DIFFRACTION OF LIGHT BY SUPERSONIC WAVES—PART I

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Received March 10, 1939
(Communicated by Sir C. V. Raman, Kt., F.R.S., N.L.)

Introduction

THE problem of the diffraction of light by supersonic waves has been rigorously investigated by Raman and Nath in their papers (R.N., IV and V) and later by N. S. N. Nath (N.I.). The starting point for the rigorous treatment is the wave equation for the propagation of a plane wave in a quasi-homogeneous medium. Working the solution by the help of fourier analysis, the emerging wave is decomposed into a number of orders governed by an infinite set of differential equations. Three methods of attack have been tried to obtain expressions for the amplitudes of the diffracted orders. The first one is the usual method of series and has been adopted by N. S. N. Nath (N.I.). He has calculated for the first few orders, the terms occurring in the power series, and squared them to obtain expressions for the intensities of the several orders. The second method of using Bessel-functions though implicit in the first, simplifies the solution considerably. This has been tried by Van Cittert. The basis of the third method consists in neglecting higher orders and solving the simultaneous differential equations thus obtained. This method has been tried by Van Cittert and recently by N. S. N. Nath for the case of normal incidence. The numerical evaluation of the intensities of the diffracted orders for the cases of normal and oblique incidences is a problem of great interest and the present investigation has been undertaken with that object in view. In the present paper, the mathematical formalism of the three methods is developed in great detail, with an appendix indicating the method followed in simplification, and summation processes. In the second part of the paper detailed calculations will be published and a discussion of the results on a comparative basis will be given.

Notation

μ_0 = Refractive index of the medium in the undisturbed state.

μ = maximum variation of the refractive index from μ_0.

λ = Wave-length of incident light.

$\lambda^{\times}$ = Wave-length of sound wave.

$\nu^{\times}$ = Frequency of the sound wave.

In all the signs of summation 'Σ', it is understood that all like terms are to be counted once only unless it is stated to the contrary.

1. *The Series Method*

The equation governing the propagation of a light wave in a quasi-homogeneous medium is given by

$$\nabla^2\psi = \left\{\frac{\mu\ (x, y, z, t)}{C}\right\}^2 \frac{\partial^2\psi}{\partial t^2}.$$

(Here ψ is the wave-function and $\mu\ (x, y, z, t)$ the refractive index of the medium, C the velocity of light). This equation after considerable simplification and reduction leads to the following difference-differential equation. (*Ref.* N.I.)

$$\boxed{2\frac{d\psi_n}{d\xi} - (\psi_{n-1} - \psi_{n+1}) = C_n\ \psi_n.} \tag{1}$$

(ψ_n being the amplitude of the nth diffracted order and $C_n = i\ \rho\ (n^2 + an)$ where $\rho = \dfrac{\lambda^2}{\mu_0\ \mu\lambda^{\times 2}}$ and $a\rho = -2na\ \sin\ \phi$, to the first powers of ϕ.)

The initial intensity being 1, we have

$$\sum_{-\infty}^{+\infty} |\ \psi_n\ |^2 = 1.$$

$$\therefore\ |\psi_n|^2 \leqslant 1.$$

We are therefore justified in solving the equation by the help of power series.

Let us consider equation (1). Putting

$$\psi_n = \left(\frac{\xi}{2}\right)^n \frac{1}{\underline{|n}} \left\{\sum_0^\infty A_{n,\,r}\ \xi^r\right\}, \quad \text{as in N. I.}$$

it is found to lead to the following recurrence relations.

$$\begin{cases} A_{0,\,1} = 0 \\ n\ A_{n,0} - n\ A_{n-1,0} = 0 \\ (n+1)\ A_{n,\,1} - n\ A_{n-1,\,1} = \frac{1}{2}\ C_n & (2) \\ 2\ (r+1)\ A_{0,\,r} - \frac{1}{2}\ \{A_{-1,r-1} - A_{1,r-1}\} = 0 \\ (n+r+1)\ A_{n,r+1} - n\ A_{n-1,\,r+1} \\ \qquad + \dfrac{1}{4\ (n+1)}\ A_{n+1,\,r-1} = \frac{1}{2}\ C_n\ A_{n,\,r}\ (\text{for } r \geqslant 1) & (2a) \end{cases}$$

$$A_{n,0} = A_{n-1,0} = A_{n-2,0} = \cdots\cdots = A_{0,0} = 1.$$
$$A_{n,0} = 1 \quad \text{for all } n \geqslant 0.$$
$$(n+1)\,A_{n,1} - n\,A_{n-1,1} = \tfrac{1}{2}\,C_n$$
$$n\,A_{n-1,1} - (n-1)\,A_{n-2,1} = \tfrac{1}{2}\,C_{n-1}$$
$$2\,A_{1,1} - A_{0,1} = \tfrac{1}{2}\,C_1.$$

On adding, we have

$$(n+1)\,A_{n,1} = \frac{\sum_1^n C_r}{2}$$

i.e.,

$$\left.\begin{aligned} A_{n,1} &= \frac{\sum_1^n C_r}{2\,(n+1)}. \quad \text{for } n \geqslant 1. \\ A_{-n,1} &= (-1)^n \frac{\sum_1^n C_{-r}}{2\,(n+1)}. \end{aligned}\right\} \qquad (3)$$

For $r = 1$, we have from (2 a)

$$(n+2)\,A_{n,2} - n\,A_{n-1,2} + \frac{1}{4\,(n+1)}\,A_{n+1,0} = \tfrac{1}{2}\,C_n\,A_{n,1}.$$

Multiplying both sides by $(n+1)$ we have

$$(n+2)\,(n+1)\,A_{n,2} - (n+1)\,n\,A_{n-1,2} + \frac{1}{4}\,A_{n+1,0} = \tfrac{1}{2}\,C_n\,(n+1)\,A_{n,1}$$

$$\therefore \quad (n+2)\,(n+1)\,A_{n,2} - (n+1)\,n\,A_{n-1,2} = \frac{C_n\,S_n}{4} - \frac{1}{4}.$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$3\cdot 2\,A_{1,2} - 2\cdot 1\cdot A_{0,2} = \frac{C_1\,S_1}{4} - \frac{1}{4}.$$

On adding, we get

$$(n+2)\,(n+1)\,A_{n,2} = 2\,A_{0,2} + \frac{1}{4}\sum_1^n C_r\,S_r - \frac{n}{4}$$
$$= \frac{1}{4}\sum_1^n C_r\,S_r - \left(\frac{n}{4} + \frac{1}{2}\right).$$

$$\left.\begin{aligned} \therefore \quad & \boxed{A_{n,2} = -\frac{1}{4\,(n+1)} + \frac{1}{4\,(n+1)\,(n+2)}\sum_1^n C_r\,S_r} \\ (-1)^n A_{-n,2} &= -\frac{1}{4\,(n+1)} + \frac{1}{4\,(n+1)\,(n+2)}\sum_1^n C_{-r}\Big(\sum_1^n C_{-r}\Big). \end{aligned}\right\} \qquad (4)$$

For $r = 2$, the difference equation (2 a) becomes

$$(n+3)\,A_{n,3} - n\,A_{n-1,3} + \frac{1}{4\,(n+1)}\,A_{n+1,1} = \tfrac{1}{2}\,C_n\,A_{n,2}.$$

$$\therefore \quad (n+3)\,A_{n,3} - n\,A_{n-1,3} = \tfrac{1}{2}C_n\left\{-\frac{1}{4(n+1)} + \frac{1}{4(n+1)(n+2)}\sum_1^n C_r C_s\right\} - \frac{1}{8(n+1)(n+2)}\sum_1^{n+1} C_r.$$

