THEORY OF THE INDIAN MUSICAL DRUMS—PART I.

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1. *Introduction.*

A special type of musical drum (Mridanga or Thabala) which is widely used in India, has interesting acoustical properties which have been observed and reported on by C. V. Raman (1920, 1934). The instrument consists of a heavy wooden shell over which the drum-skin is stretched; the latter can be adjusted to any desired uniform tension, by sixteen leather thongs placed at equal intervals along its circumference. The drum-head is loaded symmetrically by an adherent dark material composed of iron-oxide, charcoal, starch and gum; the thickness of the load is greatest at the centre, and decreases in successive steps to zero at a distance of about half the radius from the centre. There is also a second membrane in the form of a ring superimposed on the drum-head round its margin. One effect of the central load is to prolong the vibrations of the membrane, especially in its graver modes. The most remarkable property of the drum-head is, however, that its proper vibrations form a harmonic sequence of five tones. Exactly how this result is attained will be seen from the diagrams (Figs. 1 and 2) reproduced from C. V. Raman's paper (1934), showing the modes and frequencies of vibration of the loaded drum-head as compared with those of an ordinary unloaded membrane. It will be seen that the first nine modes of vibration fall into groups having five different frequencies. It is obviously of considerable interest to investigate the law of symmetrical loading of a membrane which would give the results obtained in the Mridanga. In the

