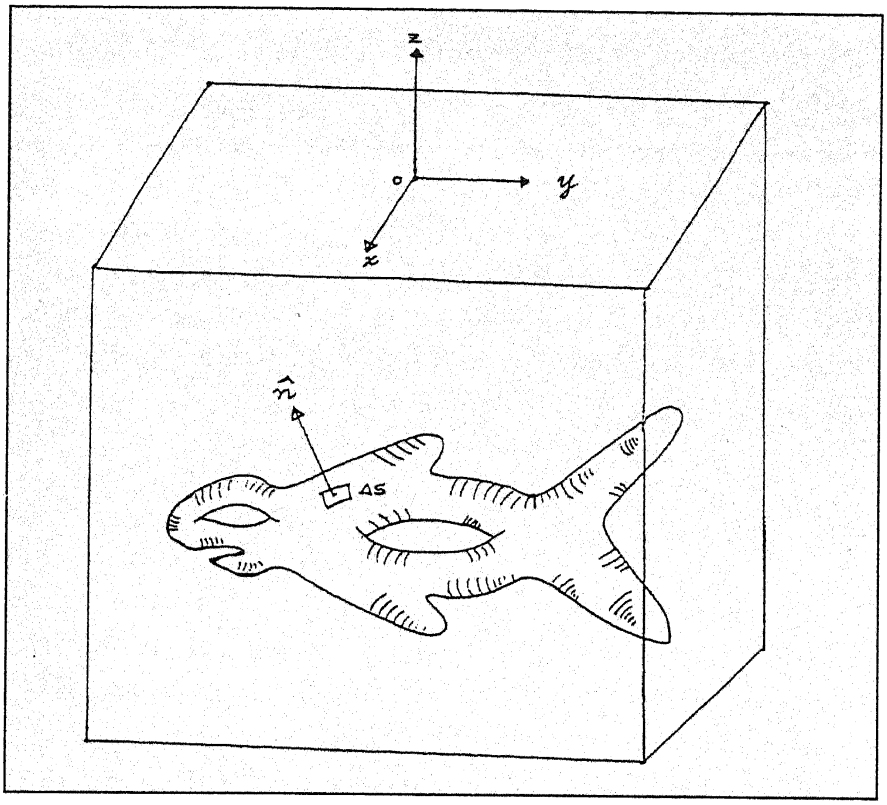

Figure 1. Immersed body of topological genus 2 (a bit fishy?).

to be a function $\rho = \rho(x, y, z)$. (For technical reasons one must assume that ρ has continuous partial derivatives with respect to the first two variables, while ρ is merely assumed to be continuous in the z variable.) If we now assume further that the density ρ depends only on the depth z (and not on x, y), we clearly have the same equations (i.e. equations 8 to 16) valid throughout. (since, in this case, $\mu_x = \mu_y = 0$.) Thus even in this generality we derive the same conclusion that *the final effect of all the pressure is mechanically equivalent to a single force of magnitude equal to the weight of displaced fluid, acting upward along the vertical line passing through the center of mass of the fluid in the region displaced. Remarkably enough, there is no residual turning moment!*

One may also note that the dot product $\vec{\mathbf{R}} \cdot \mathbf{Torq}(O)$ has a very suggestive formula even with the most general density function ρ. We leave the interested reader to pursue further that line of investigation.

I wish you many happy returns of 'Eureka'!

Suggested Reading

[1] R Courant. *Differential and Integral Calculus*. vol.II, (Translated by E McShane), Nordeman Publishing, Inc., New York, 1936.

[2] A P French. *Newtonian Mechanics*. MIT Introductory Physics Series. WW Norton and Co., New York, 1971.

[3] M Spivak. *Calculus on Manifolds*. W A Benjamin Inc., New York, 1965.

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