Multiplying both sides of the equation by $(n+1)(n+2)$ we get

$$(n+3)(n+2)(n+1)\,A_{n,3} - (n+2)(n+1)\,n\,A_{n-1,3} = -\tfrac{1}{8}S_{n+1} + \tfrac{1}{2}C_n\left\{\sum_1^n \frac{C_r C_s}{4} - \frac{(n+2)}{4}\right\}$$

$$\therefore \quad (n+3)(n+2)(n+1)\,A_{n,3}$$

$$= 6\,A_{0,3} - \sum_1^n \frac{S_{r+1}}{8} + \frac{1}{8}\sum_1^n C_n\left\{\sum_1^n C_n \sum_1^n C_n - (n+2)\right\}$$

$$= \frac{1}{8}\sum_1^n C_n \sum_1^n C_n \sum_1^n C_n - \frac{1}{8}\left\{C_{-1} + (n+4)\sum_1^n C_r + C_{n+1}\right\}$$

$$\left.\begin{array}{l} \boxed{\begin{array}{l} A_{n,3} = \dfrac{1}{8(n+1)(n+2)(n+3)} \\ \quad\left\{\begin{array}{l} -[C_{-1} + (n+4)\,S_n + C_{n+1}] + \sum_1^n r,s,t\,C_r C_s C_t \\ \qquad\qquad \text{all like terms to be written once only} \end{array}\right\} \end{array}} \\ (-1)^n A_{-n,3} = \dfrac{1}{8(n+1)(n+2)(n+3)} \\ \quad\left\{-[C_1 + (n+4)\sum_1^n C_{-r} + C_{-(n+1)}] + \sum_1^n r,s,t\,C_{-r}C_{-s}C_{-t}\right\} \end{array}\right\} \quad (5)$$

For $r = 3$, we have from $(2\,a)$

$$(n+4)\,A_{n,4} - n\,A_{n-1,4} + \frac{1}{4(n+1)}A_{n+1,2} = \tfrac{1}{2}C_n A_{n,3}$$

$$(n+4)\,A_{n,4} - n\,A_{n-1,4} = -\frac{1}{4(n+1)}\left\{-\frac{1}{4(n+2)} + \frac{1}{4(n+2)(n+3)}\sum_1^{n+1} C_r S_r\right\} + \tfrac{1}{2}C_n A_{n,3}.$$

$$\therefore \quad (n+1)(n+2)(n+3)(n+4)\,A_{n,4} - n(n+1)(n+2)(n+3)\,A_{n-1,4}$$

$$= \frac{1}{16}(n+3) - \frac{1}{16}\sum_1^{n+1} C_r S_r + \frac{1}{16}C_n\left\{\sum_1^n C_r C_s C_t - [C_{-1} + (n+4)\,S_n + C_{n+1}]\right\}.$$

$$\therefore \quad (n+1)(n+2)(n+3)(n+4)\,A_{n,4}$$

$$= 24\,A_{0,4} + \frac{1}{16}\sum_1^n (r+3) - \frac{1}{16}\sum_1^n\left\{\sum_1^{n+1} C_r S_r - C_n[C_{-1} + (n+4)\,S_n + C_{n+1}]\right\} + \frac{1}{16}\sum_1^n C_n \sum_1^n C_n \sum_1^n C_n \sum_1^n C_n.$$

$$= \frac{1}{16}\left\{\frac{n(n+1)}{2} + 3n\right\} - \left\{-\frac{3}{8} + \frac{1}{16}(C^2_{-1} + C_1^2) + \sum_1^n\left(\sum_1^{n+1} C_r S_r\right)\right\}$$

$$+ \frac{1}{16}\left\{C_{-1}S_n + \sum_1^n (n+4) S_n + \sum_1^n C_n C_{n+1}\right\} + \frac{1}{16}\sum_1^n {}_{r,s,t,u.}\, C_r C_s C_t C_u.$$

(all like terms are to be written once only)

$$\therefore \quad A_{n,4} = \frac{1}{32(n+1)(n+2)}$$

$$- \frac{1}{16(n+1)(n+2)(n+3)(n+4)}\left\{C^2_{-1}\right.$$

$$+ C_{-1}S_n + (n+6)\sum_1^n C_r C_s + C_{n+1}S_{n+1} + \sum_1^n C_r C_{r+1}$$

$$\left. - \left[\sum_1^n C_r C_s C_t C_u\right]\right\}.$$

Consider the expression

$$C^2_{-1} + C_{-1}S_n + (n+6)\sum_1^n C_r C_s + C_{n+1}S_{n+1} + \sum_1^n C_r C_{r+1}$$

$$= C^2_{-1} + C_{-1}S_n + (n+6)\left\{\sum_1^n C_r^2 + \sum_{\substack{1\\ r \neq s}}^n C_r C_s\right\} + C_{n+1}(C_1 + C_2 + \cdots + C_n)$$

$$+ C^2_{n+1} + \sum_1^n C_n C_{n+1}$$

$$= C^2_{-1} + C_{-1}S_n + (n+6)\sum_1^n C_r^2 + C^2_{n+1} + (n+7)\sum_1^n C_n C_{n-1}$$

$$+ (n+6)\sum_{\substack{1\\ r \neq s\\ r \sim s \neq 1}}^n C_r C_s + C_{n+1}(C_1 + C_2 + \cdots + C_{n-1} + 2C_n).$$

$$= C^2_{-1} + (n+6)\sum_1^n C_r^2 + C^2_{n+1} + (n+7)\sum_1^n C_{n-1}C_n + (n+6)\sum_{\substack{1\\ r \neq s\\ r \sim s \neq 1}}^n C_r C_s$$

$$+ C_{-1}(C_1 + C_2 + \cdot + C_n) + C_{n+1}(C_1 + C_2 + \cdot + C_{n-1} + 2C_n)$$

$$\left\{\begin{aligned} A_{n,4} &= \frac{1}{32(n+1)(n+2)} \\ &\quad - \frac{1}{16(n+1)(n+2)(n+3)(n+4)}\left\{C^2_{-1}\right. \\ &\quad + C_{-1}S_n + (n+6)\sum_1^n C_r^2 + C^2_{n+1} + (n+7)\sum_1^n C_n C_{n-1} \\ &\quad \left. + (n+6)\sum_{\substack{1\\ r \neq s\\ r \sim s \neq 1}}^n C_r C_s + C_{n+1}(C_1 + C_2 + \cdots + C_{n-1} + 2C_n)\right\}. \\ &\quad + \frac{1}{16(n+1)(n+2)(n+3)(n+4)}\sum_1^n C_r C_s C_t C_u. \end{aligned}\right. \tag{6}$$

Proceeding as before, we have,

$$(n+1)(n+2)(n+3)(n+4)\,A_{n,5}$$

$$= 120\,A_{0,5} - \tfrac{1}{4}\sum_{1}^{n}(n+4)(n+2)(n+3)\,A_{n+1,3}$$

$$+ \tfrac{1}{2}\sum_{1}^{n} C_n A_{n,4}(n+1)(n+2)(n+3)(n+4)$$

$$= -\frac{1}{32}\left\{C_1^3 + C^3_{-1} - 6(C_1 + C_{-1}) - C_2 - C_{-2}\right\}$$

$$-\frac{1}{32}\sum_{1}^{n}\left\{-[C_{-1} + (n+5)S_{n+1} + C_{n+2}] + \sum_{1}^{n+1} C_r C_s C_t\right\}$$

$$+\frac{1}{64}\sum_{1}^{n}(n+3)(n+4)\,C_n$$

$$-\frac{1}{32}\sum_{1}^{n} C_n\left\{C^2_{-1} + C_{-1}S_n + (n+6)\sum_{1}^{n} C_r S_r + C_{n+1}S_{n+1} + \sum_{1}^{n} C_n C_{n+1}\right\}$$

$$+\frac{1}{32}\sum_{1}^{n} C_n\left\{\sum_{1}^{n} C_r C_s C_t C_u\right\}.$$

Now consider the expression

$$C_{-2} + C_2 + 6(C_{-1} + C_1) + n\,C_{-1} + \sum_{1}^{n}(n+5)\,S_{n+1} + \sum_{1}^{n} C_{n+2}$$

$$+\sum_{1}^{n}\frac{(n+3)(n+4)}{2}\,C_n.$$

$$\sum_{0}^{n}(n+5)\,S_{n+1} = \sum_{r=1}^{n+1}\left\{\frac{(n+5)(n+6) - (r+3)(r+4)}{2}\,C_r\right\}$$