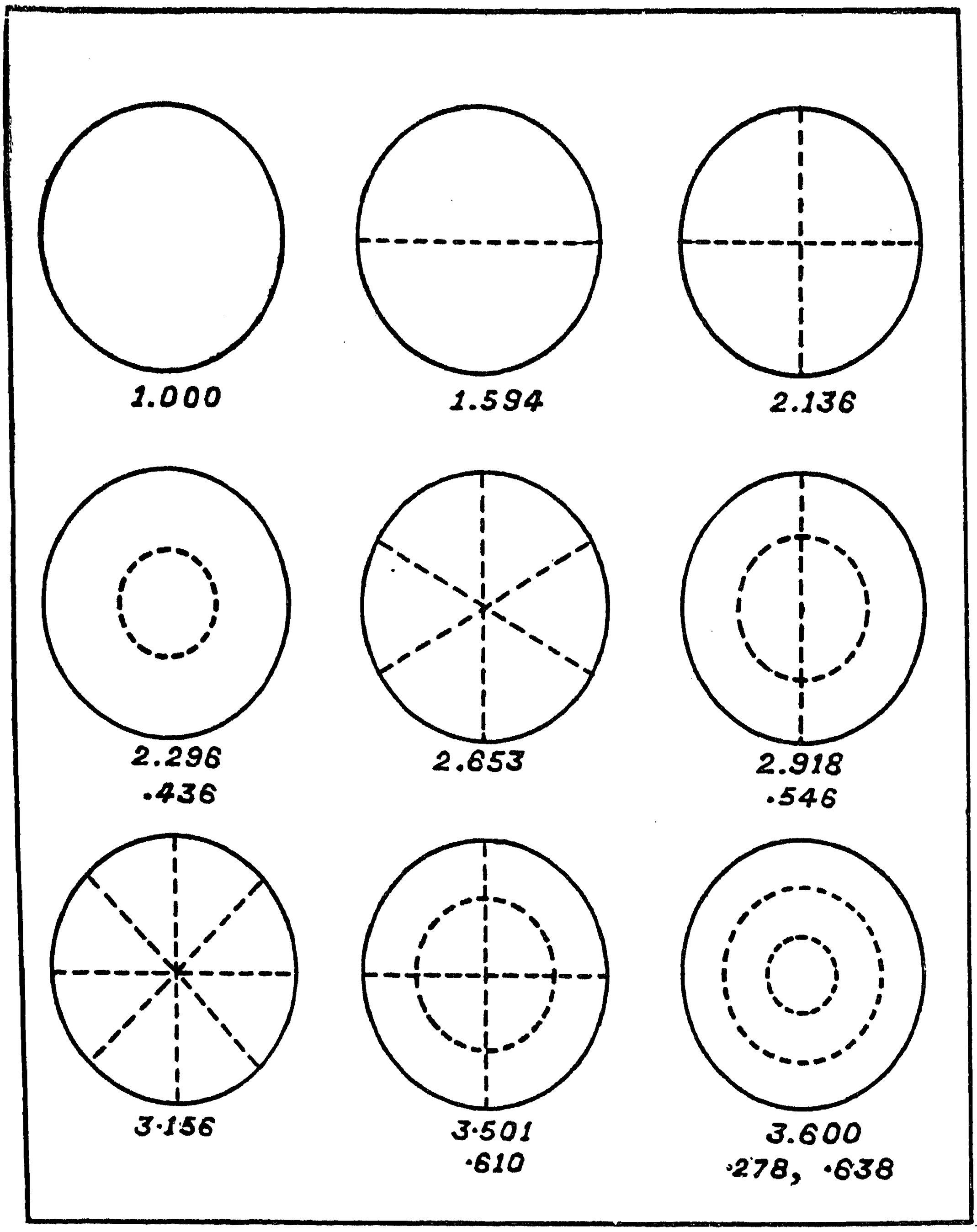

FIG. 1.

Normal Modes of Uniform Membrane.

present paper, the case is considered of a circular membrane the density of which varies with the radial distance from the centre, according to the law r^{-q}. By giving various values to q, we can determine the effect on the vibrations of the membrane of a gradually increasing concentration of load at the centre. It is obvious that this assumed law of loading is not the one actually adopted in the drum. Nevertheless the discussion is interesting as it gives an insight into the problem the practical solution of which is embodied in the Mridanga.