$$\therefore\quad C_{-2} + C_2 + 6(C_{-1} + C_1) + n\,C_{-1} + \sum_{1}^{n}(n+5)\,S_{n+1} + \sum_{1}^{n} C_{n+2}$$

$$+\sum_{1}^{n}\frac{(n+3)(n+4)}{2}\,C_n$$

$$= C_{-2} + (n+6)\,C_{-1} + \sum_{0}^{n}(n+5)\,S_{r+1} + \sum_{1}^{n+2} C_r + \sum_{1}^{n}\frac{(r+3)(r+4)}{2}\,C_n.$$

$$= C_{-2} + C_{n+2} + (n+6)\{C_{-1} + C_{n+1}\} + \left\{\frac{(n+5)(n+6)}{2} + 1\right\}\left(\sum_{1}^{n} C_r\right).$$

$$\therefore\quad A_{n,5}\cdot(n+1)(n+2)(n+3)(n+4)$$

$$= \frac{1}{32}\sum_{1}^{n} C_r C_s C_t C_u C_v$$

$$-\frac{1}{32}\left\{C^3_{-1} + C^2_{-1}\sum_{1}^{n} C_r + C_{-1}\sum_{1}^{n} C_r S_r\right\} \tag{7}$$

$$-\frac{1}{32}\left\{C_1^3 + \sum_{1}^{n}\left(\sum_{1}^{n+1} C_r C_s C_t\right) + \sum_{1}^{n}(n+6)\,C_n\left(\sum_{1}^{n} C_r S_r\right) + \sum_{1}^{n} C_n C_{n+1} S_{n+1}\right.$$

$$+\sum_1^n C_n \left(\sum_1^n C_n C_{n+1}\right)\Big\} + \frac{1}{32}\Big\{C_{-2} + C_{n+2} + (n+6)(C_{-1} + C_{n+1}) + \left[\frac{(n+5)(n+6)}{2} + 1\right] S_n\Big\}.$$

We shall now consider the general equation and effect some simplification in its solution. The general difference equation is given by

$$(n+r) A_{n,r} - n A_{n-1,r} + \frac{1}{4(n+1)} A_{n+1,r-2} = \tfrac{1}{2} C_n A_{n,r-1} (r \geqslant 1). \quad (2a)$$

To solve this, we multiply both sides of the equation by

$$(n+1)(n+2)\cdots(n+r-1).$$

$$\therefore (n+1)(n+2)\cdots(n+r) A_{n,r} - n(n+1)(n+2)\cdots(n+r-1) A_{n-1,r}$$
$$= \tfrac{1}{2}(n+1)(n+2)\cdots(n+r-1) C_n A_{n,r-1} - \tfrac{1}{4}(n+2)(n+3)\cdots(n+r-1) A_{n+1,r-2}.$$

Let $(n+1)(n+2)\cdots(n+r) A_{n,r} = U_{n,r}$. We have then

$$U_{n,r} - U_{n-1,r} = \tfrac{1}{2} C_n U_{n,r-1} - \tfrac{1}{4} U_{n+1,r-2}$$
$$U_{n-1,r} - U_{n-2,r} = \tfrac{1}{2} C_{n-1} U_{n-1,r-1} - \tfrac{1}{4} U_{n,r-2}$$

....

$$U_{1,r} - U_{0,r} = \tfrac{1}{2} C_1 U_{1,r-1} - \tfrac{1}{4} U_{2,r-2}$$

$\therefore$ On adding, we have

$$\boxed{U_{n,r} = U_{0,r} + \tfrac{1}{2}\sum_1^n C_n U_{n,r-1} - \tfrac{1}{4}\sum_1^n U_{n+1,r-2}} \quad (8)$$

But $$U_{n,r} = \frac{\Gamma(n+r+1)}{\Gamma(n+1)} A_{n,r}$$

$$\therefore \frac{\Gamma(n+r+1)}{\Gamma(n+1)} A_{n,r} = \Gamma(r+1) A_{0,r} + \sum_1^n \left\{\tfrac{1}{2} C_n A_{n,r-1} \frac{\Gamma(n+r)}{\Gamma(n+1)} - \tfrac{1}{4}\frac{\Gamma(n+r-1)}{\Gamma(n+1)} A_{n+1,r-2}\right\}$$

$$= \Gamma(r+1) A_{0,r} + \frac{1}{\Gamma(n+1)} \sum_1^n \left\{\frac{\Gamma(n+r) C_n A_{n,r-1}}{2} - \frac{\Gamma(n+r-1) A_{n+1,r-2}}{4}\right\}$$

$$A_{n,r} = \frac{\Gamma(r+1)\Gamma(n+1)}{\Gamma(n+r+1)} A_{0,r} + \frac{1}{\Gamma(n+1)} \sum_1^n \left\{\frac{\Gamma(n+r) C_n A_{n,r-1}}{2} - \frac{\Gamma(n+r-1)}{4} A_{n+1,r-2}\right\} \quad (9)$$

We shall now express the coefficients $A_{n,\,r}$ in terms of n, r and α.

$$A_{n,\,1} = \frac{\sum_{1}^{n} C_r}{2\,(n+1)} = \frac{i\rho}{2}\left\{\frac{n\,(n+1)\,(2n+1)}{6} + \frac{\alpha}{2}\,n\,(n+1)\right\}\frac{1}{(n+1)}$$

$$\therefore \quad \boxed{A_{n,\,1} = i\rho\left\{\frac{n\,(2n+1)}{12} + \frac{\alpha}{4}\,n\right\}} \qquad (10)$$

$$A_{1,\,1} = \frac{i\rho}{4}\,(1+\alpha) \qquad A_{-1,\,1} = -\frac{i\rho}{4}\,(1-\alpha)$$

$$A_{2,\,1} = \frac{i\rho}{6}\,(5+3\alpha) \qquad A_{-2,\,1} = \frac{i\rho}{6}\,(5-3\alpha)$$

$$A_{3,\,1} = \frac{i\rho}{4}\,(7+3\alpha) \qquad A_{-3,\,1} = -\frac{i\rho}{4}\,(7-3\alpha)$$

$$A_{4,\,1} = i\rho\,(3+\alpha) \qquad A_{-4,\,1} = i\rho\,(3-\alpha)$$

$$A_{5,\,1} = \frac{i\rho}{2}\,(55+15\alpha) \qquad A_{-5,\,1} = -\frac{i\rho}{2}\,(55-15\alpha)$$

$$A_{6,\,1} = \frac{i\rho}{2}\,(13+3\alpha) \qquad A_{-6,\,1} = \frac{i\rho}{2}\,(13-3\alpha)$$

$$A_{7,\,1} = \frac{i\rho}{4}\,(35+7\alpha) \qquad A_{-7,\,1} = -\frac{i\rho}{4}\,(35-7\alpha)$$

$$A_{8,\,1} = i\rho\left(\frac{34}{3} + 2\alpha\right) \qquad A_{-8,\,1} = i\rho\left(\frac{34}{3} - 2\alpha\right)$$

$$A_{9,\,1} = \frac{i\rho}{4}\,(57+9\alpha) \qquad A_{-9,\,1} = \frac{i\rho}{4}\,(57-9\alpha)$$

$$A_{10,\,1} = \frac{i\rho}{2}\,(35+5\alpha) \qquad A_{-10,\,1} = -\frac{i\rho}{2}\,(35-5\alpha)$$

$$A_{n,\,2} = -\frac{1}{4\,(n+1)} + \frac{1}{4\,(n+1)\,(n+2)}\sum_{1}^{n}{}_{r,\,s}\, C_r\, C_s.$$

$$= -\frac{1}{4\,(n+1)}$$

$$-\frac{\rho^2}{8\,(n+1)\,(n+2)}\left\{\frac{n\,(n+1)\,(2n+1)\,(2n+3)\,(5n-1)\,(n+2)}{180}\right.$$

$$\left. + \frac{\alpha}{3}\,n^2\,(n+1)^2\,(n+2) + \alpha^2\,\frac{n\,(n+1)\,(n+2)\,(3n+1)}{12}\right\}$$

$$A_{n,\,2} = -\frac{1}{4\,(n+1)} - \frac{\rho^2}{8}\left\{\frac{n\,(2n+1)\,(2n+3)\,(5n-1)}{180} + \frac{\alpha}{3}\,n^2\,(n+1)\right.$$

$$\left. + \frac{\alpha^2}{12}\,n\,(3n+1)\right\} \qquad (11)$$

$$A_{1,2} = -\frac{1}{8} - \frac{\rho^2}{24}(1+\alpha)^2 \qquad -A_{-1,2} = -\frac{1}{8} - \frac{\rho^2}{24}(1-\alpha)^2$$

$$A_{2,2} = -\frac{1}{12} - \frac{\rho^2}{48}(21+24\alpha+7\alpha^2) \qquad A_{-2,2} = -\frac{1}{12} - \frac{\rho^2}{48}(21-24\alpha+7\alpha^2)$$

$$A_{3,2} = -\frac{1}{16} - \frac{\rho^2}{80}(147+120\alpha+25\alpha^2) \qquad -A_{-3,2} = -\frac{1}{16} - \frac{\rho^2}{80}(147-120\alpha+25\alpha^2)$$

$$A_{4,2} = -\frac{1}{20} - \frac{\rho^2}{120}(627+400\alpha+65\alpha^2) \qquad A_{-4,2} = -\frac{1}{20} - \frac{\rho^2}{120}(627-400\alpha+65\alpha^2)$$

$$A_{5,2} = -\frac{1}{24} - \frac{\rho^2}{12}(143+75\alpha+10\alpha^2) \qquad -A_{-5,2} = -\frac{1}{24} - \frac{\rho^2}{12}(143-75\alpha+10\alpha^2)$$

$$A_{6,2} = -\frac{1}{28} - \frac{\rho^2}{16}(377+168\alpha+19\alpha^2) \qquad A_{-6,2} = -\frac{1}{28} - \frac{\rho^2}{16}(377-168\alpha+19\alpha^2)$$

$$A_{7,2} = -\frac{1}{32} - \frac{\rho^2}{48}(2023+294\alpha+77\alpha^2) \qquad -A_{-7,2} = -\frac{1}{32} - \frac{\rho^2}{48}(2023-294\alpha+77\alpha^2)$$

$$A_{8,2} = -\frac{1}{36} - \frac{\rho^2}{60}(3899+1440\alpha+125\alpha^2) \qquad A_{-8,2} = -\frac{1}{36} - \frac{\rho^2}{60}(3899-1440\alpha+125\alpha^2)$$

$$A_{9,2} = -\frac{1}{40} - \frac{\rho^2}{40}(1463+1350\alpha+105\alpha^2) \qquad -A_{-9,2} = -\frac{1}{40} - \frac{\rho^2}{40}(1463-1350\alpha+105\alpha^2)$$

$$A_{10,2} = -\frac{1}{44} - \frac{\rho^2}{48}(7889+2200\alpha+155\alpha^2) \qquad A_{10,2} = -\frac{1}{44} - \frac{\rho^2}{48}(7889-2200\alpha+155\alpha^2)$$

$$\begin{aligned}
A_{n,3} &= \frac{1}{8(n+1)(n+2)(n+3)}\left\{-\left[C_{-1}+C_{n+1}+(n+4)\sum_{1}^{n} C_r\right] + \sum_{1}^{n} C_r C_s C_t\right\} \\
&= -\frac{i\rho}{8(n+1)(n+2)(n+3)}\left\{-\left[\frac{n(n+2)(n+3)}{2}\alpha + \frac{n(n+1)(n+2)(2n+5)}{6} + (n+2)\right]\right. \\
&\quad + \frac{\rho^2}{6}\left[\left\{\frac{n(n+1)(2n+1)}{6} + \frac{\alpha}{2}n(n+1)\right\}^3\right. \\
&\qquad + 2\left\{\frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}\right. \\
&\quad + \frac{\alpha}{4}n^2(n+1)^2(2n^2+2n-1) + \frac{\alpha^2}{10}n(n+1)(2n+1)(3n^2+3n-1) \\
&\quad \left. + \frac{\alpha^3}{4}n^2(n+1)^2\right\} + 3\left\{\frac{n(n+1)(2n+1)}{6} + \frac{\alpha}{2}n(n+1)\right\}\times \\
&\quad \left\{\frac{n(n+1)(2n+1)(3n^2+3n-1)}{180} + \frac{\alpha}{2}n^2(n+1)^2\right. \\
&\quad \left.\left.\left. + \alpha^2\frac{n(n+1)(2n+1)}{6}\right]\right\}
\end{aligned} \tag{12}$$

$$A_{1,3} = -\frac{i\rho}{192}\left\{10 + 6\alpha + \rho^2(1+\alpha)^3\right\}$$

$$A_{2,3} = -\frac{i\rho}{480}\left\{(40+20\alpha) + \rho^2(85 + 141\alpha + 79\alpha^2 + 15\alpha^3\right\}$$

$$A_{3,3} = -\frac{i\rho}{960}\left\{(115+45\alpha) + \rho^2(1408 + 1662\alpha + 664\alpha^2 + 90\alpha^3)\right\}$$

$$A_{4,3} = -\frac{i\rho}{1680}\left\{(266+84\alpha) + \rho^2(11440 + 10570\alpha + 3304\alpha^2 + 350\alpha^3)\right\}$$

$$A_{5,3} = -\frac{i\rho}{2688}\left\{(532+140\alpha) + \rho^2(61490 + 46830\alpha + 12054\alpha^2 + 1050\alpha^3)\right\}$$

$$A_{6,3} = -\frac{i\rho}{4032}\left\{(960+216\alpha) + \rho^2(251498 + 163170\alpha + 35742\alpha^2 + 2646\alpha^3)\right\}$$

$$A_{7,3} = -\frac{i\rho}{5760}\left\{(1605+315\alpha) + \rho^2(906200 + 475298\alpha + 91308\alpha^2 + 5880\alpha^3)\right\}$$

$$A_{-1,3} = \frac{i\rho}{192}\left\{10 - 6\alpha + \rho^2(1-\alpha)^3\right\}$$

$$A_{-2,3} = -\frac{i\rho}{480}\left\{(40-20\alpha) + \rho^2(85 - 141\alpha + 79\alpha^2 - 15\alpha^3)\right\}$$

$$A_{-3,3} = \frac{i\rho}{960}\left\{(115-45\alpha) + \rho^2(1408 - 1662\alpha + 664\alpha^2 - 90\,\alpha^3)\right\}$$

$$A_{-4,3} = \frac{i\rho}{1680}\left\{(266-84\alpha) + \rho^2(11440 - 10570\alpha + 3304\alpha^2 - 350\,\alpha^3)\right\}$$

$$A_{-5,3} = \frac{i\rho}{2688}\left\{(532-140\alpha) + \rho^2(61490 - 46830\alpha + 12054\alpha^2 - 1050\alpha^3)\right\}$$

$$A_{-6,3} = -\frac{i\rho}{4032}\left\{(960-216\alpha) + \rho^2(251498 - 163170\alpha + 35742\alpha^2 - 2646\alpha^3)\right\}$$

$$A_{-7,3} = \frac{i\rho}{5760}\left\{(1605-315\alpha) + \rho^2(906200 - 475298\alpha + 91308\alpha^2 - 5880\alpha^3)\right\}$$