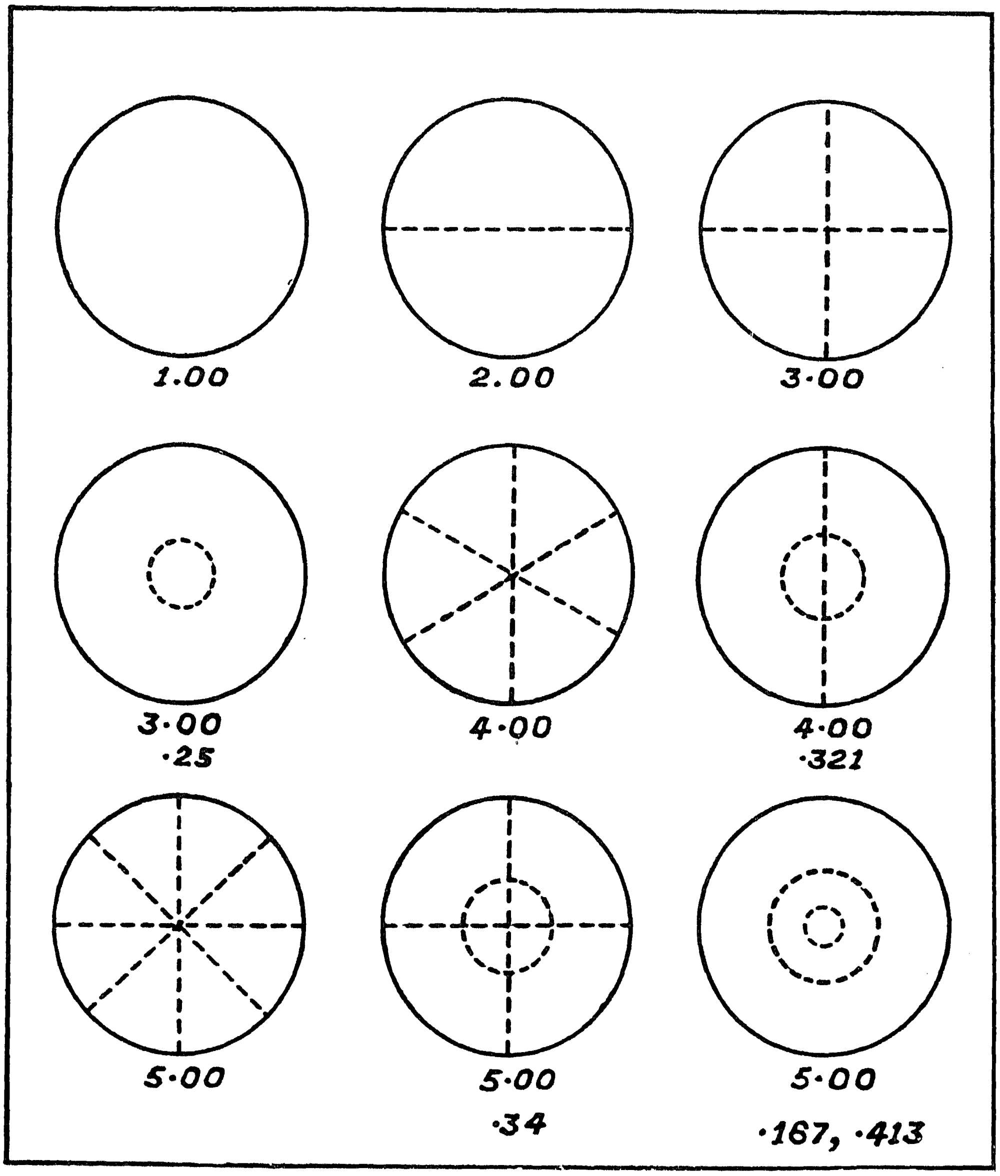

Fig. 2.

Normal Modes of Harmonic Drum.

2. *The Normal Modes of Vibration.*

The equation of motion of a membrane in polar co-ordinates is given by

$$\rho \frac{\partial^2 w}{\partial t^2} = T_1 \left\{ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right\}, \tag{1}$$

where w is the displacement, ρ is the superficial density and T_1 is the tension. Putting $w = R(r) \cos n(\theta - \beta) \cos(pt - \epsilon)$ in equation (1), we have

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left\{ k^2\rho - \frac{n^2}{r^2} \right\} R = 0, \tag{2}$$

where $k^2 = \frac{p^2}{T_1}$, n is an integer.

If $\rho = r^{-q}$ where $q = 2 - 2m$, (2) becomes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left\{ k^2 r^{2m-2} - \frac{n^2}{r^2} \right\} R = 0 \tag{3}$$

whose solution is given by

$$R = \left\{ A\, J_{n/m}\left(\frac{k}{m} r^m\right) + B\, Y_{n/m}\left(\frac{k}{m} r^m\right) \right\}.$$

In the case of the complete circular membrane, the Bessel function of the second order ($Y_{n/m}$) becomes infinite at the origin, and is hence to be omitted. Thus for a normal component vibration,

$$\boxed{w = A\, J_{n/m}\left(\frac{k}{m} r^m\right) \cos n\,(\theta - \beta) \cos (pt - \epsilon).}$$

The nodal system is given by $w = 0$, for all t.

$\therefore$ Cos $n\,(\theta - \beta) = 0$, *i.e.*, $\theta = \beta + \frac{(2m+1)\pi}{2n}$, which gives the diameters, and $J_{n/m}\left(\frac{k}{m} r^m\right) = 0$ represents a series of concentric nodal circles. As the membrane is fixed at the boundary, $r = a$, we have $J_{n/m}\left(\frac{k}{m} a^m\right) = 0$. This equation gives values for $\left(\frac{k}{m} a^m\right)$ which are infinite in number and all real. Let them be $\alpha_n^{(1)}, \alpha_n^{(2)}, \cdots, \alpha_n^{(s)}, \cdots$ arranged in order of magnitude. The frequencies are then given by $\alpha_n^{(s)} \left[\frac{\sqrt{T_1}}{2\pi} \cdot \frac{m}{a^m}\right]$.