The expressions for the amplitudes of the diffracted orders are as below:

$$\left(\psi_n \text{ denotes the amplitude of the } n\text{th order, } \xi = \frac{2\pi\mu L}{\lambda}\right).$$

$$\psi_0 = 1 - \frac{\xi^2}{4} - \frac{i\rho\xi^3}{24} + \left\{\frac{1}{64} + \frac{\rho^2(1+\alpha^2)}{192}\right\}\xi^4 + i\rho\left\{\frac{1}{192} + \frac{\rho^2(1+3\alpha^2)}{1920}\right\}\xi^5$$

$$-\left\{\frac{1}{2304} + \frac{\rho^2(11+5\alpha^2)}{7680} + \frac{\rho^4(1+6\alpha^2+\alpha^4)}{23040}\right\}\xi^6$$

$$-i\rho\left\{\frac{70 + \rho^2(120+140\alpha^2) + \rho^4(1+10\alpha^2+5\alpha^4)}{2^6\cdot 5040}\right\}\xi^7$$

$$+\left\{\frac{1}{147456} + \frac{\rho^2(17+5\alpha^2)}{184320} + \frac{\rho^4(463+970\alpha^2+63\alpha^4)}{2^7\cdot(40320)}\right.$$

$$\left.+\frac{\rho^6(1+15\alpha^2+15\alpha^4+\alpha^6)}{2^7\cdot(40320)}\right\}\xi^8 + \cdots$$

$$\psi_1 = \frac{\xi}{2} + \frac{i\rho\ (1 + \alpha)}{8}\ \xi^2 - \left\{\frac{3 + \rho^2 (1 + \alpha)^2}{48}\right\} \xi^3 - i\rho \left\{\frac{(10 + 6\alpha) + \rho^2 (1 + \alpha)^3}{384}\right\} \xi^4$$

$$+ \left\{\frac{10 + \rho^2 (33 + 40\alpha + 15\alpha^2) + \rho^4 (1 + \alpha)^4}{3840}\right\} \xi^5 + i\rho \left\{70 + 30\alpha\right.$$

$$\left.\rho^2 (120 + 214\alpha + 130\alpha^2 + 30\alpha^3) + \rho^4 (1 + \alpha)^5\right\} \frac{\xi^6}{46080} + \left\{\frac{1}{144} + \right.$$

$$\left.\frac{\rho^2 (17 + 15\alpha + 15\alpha^2)}{180} + \frac{\rho^4 (463 + 1072\alpha + 956\alpha^2 + 371\alpha^3)}{5040} + \frac{\rho^6 (1 + \alpha)^6}{5040}\right\} \frac{\xi^7}{2^7}$$

$$\psi_2 = \frac{\xi^2}{8} + \frac{i\rho\ (5 + 3\alpha)}{48}\ \xi^3 - \left\{\frac{4 + \rho^2 (21 + 24\alpha + 7\alpha^2)}{384}\right\} \xi^4$$

$$- i\rho \left\{(40 + 20\alpha) + \rho^2 (85 + 141\alpha + 79\alpha^2 + 15\alpha^3)\right\} \frac{\xi^5}{3840}$$

$$+ \left\{15 + \rho^2 (340 + 320\alpha + 80\alpha^2)\right.$$

$$\left. + \rho^4 (341 + 738\alpha + 604\alpha^2 + 222\alpha^3 + 31\alpha^4)\right\} \frac{\xi^6}{46080} + \cdots$$

$$\psi_3 = \frac{\xi^3}{8} + \frac{i\rho\ (14 + 6\alpha)}{384}\ \xi^4 - \left\{\frac{1}{24} + \frac{\rho^2 (147 + 120\alpha + 25\alpha^2)}{120}\right\} \frac{\xi^5}{32}$$

$$- i\rho \left\{\frac{84 + 36\alpha + \rho^2 (627 + 400\alpha + 65\alpha^2)}{720}\right\} \frac{\xi^6}{64}$$

$$+ \frac{\xi^7}{128} \left\{\frac{21 + \rho^2 (2002 + 1470\alpha + 280\alpha^2) + \rho^4 (13013 + 19910\alpha + 11566\alpha^2 + 3024\alpha^3 + 301\alpha^4)}{5040}\right\}$$

$$+ \cdots\cdot$$

$$\psi_4 = \frac{\xi^4}{384} + \frac{i\rho\ (\alpha + 3)}{384}\ \xi^5 - \left\{\frac{6 + \rho^2 (627 + 400\alpha + 65\alpha^2)}{64 \cdot 720}\right\} \xi^6$$

$$- i\rho \left\{\frac{(266 + 84\alpha) + \rho^2 (11440 + 10570\alpha + 3304\alpha^2 + 350\alpha^3)}{645120}\right\} \xi^7$$

$$+ \cdots\cdots$$

$$\psi_5 = \frac{\xi^5}{3840} + \frac{i\rho\ (55 + 15\alpha)}{64 \cdot 720} - \left\{\frac{1 + \rho^2 (286 + 150\alpha + 120\alpha^2)}{132160}\right\} \xi^7$$

$$- i\rho \left\{\frac{1360 + 140\alpha + \rho^2 (61490 + 46830\alpha + 12054\alpha^2 + 1050\alpha^3)}{2^8 \cdot \underline{|8}}\right\} \xi^8 + \cdots$$

$$\psi_6 = \frac{\xi^6}{46080} + \frac{i\rho\ (13 + 3\alpha)}{132160}\ \xi^7 - \left\{\frac{8 + \rho^2 (5278 + 2352\alpha + 266\alpha^2)}{2^8\, \underline{|8}}\right\} \xi^8$$

$$- i\rho \left\{\frac{960 + 216\alpha + \rho^2 (251498 + 155570\alpha + 35742\alpha^2 + 2646\alpha^3)}{2^9\, \underline{|9}}\right\} \xi^9 + \cdots\cdot$$

$$\psi_7 = \frac{\xi^7}{2^7\, \underline{|7}} + \frac{i\rho\ (140 + 28\alpha)}{2^8\, \underline{|8}}\ \xi^8 - \frac{9 + \rho^2 (12138 + 4704\alpha + 462\alpha^2)}{2^9\, \underline{|9}}\ \xi^9 -$$

$$\psi_n = \left(\frac{\xi}{2}\right)^n \frac{1}{\underline{|n}} \left\{1 + i\rho \left[\frac{n\,(2n + 1)}{12} + \frac{\alpha}{4}\, n\right] \xi\right.$$

$$
-\left[\frac{1}{4(n+1)}+\frac{\rho^2}{2}\left\{\frac{n(2n+1)(2n+3)(5n-1)}{720}+\frac{\alpha}{12}n^2(n+1)\right.\right.
$$

$$
\left.\left.+\frac{\alpha^2}{48}n(3n+1)\right\}\right]\xi^2
$$

$$
\frac{-i\rho}{8(n+1)(n+2)(n+3)}\left[\left\{\frac{n(n+2)(n+3)}{2}\alpha+\frac{n(n+1)(n+2)(2n+5)}{6}+n+2\right\}\right.
$$

$$
+\frac{\rho^2}{6}\left\{\left[\frac{n(n+1)(2n+1)}{6}+\frac{\alpha}{2}n(n+1)\right]^3\right.
$$

$$
+2\left[\frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}+\frac{\alpha}{4}n^2(n+1)^2(2n^2+2n-1)\right.
$$

$$
\left.+\frac{\alpha^2}{10}n(n+1)(2n+1)(3n^2+3n-1)+\frac{\alpha^3}{4}n^2(n+1)^2\right]
$$

$$
+3\left[\frac{n(n+1)(2n+1)}{6}+\frac{\alpha}{2}n(n+1)\right]\left[\frac{n(n+1)(2n+1)(3n^2+3n-1)}{180}\right.
$$

$$
\left.\left.\left.+\frac{\alpha}{2}n^2(n+1)^2+\frac{\alpha^2}{6}n(n+1)(2n+1)\right]\right\}\right]\xi^3+\cdots\cdots
$$

$$
\psi_{-1}=\frac{\xi}{2}+\frac{i\rho(1-\alpha)}{8}\xi^2-\left\{\frac{3+\rho^2(1-\alpha)^2}{48}\right\}\xi^3-i\rho\left\{\frac{10-6\alpha+\rho^2(1-\alpha)^3}{384}\right\}\xi^4
$$

$$
+\left\{\frac{10+\rho^2(33-40\alpha+15\alpha^2)+\rho^4(1-\alpha)^4}{3840}\right\}\xi^5
$$

$$
+i\rho\left\{70-30\alpha+\rho^2(120-214\alpha+130\alpha^2-30\alpha^3)+\rho^4(1-\alpha)^5\right\}\frac{\xi^6}{46080}
$$

$$
+\left\{\frac{1}{144}+\frac{\rho^2(17-15\alpha+15\alpha^2)}{180}+\frac{\rho^4(463-1072\alpha+956\alpha^2-371\alpha^3)}{5040}\right.
$$

$$
\left.+\frac{\rho^6(1-\alpha)^6}{5040}\right\}\frac{\xi^7}{2^7}-\cdots\cdot
$$

$$
\psi_{-2}=\frac{\xi^2}{8}+\frac{i\rho(5-3\alpha)}{48}\xi^3-\left\{\frac{4+\rho^2(21-24\alpha+7\alpha^2)}{384}\right\}\xi^4
$$

$$
-i\rho\left\{(40-20\alpha)+\rho^2(85-141\alpha+79\alpha^2-15\alpha^3)\right\}\frac{\xi^5}{3840}
$$

$$
+\left\{15+\rho^2(340-320\alpha+80\alpha^2)+\rho^4(341-738\alpha+604\alpha^2-222\alpha^3+31\alpha^4)\right\}
$$

$$
\frac{\xi^6}{46080}+\cdots
$$

$$
\psi_{-3}=\frac{\xi^3}{8}+\frac{i\rho(14-6\alpha)}{384}\xi^4-\left\{\frac{1}{24}+\frac{\rho^2(147-120\alpha+25\alpha^2)}{120}\right\}\frac{\xi^5}{32}
$$

$$
-i\rho\left\{\frac{84-36\alpha+\rho^2(627-400\alpha+65\alpha^2)}{720}\right\}\frac{\xi^6}{64}
$$

$$
+\frac{\xi^7}{128}\left\{\frac{21+\rho^2(2002-1470\alpha+280\alpha^2)+\rho^4(13013-19910\alpha+11566\alpha^2-3024\alpha^3+301\alpha^4)}{5040}\right\}
$$

$$\psi_{-4} = \frac{\xi^4}{384} + \frac{i\rho(3-\alpha)}{384}\xi^5 - \left\{\frac{6+\rho^2(627-400\alpha+65\alpha^2)}{46080}\right\}\xi^6$$
$$- i\rho\left\{\frac{(266-84\alpha)+\rho^2(11440-10570\alpha+3304\alpha^2-350\alpha^3)}{645120}\right\}\xi^7 + \cdots$$

$$\psi_{-5} = \frac{\xi^5}{3840} + \frac{i\rho(55-15\alpha)}{46080}\xi^6 - \left\{\frac{1+\rho^2(286-150\alpha+120\alpha^2)}{9216}\right\}\xi^7$$
$$- i\rho\left\{\frac{1360-140\alpha+\rho^2(61490-46830\alpha+12054\alpha^2-1050\alpha^3)}{2^8\cdot\underline{|8}}\right\}\xi^8 + \cdots$$

$$\psi_{-6} = \frac{\xi^6}{46080} + \frac{i\rho(13-3\alpha)}{92160}\xi^7 - \left\{\frac{8+\rho^2(5278-2352\alpha+266\alpha^2)}{2^8\cdot\underline{|8}}\right\}\xi^8$$
$$- i\rho\left\{\frac{960-216^{\alpha}+\rho^2(251498-155570\alpha+35742\alpha^2-2646\alpha^3)}{2^9\cdot\underline{|9}}\right\}\xi^9 + \cdots$$

$$\psi_{-7} = \frac{\xi^7}{2^7\underline{|7}} - \frac{i\rho(140-28\alpha)}{2^8\cdot\underline{|8}}\xi^8 - \frac{9+\rho^2(12138-4704\alpha+462\alpha^2)}{2^9\cdot\underline{|9}}\Big\}\xi^9\cdots$$

$$\psi_{-n} = (-1)^n\left(\frac{\xi}{2}\right)^n\frac{1}{\underline{|n}}\left\{1 + i\rho\left[\frac{n(n+1)(2n+1)}{6} - \frac{\alpha}{2}n(n+1)\right]\xi\right.$$
$$- \left[\frac{1}{4(n+1)} + \frac{\rho^2}{2}\left\{\frac{n(2n+1)(2n+3)(5n-1)}{720} - \frac{\alpha}{12}n^2(n+1)\right.\right.$$
$$\left.\left. + \frac{\alpha^2}{48}n(3n+1)\right\}\right]\xi^2$$
$$- \frac{i\rho}{8(n+1)(n+2)(n+3)}\left[\left\{-\frac{n(n+2)(n+3)}{2}\alpha\right.\right.$$
$$\left. + \frac{n(n+1)(n+2)(2n+5)}{6} + (n+2)\right\} + \frac{\rho^2}{6}\left\{\left[\frac{n(n+1)(2n+1)}{6}\right.\right.$$
$$\left. - \frac{\alpha}{2}n(n+1)\right]^3 + 2\left[\frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}\right.$$
$$- \frac{\alpha}{4}n^2(n+1)^2(2n^2+2n-1) + \frac{\alpha^2}{10}n(n+1)(2n+1)(3n^2+3n-1)$$
$$\left. - \frac{\alpha^3}{4}n^2(n+1)^2\right] + 3\left[\frac{n(n+1)(2n+1)}{6} - \frac{\alpha}{2}n(n+1)\right]$$
$$\left.\left.\left.\left[\frac{n(n+1)(2n+1)(3n^2+3n-1)}{180} - \frac{\alpha}{2}n^2(n+1)^2 + \frac{\alpha^2}{6}n(n+1)(2n+1)\right]\right\}\right]\xi^3 + \cdots$$

2. *Bessel Function Method*

Let us consider equation (1)

$$2\frac{d\psi_p}{d\xi} - (\psi_{p-1} - \psi_{p+1}) = C_p\,\psi_p. \qquad (1)$$

Putting $\psi_p = J_p + \sum_1^\infty a_{p,\,p+n} J_{p+n}$, where $J_n = J_n(\xi)$, we have the following recurrence relations

$$\begin{cases} a_{0,1} = 0, \quad a_{0,2} = 0 \\ a_{0,n+1} - a_{0,n-1} = -\{a_{+1,n} - a_{-1,n}\} \\ a_{p,p+1} - a_{p-1,p} = C_p \\ a_{p,p+n+1} - a_{p-1,p+n} + a_{p+1,p+n} - a_{p,p+n-1} = C_p\, a_{p,p+n} \end{cases} \quad \begin{matrix} \\ (13) \\ \\ (13\,a) \end{matrix}$$

From (13 *a*), we have, by changing p to $p-1, \cdots 1$, and adding,

$$\left.\begin{aligned} a_{p,p+n+1} &= a_{0,n+1} - a_{p+1,p+n} + a_{1,n} + \sum_1^p C_p\,(a_{p,p+n}). \\ &= a_{0,n-1} - a_{p+1,p+n} + a_{-1,n} + \sum_1^p C_p\,(a_{p,p+n}). \end{aligned}\right\} \quad (14)$$

$$a_{p,p+1} - a_{p-1,p} = C_p$$

$$\boxed{a_{p,p+1} = \sum_1^p C_p.} \quad (15)$$

$$a_{p,p+2} = \sum_1^p C_p \left\{\sum_1^p C_p\right\} = \sum_1^p {}^{r,s'} C_r\, C_s \text{ (where like terms are written once only).}$$

$$\therefore \quad a_{p,p+2} = \sum_1^p {}^{r,s}\, C_r\, C_s. \quad (16)$$

From (14) for $n = 2$

$$a_{p,p+3} = a_{0,1} - a_{p+1,p+2} + a_{-1,2} + \sum_1^p C_p \left\{\sum_1^p C_p \sum_1^p C_p\right\}$$

$$a_{p,p+3} = -\left\{C_{-1} + \sum_1^{p+1}{}' C_r\right\} + \sum_1^p C_r\, C_s\, C_t. \quad (17)$$

(Details regarding summation, etc., are given in the appendix.)