3. *Effect of Loading on the Frequencies.*

To find the effect of varying the law of loading on the frequencies, it is convenient to regard the density at the boundary as invariable. This is secured by taking $a = 1$, when $\rho = 1$. The frequency equation is then $J_{n/m}\left(\frac{k}{m}\right) = 0$, which becomes for the case of the symmetrical vibrations,

$$J_0\left(\frac{k}{m}\right) = 0.$$

Let $\alpha_0^{(1)}, \alpha_0^{(2)}, \cdots$ be the roots of this equation. The frequencies of vibration are given by

$$\frac{\sqrt{T_1}}{2\pi}\left\{m\alpha_0^{(1)},\ m\alpha_0^{(2)},\ \cdots,\ m\alpha_0^{(s)},\ \cdots\right\},$$

where $0 < m \leqslant 1$.

If $m = 1, \rho = 1$, which is the case of the unloaded membrane.

If $m = 0, \rho = \frac{1}{r^2}$, and the mass of the membrane becomes infinite.

Since m is a fraction, as it diminishes from 1 to 0, the frequency falls off, varying as m. The period varies inversely as m and thus tends to infinity as $m \to 0$. The limiting case $m = 0$ is to be omitted from the range of variation for m, as the mass is then infinite and a continuous solution for R does not exist at the origin.

It is as well to remark here, that while the frequency changes in its absolute value, the ratios of the overtone frequencies of the symmetric modes to the fundamental frequency remain unaltered with the variation of m. $(m \neq 0.)$

Proof. For $m = m_1$, let $\begin{cases} \alpha_0^{(1)}, \alpha_0^{(2)}, \cdots \alpha_0^{(n)}, \cdots \text{be the roots of } J_0\left(k/_{m_1}\, a^{m_1}\right) = 0 \\ \text{and } f_1, f_2, \cdots f_n, \text{ be the corresponding frequencies.} \end{cases}$

For $m = m_2$

$$\begin{cases} \alpha_0^{(1)}, \alpha_0^{(2)}, \cdots \alpha_0^{(n)}, \cdots \text{ continue to be the roots of } J_0\left(\frac{k}{m^2}\, a^{m_2}\right) = 0 \\ \text{and } \phi_1, \phi_2, \cdots \phi_n, \cdots \text{ are the corresponding frequencies.} \end{cases}$$

We know that

$$\alpha_0^{(1)} : \alpha_0^{(2)} : \cdots : \alpha_0^{(n)} : \cdots :: f_1 : f_2 : \cdots : f_n : \cdots$$

$$\alpha_0^{(1)} : \alpha_0^{(2)} : \cdots : \alpha_0^{(n)} : \cdots :: \phi_1 : \phi_2 : \cdots : \phi_n : \cdots$$

$$\therefore\ f_1 : f_2 : f_3 : \cdots : f_n : \cdots :: \phi_1 : \phi_2 : \phi_3 : \cdots : \phi_n : \cdots$$

from which we have $\left\{\frac{f_2}{f_1} = \frac{\phi_2}{\phi_1}, \frac{f_3}{f_1} = \frac{\phi_3}{f_1}, \cdots \frac{f_n}{f_1} = \frac{\phi_n}{\phi_1} \cdots\right\}$,

which is the result stated above.

Taking the general frequency equation $J_{n/m}\left(\frac{k}{m}\right) = 0$, it can readily be shown that the frequency decreases from $m = 1$, when $m \to 0$, though not according to such a simple law as in the case of the symmetrical vibrations.

For $J_{n/m}\left(\frac{k}{m}\right) = 0$, we have

$$\alpha_n^{(s)} \approx \left\{\frac{n}{m} - \tfrac{1}{2} + 2s\right\} + \text{a fraction},$$

where n represents the number of nodal diameters and $(s-1)$ the number of internal nodal circles.

$$\therefore \quad m.\, \alpha_n^{(s)} \backsimeq \{n + (2s - \tfrac{1}{2})\, m\} + m. \text{ a fraction}$$

Hence $m.\, \alpha_n^{(s)}$, *i.e.*, the frequency decreases as m tends to zero.