From (14) for $n = 3$,

$$a_{p,p+4} = a_{0,2} + a_{-1,3} - a_{p+1,p+3} + \sum_1^p C_p\, a_{p,p+3}.$$

$$\therefore\ a_{p,p+4} = \sum_1^p C_r\, C_s\, C_t\, C_u - \left\{C^2_{-1} + C_{-1}\sum_1^p C_p + \sum_1^{p+1} C_r C_s + \sum_1^p C_p S_{p+1}\right\} \quad (18)$$

From (14) for $n = 4$, we have,

$$a_{p,p+5} = a_{0,3} + a_{-1,4} - a_{p+1,p+4} + \sum_1^p C_p\, a_{p,p+4}$$

$$= -(C_1 + C_{-1}) - \left\{C_{-1}^3 - (C_1 + C_{-1} + C_{-2})\right\} - \left\{\sum_1^{p+1} C_r C_s C_t - \left[C_{-1} + \sum_1^{p+2} C_r\right]\right\}$$

$$+ \sum_1^p C_p \left\{\sum_1^p C_r\, C_s\, C_t\, C_u - \left[C_{-1}^2 + C_{-1}\sum_1^p C_p + \sum_1^{p+1} C_r\, C_s + \sum_1^p C_p\, S_{p+1}\right]\right\}$$

$$a_{p,p+5} = \sum_1^p C_r\, C_s\, C_t\, C_u\, C_v - \left\{C^3_{-1} + C_{-1}^2 \sum_1^p C_r + C_{-1}\sum_1^p C_r\, C_s + \sum_1^{p+1} C_r\, C_s\, C_t\right.$$

$$+\sum_1^p C_p\left[\sum_1^{p+1} C_r C_s\right]+\sum_1^p C_p\left[\sum_1^p C_p S_{p+1}\right]\Big\}+\left\{C_{-1}+C_{-2}+\sum_1^{p+2} C_r\right\}.$$

$$=\sum_1^p C_r C_s C_t C_u C_v-\Big\{C_{-1}^3+C_{-1}^2\sum_1^p C_r+C_{-1}\sum_1^p C_r C_s+\sum_1^{p+1} C_r C_s C_t$$
$$+2\sum_1^p C_r C_s C_t+\sum_1^p C_p C_{p+1}\left[C_1+C_2+\cdots+C_{p-1}+2C_p+C_{p+1}\right]\Big\}$$
$$+\left\{C_{-1}+C_{-2}+\sum_1^{p+2} C_r\right\}. \tag{19}$$

$$a_{p,\,p+6}=\sum_1^p C_q C_r C_s C_t C_u C_v-\Big\{C_{-1}^4+C_{-1}^3\sum_1^p C_r+C_{-1}^2\sum_1^p C_r C_s$$
$$+C_{-1}\sum_1^p C_r C_s C_t+\sum_1^{p+1} C_r C_s C_t C_q\Big\}+\Big\{(C_{-1}+C_{-2})S_p+\sum_1^p C_p S_{p+2}$$
$$+(C_{-1}+C_{-2})^2+C_{-1}C_1\Big\}-\Big\{\sum_1^p C_p\sum_1^p C_p\sum_1^{p+1} C_r C_s$$
$$+\sum_1^p C_p\left[\sum_1^p C_p\sum_1^p C_p C_{p+1}\right]+\sum_1^p C_r C_s C_t C_q\Big\}. \tag{20}$$

We have from the equation

$$a_{0,\,n+1}-a_{0,\,n-1}=-\{a_{1,\,n}-a_{-1,\,n}\}$$

the following relations

$$\begin{cases} a_{0,\,1}=0 \\ a_{0,\,2}=0 \\ a_{0,\,3}=-\{C_1+C_{-1}\} \\ a_{0,\,4}=-\{C_{-1}^2+C_{+1}^2\} \\ a_{0,\,5}=-\{C_{-1}^3+C_1^3\}+\{C_{-1}+C_{-2}+C_1+C_5\} \\ a_{0,\,6}=-\{C_{-1}^4+C_1^4\}+\{(C_{-1}+C_1)^2+(C_{-2}+C_2)^2+(C_1+C_2)^2\} \\ a_{0,\,7}=-(C_1^5+C_{-1}^5)+\{3(C_1^3+C_{-1}^3)+(C_{-2}^3+C_2^3) \\ \quad +2(C_{-1}^2C_1+C_{-1}C_1^2)+2(C_2^2C_1+C_{-2}^2C_{-1}) \\ \quad +3(C_1^2C_2+C_{-1}^2C_{-2})-\{C_{-1}+C_{-2}+C_{-3}+C_1+C_2+C_3\} \end{cases} \tag{21}$$

In the expression for $a_{0,\,2n+1}$, the leading and the last terms are

$$a_{0,\,2n+1}=-\{C_1^{2n-1}+C_{-1}^{2n-1}\}+\{\qquad\qquad\qquad\}$$
$$+(-1)^n\left\{\sum_{-n}^{+n} C_r\right\}.$$

The order of this term for large n is governed by the first term

$$(C_1^{2n-1}+C_{-1}^{2n-1})$$

$$C_1^{2n-1}+C_{-1}^{2n-1}=(i\rho)^{2n-1}\{(1+\alpha)^{2n-1}+(1-\alpha)^{2n-1}\}.$$
$$=2(i\rho)^{2n-1}\{1+(2n-1)C_2\alpha^2+(2n-1)C_4\alpha^4+\cdots\}$$

If $|\alpha|<1$ only one of the terms need be considered.

The expressions for the amplitudes in terms of Bessel functions have been given in a previous paper,* by the author and here the expression for ψ_p only will be given;

$$\psi_p = J_p + i\rho\left\{\frac{p(p+1)(2p+1)}{6} + \frac{a}{2}p(p+1)\right\}J_{p+1}$$

$$-\frac{\rho^2}{2}\left\{\frac{p(p+1)(2p+1)(p+2)(2p+3)(5p-1)}{180} + \frac{a}{3}p^2(p+1)^2(p+2)\right.$$

$$\left.+\frac{a^2}{12}p(p+1)(p+2)(3p+1)\right\}J_{p+2}$$

$$+\left\{-i\rho\left[\frac{(p+1)(p+2)(2p+3)}{6} + 1 + \frac{a}{2}p(p+3)\right]\right.$$

$$-\frac{i\rho^3}{6}\left[\left\{\frac{p(p+1)(2p+1)}{6} + \frac{a}{2}p(p+1)\right\}^3\right.$$

$$+2\left\{\frac{p(p+1)(2p+1)(3p^4+6p^3-3p+1)}{42}\right.$$

$$+\frac{a}{4}p^2(p+1)^2(2p^2+2p-1) + \frac{a^2}{10}p(p+1)(2p+1)(3p^2+3p-1)$$

$$\left.+\frac{a^3}{4}p^2(p+1)^2\right\} + 3\left\{\frac{p(p+1)(2p+1)}{6} + \frac{a}{2}p(p+1)\right\}\times$$

$$\left\{\frac{p(p+1)(2p+1)(3p^2+3p-1)}{180} + \frac{a}{2}p^2(p+1)^2\right.$$

$$\left.\left.\left.+\frac{a^2}{6}p(p+1)(2p+1)\right\}\right]\right\}J_{p+3} + \cdots \quad (22)$$

3. *Closed Expression Method*

We obtain closed expressions for the amplitudes of the diffracted orders, by neglecting higher orders.