When there are no internal nodal circles, we have

$$m.\, \alpha_n^{(1)} \backsimeq \{n + \tfrac{3}{2}\, m\} \frac{\pi}{2} + m \cdot \text{ a fraction.}$$

From this equation it is clear that the decrease in frequency is less than when there are both nodal circles and nodal diameters. Further, as n increases (*i.e.*, the number of nodal diameters increases), the effect of variation of m on the frequency becomes relatively smaller.

4. *Radii of the Nodal Circles.*

Consider the sth root $\alpha_n^{(s)}$ of $J_{n/m}\left(\frac{k}{m}\; a^m\right) = 0$. The corresponding solution for w is given by

$$w = A_n^{(s)}\, J_{n/m}\left\{\alpha_n^{(s)} \left(\frac{r}{a}\right)^m\right\} \cos n\,(\theta - \beta_n^{(s)}) \cdot \cos\left\{\left(\frac{\sqrt{T_1} \cdot m}{a^m}\right) \alpha_n^{(s)} \cdot t - \epsilon_n^{(s)}\right\}.$$

$\alpha_n^{(1)}$ corresponds to the lowest tone of the group n.

Consider $J_{n/m}\left\{\alpha_n^{(s)} \left(\frac{r}{a}\right)^m\right\}$. This vanishes not only when $r = a$, but also when

$\alpha_n^{(s)} \left(\frac{r}{a}\right)^m = \alpha_n^{(1)},\ \alpha_n^{(2)},\ \cdots\ \alpha_n^{(s-1)}$, showing that there are $(s-1)$ internal nodal circles whose radii are given by

$$r = a \left\{\frac{\alpha_n^{(1)}}{\alpha_n^{(s)}}\right\}^{\frac{1}{m}},\ a \left\{\frac{\alpha_n^{(2)}}{\alpha_n^{(s)}}\right\}^{\frac{1}{m}},\ \cdots,\ a \left\{\frac{\alpha_n^{(s-1)}}{\alpha_n^{(s)}}\right\}^{\frac{1}{m}}.$$

In the particular case of the symmetrical vibrations, the radii of the internal nodal circles are given by

$$r = a \left\{\frac{\alpha_0^{(1)}}{\alpha_0^{(s)}}\right\}^{\frac{1}{m}},\ a \left\{\frac{\alpha_0^{(2)}}{\alpha_0^{(s)}}\right\}^{\frac{1}{m}},\ \cdots,\ a \left\{\frac{\alpha_0^{(s-1)}}{\alpha_0^{(s)}}\right\}^{\frac{1}{m}}.$$

$$\left\{\frac{\alpha_0^{(n)}}{\alpha_0^{(s)}}\right\} < 1 \quad \text{for } n < s. \qquad 0 < m \leqslant 1.$$

$$\therefore \quad \left\{\frac{\alpha_0^{(n)}}{\alpha_0^{(s)}}\right\}^{\frac{1}{m}} < 1, \text{ and tends to zero as } m \rightarrow 0.$$

In the case of the symmetrical vibrations, the radii contract according to a definite law and $\rightarrow 0$ as $m \rightarrow 0$.

For modes with both nodal diameters and nodal circles, the latter contract as $m \to 0$, but not so rapidly as in the case when there are no nodal diameters.

5. *Discussion of the Results.*

The ratios of frequencies of the overtones to the fundamental, and the radii of the internal nodal circles are shown in Table I for various values of

TABLE I.

$q \to$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{8}{9}$	1	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$
N.D. 1 N.C. 0	1·593	1·65	1·78	1·88	1·92	2	2	2·15	2·65	3·15	3·65
N.D. 2 N.C. 0	2·135	2·35	2·45	2·66	2·8	2·94	3	3·16	4·13	5·07	6·02
N.D. 3 N.C. 0	2·653	2·99	3·16	3·41	3·58	3·85	3·9	4·13	5·54	6·94	
N.D. 4 N.C. 0	3·156	3·46	3·85	4·14	4·33	4·68	4·75	5·09	6·2		
N.D. 1 N.C. 1	2·918 *·546*	3	3·18 *·46*	3·23	3·28 *·423*	3·4 *·4*	3·43	3·51 *·37*	4·06	5·124	
N.D. 2 N.C. 1	3·501 *·61*	3·7	3·9 *·543*	4·06	4·2 *·523*	4·37 *·5*	4·4	4·6 *·43*	5·66	6·67	7·66
N.C. 1 N.D. 0	2·3 *·436*	2·3	2·3 *·329*	2·3	2·3 *·264*	2·3 *·228*	2·3 *·223*	2·3 *·19*	2·3	2·3	2·3
N.D. 0 N.C. 2	3·6 *·278* *·638*	3·6	3·6 *·181* *·548*	3·6	3·6 *·129* *·49*	3·6 *·103* *·451*	3·6 *·0997* *·441*	3·6 *·08* *·41*	3·6	3·6	3·6

(Figures in italics indicate the radii of the internal nodal circles.)

$q = (2 - 2m)$ in $\rho = \frac{1}{r^q}$, and are also represented in Figs. 3 and 4 respectively for modes of vibration with nodal diameters only, and for modes having nodal circles but with or without nodal diameters.

It is seen from Fig. 3 that when ρ varies nearly as $\frac{1}{r}$, *e.g.*, as $\left(\frac{1}{r^{\frac{7}{8}}}, \frac{1}{r^{\frac{8}{9}}}\right)$ the first four modes with nodal diameters only, approach the desired harmonic sequence. The laws of density necessary to bring the four respectively into exact harmonic relationship with the fundamental, are not, however, quite identical. As already remarked, the frequency ratios of the symmetric

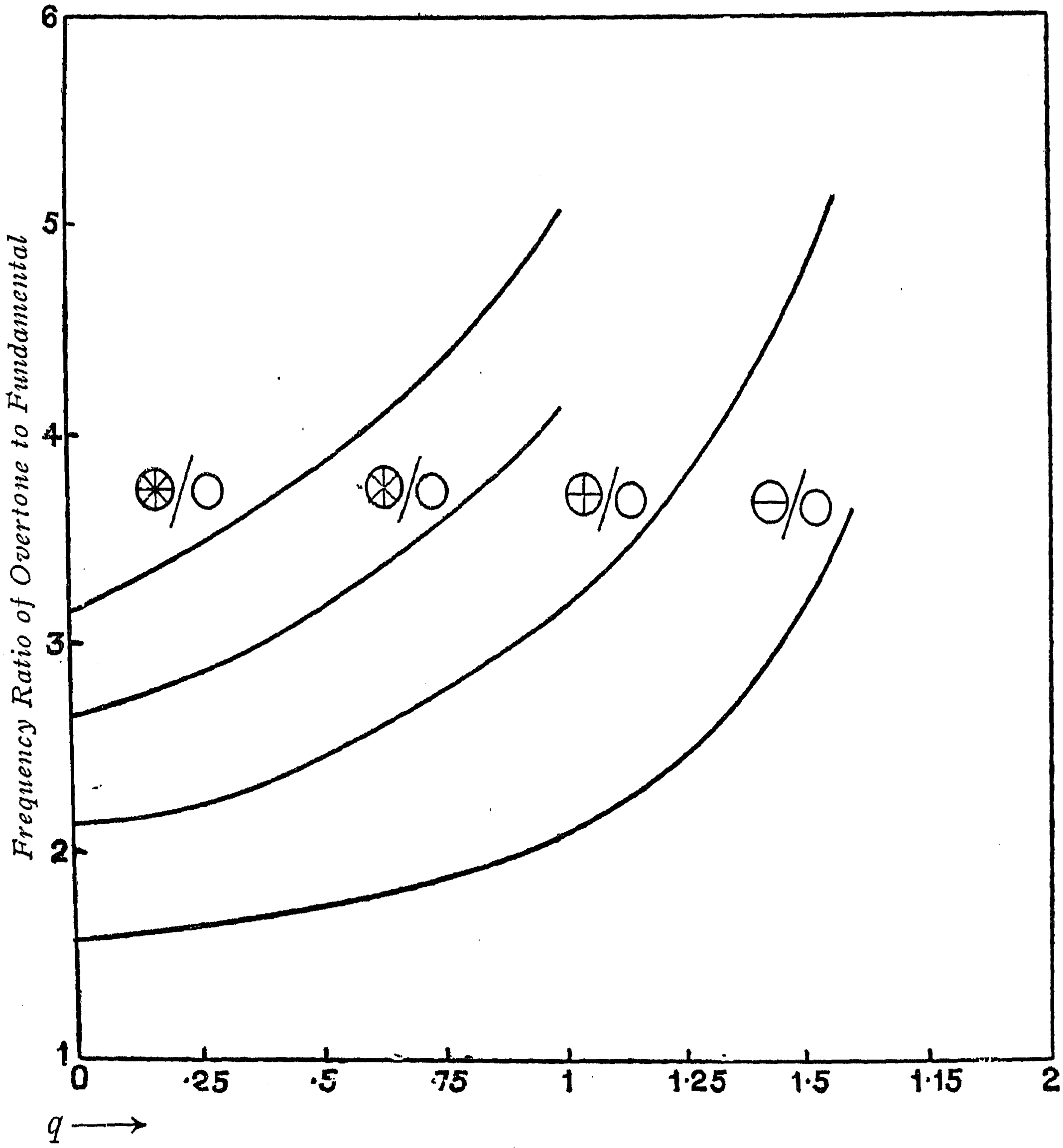

Fig. 3.

modes remain unaffected with increasing load, and hence their graphs appear in Fig. 4 as straight lines. The graphs in Fig. 4 for modes of vibration having one nodal diameter with one internal nodal circle, and two nodal diameters with one internal nodal circle, show that when $\rho = \frac{1}{r^{1 \cdot 25}}$ for the first, and $\rho = \frac{1}{r^{1 \cdot 15}}$ for the second, the frequency ratios become four and five respectively. The loading necessary is thus much greater than for modes with nodal diameters only.

A study of the results thus shows clearly that it is not possible for any value of q in $\rho = \frac{1}{r^q}$ to bring the first nine normal modes into the desired harmonic sequence as in Fig. 2. The failure is specially marked in the case of the symmetrical modes of which indeed the ratios of the frequencies

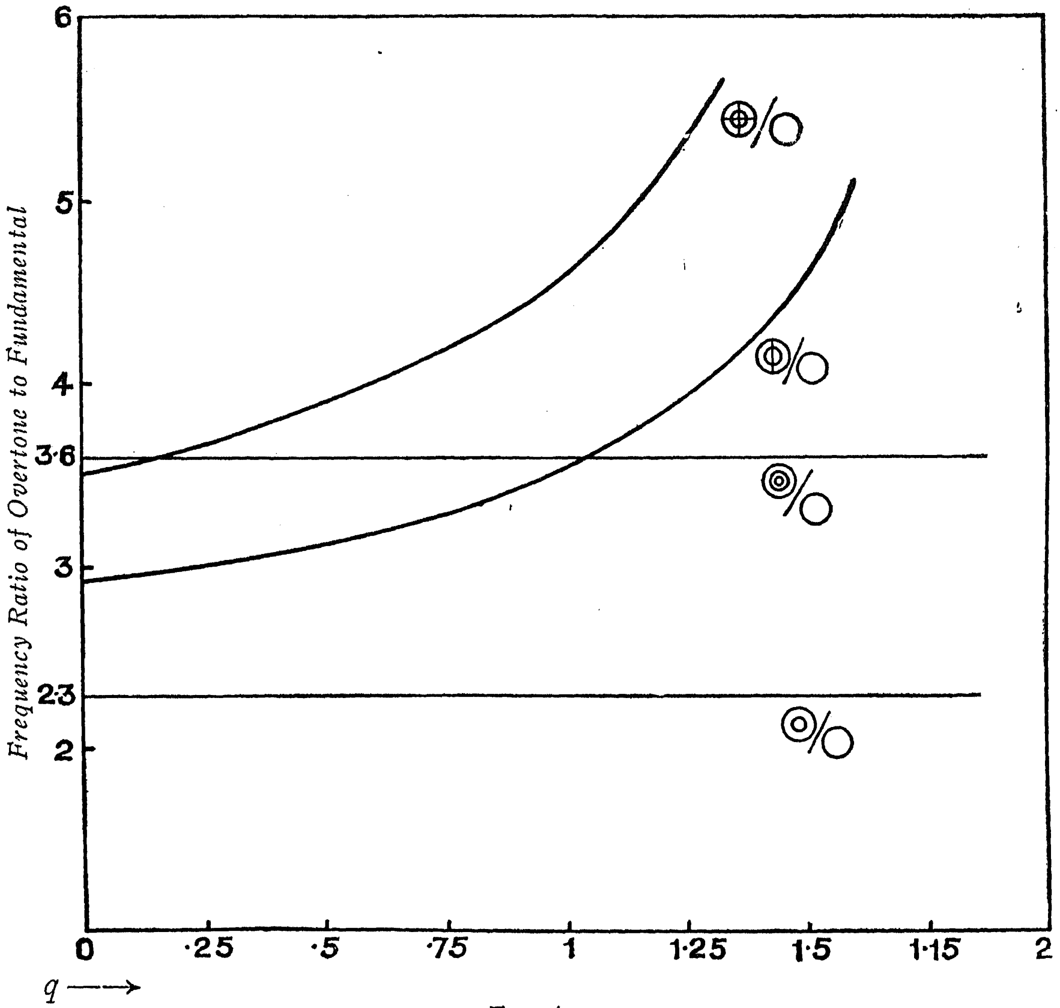

FIG. 4.

remain unaltered by the type of loading considered. It is characteristic of the law of density discussed in the paper that the density of the membrane is greatest at its centre. If instead of such a relatively concentrated loading, we have, as in the actual drum, a more widely distributed type of load, the latter may be expected to influence frequencies of the symmetrical modes differently in each case; the lowering of the frequency should then be more marked for the fundamental and less marked for the overtones. Such a distributed load should therefore favour the adjustment of frequencies of the symmetric modes into a harmonic sequence. On the other hand, the frequencies of the modes with nodal diameters would not depend greatly on whether the load at the centre is a concentrated or a distributed one. It should therefore be possible with a suitably distributed load to get much the same result as with the law of density $\frac{1}{r^q}$ considered above, and thus to put these modes into a harmonic sequence. These considerations, however, are only qualitative and remain to be established by calculation of the frequencies of the membrane for symmetrically distributed types of load having an appropriate law of density.

In conclusion, I wish to express my heartfelt thanks to Professor Sir C. V. Raman for suggesting the problem and for the valuable guidance given to me throughout the work.

Summary.

The paper contains a discussion of the normal modes of vibration of a symmetrically loaded membrane in which the surface density varies as an inverse fractional power of the radial distance from the centre, with a view to ascertain how far a law of density of this type enables the harmonic sequence of tones observed in the Indian musical drum to be approximated to. It is shown that for a law of density varying in inverse proportion to the radius, the first four modes with nodal diameters only, form an approximate harmonic sequence with the fundamental. With this law of loading, however, the relative frequencies of the symmetrical modes remain entirely unaffected (though their interior nodal circles contract), while the frequencies of the modes having both nodal diameters and nodal circles, require a greater degree of loading than that stated above to fall into the same harmonic sequence. The results thus show clearly that a type of loading so highly concentrated at the centre cannot succeed in reproducing completely the observed results. The indications are that a more widely distributed load such as is actually employed should theoretically be necessary to achieve the desired purpose.

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