The difference-differential equation is

$$2\frac{d\psi_n}{d\xi} - (\psi_{n-1} - \psi_{n+1}) = C_n\psi_n. \quad (1)$$

Neglecting higher orders than the first on either side of the central order, we have to solve the following simultaneous differential equations.

$$\left.\begin{aligned} 2\frac{d\psi_{-1}}{d\xi} + \psi_0 &= C_{-1}\psi_{-1} \\ 2\frac{d\psi_0}{d\xi} - (\psi_{-1} - \psi_1) &= 0 \\ 2\frac{d\psi_1}{d\xi} - \psi_0 &= C_1\psi_1 \end{aligned}\right\} \quad (23)$$

$$\therefore\ 2\frac{d^2\psi_0}{d\xi^2} = \frac{d\psi_{-1}}{d\xi} - \frac{d\psi_1}{d\xi}\cdot\left(\frac{d\psi_1}{d\xi}\right)_{\xi=0} = \tfrac{1}{2}\text{ and }\left(\frac{d\psi_{-1}}{d\xi}\right)_{\xi=0} = -\tfrac{1}{2}.$$

$$\therefore\ \left(\frac{d^2\psi_0}{d\xi^2}\right)_{\xi=0} = -\tfrac{1}{2}.$$

The boundary conditions of the problem are

$$\begin{cases} \psi_n(0) = 0 & \text{for } n \neq 0 \\ \psi_0(0) = 1. \end{cases}$$ With the further conditions

$$\left(\frac{d\psi_0}{d\xi}\right)_{\xi=0} = 0 \text{ and } \left(\frac{d^2\psi_0}{d\xi^2}\right)_{\xi=0} = -\tfrac{1}{2}.$$

We have $\begin{vmatrix} 2D - C_{-1} & 1 & 0 \\ -1 & 2D & 1 \\ 0 & -1 & 2D - C_1 \end{vmatrix} S_0 = 0.$

$\therefore \quad S_0 = \sum_1^3 A_r e^{i\theta_r\xi}$, where $i(\theta_1, \theta_2, \theta_3)$ are the roots of the equation

$$8D^3 - 4D^2(C_1 + C_{-1}) + 2D(2 + C_1C_{-1}) - (C_1 + C_{-1}) = 0.$$

Putting $D = i\theta$, this equation simplifies to

$$\boxed{\theta^3 - \theta^2\rho + \theta\left\{\frac{\rho^2(1-\alpha^2) - 2}{4}\right\} + \frac{\rho}{4} = 0} \tag{24}$$

whose roots are now θ_1, θ_2 and θ_3.

When $\rho = 1$, this equation becomes

$$\boxed{\theta^3 - \theta^2 - \theta\left(\frac{1+\alpha^2}{4}\right) + \tfrac{1}{4} = 0} \tag{24 a}$$

The roots of this equation are all real. It can easily be seen that one of the roots always lies between 0 and 1.

1st approximation to the root between 0 and 1. $\left(\dfrac{1}{1+\alpha^2}\right)$

2nd $\cdots\cdots\cdots = \dfrac{\lambda^3 + 4\lambda - 8}{\lambda(\lambda^3 + 8\lambda - 12)}$, where $1 + \alpha^2 = \lambda$.

The expressions for the amplitudes S_0, S_1 and S_{-1} are

$$\begin{cases} S_0 = \sum_1^3 A_r e^{i\theta_r\xi} \\ S_1 = \sum_1^3 \dfrac{A_r}{(2i\theta_r - C_1)} e^{i\theta_r\xi} + a_1 e^{\frac{C_1}{2}\xi} \\ S_{-1} = -\sum_1^3 \dfrac{A_r}{(2i\theta_r - C_{-1})} e^{i\theta_r\xi} - a_2 e^{\frac{C_{-1}}{2}\xi} \end{cases}$$

where $a_1 = -\sum_1^3 \dfrac{A_r}{(2i\theta_r - C_1)}$ and $a_2 = -\sum_1^3 \dfrac{A_r}{2i\theta_r - C_{-1}}$.

Making use of the boundary conditions, we have

$$\begin{cases} \sum_{1}^{3} A_r = 1 \\ \sum_{1}^{3} A_r \theta_r = 0 \\ \sum_{1}^{3} A_r \theta_r^2 = \tfrac{1}{2} = \phi. \end{cases}$$

Solving for A_1, A_2, A_3 from these relations, we have

$$\left\{ A_r = \frac{\phi + \theta_s \theta_t}{(\theta_r - \theta_s)(\theta_r - \theta_t)} \right\} \quad r, s, t \text{ taking the values 1 to 3.}$$

To find the value of $\sum_{1}^{3} \dfrac{A_r}{2i\theta_r - C}$, where $C = C_{+1}$ or C_{-1}.

$$\sum_{1}^{3} \frac{A_r}{2i\theta_r - C} = -\Sigma \frac{\frac{1}{2} + \theta_2 \theta_3}{(\theta_1 - \theta_2)(\theta_3 - \theta_1)(2i\theta_1 - C)}$$

$$= -\frac{1}{\Pi(\theta_1 - \theta_2)\cdot \Pi(2i\theta_r - C)} \left\{ \Sigma(\tfrac{1}{2} + \theta_2\theta_3)(\theta_2 - \theta_3)(2i\theta_2 - C)(2i\theta_3 - C) \right\}.$$

$$\Sigma(\tfrac{1}{2} + \theta_2\theta_3)(\theta_2 - \theta_3)\{C^2 - 2iC(\theta_2 + \theta_3) - (4\theta_2\theta_3)\}$$

$$= \tfrac{1}{2}\Sigma(\theta_2 - \theta_3)(C^2 - 2iC\,\overline{\theta_2 + \theta_3} - 4\theta_2\theta_3)$$

$$+ \Sigma\theta_2\theta_3(\theta_2 - \theta_3)(C^2 - 2iC\,\overline{\theta_2 + \theta_3} - 4\theta_2\theta_3)$$

$$= \tfrac{1}{2}\{C^2\Sigma(\theta_2 - \theta_3) - 2iC\Sigma(\theta_2^2 - \theta_3^2) - 4\Sigma\theta_2\theta_3(\theta_2 - \theta_3)\}$$

$$+ \{C^2\Sigma\theta_2\theta_3(\theta_2 - \theta_3) - 2iC\Sigma\theta_2\theta_3(\theta_2^2 - \theta_3^2) - 4\Sigma\theta_2^2\theta_3^2(\theta_2 - \theta_3)\}$$

$$= 2\Pi(\theta_1 - \theta_2) + \{-C^2\Pi(\theta_1 - \theta_2) + 2iC\Pi(\theta_1 - \theta_2).\ \sum_{1}^{3}\theta_r$$

$$+ 4\Pi(\theta_1 - \theta_2)\cdot\Sigma\theta_2\theta_3\}$$

$$= \Pi(\theta_1 - \theta_2)\cdot\{2 - C^2 + 2iC\sum_{1}^{3}\theta_r + 4\Sigma\theta_2\theta_3\}.$$

$$i\sum_{1}^{3}\theta_r = \frac{C_1 + C_{-1}}{2}$$

$$i^2\Sigma\theta_2\theta_3 = \frac{(2 + C_{-1}C_1)}{4}$$

$$2 - C^2 + 2Ci\sum_{1}^{3}\theta_r + 4\Sigma\theta_2\theta_3$$

$$= 2 - C^2 + 2C\left(\frac{C_1 + C_{-1}}{2}\right) - (2 + C_{-1}C_1)$$

$$= -C^2 + C(C_{-1} + C_1) - C_{-1}C_1$$

$= 0$ when $C = C_{+1}$ or C_{-1}.

$$\therefore \quad \boxed{\sum_{1}^{3} \frac{A_r}{2i\theta_r - C_{+1}} = 0} \qquad (25)$$

The expressions for amplitudes are as below :

$$\left\{\begin{aligned} S_0 &= \sum_1^3 A_r\, e^{i\theta_r\xi} \\ S_1 &= \sum_1^3 \frac{A_r}{2i\theta_r - C_1}\, e^{i\theta_r\xi}, \qquad \text{where } A_r = \frac{\frac{1}{2} + \theta_s\,\theta_t}{(\theta_r - \theta_s)(\theta_r - \theta_t)} \\ S_{-1} &= \sum_1^3 \frac{A_r}{2i\theta_r - C_{-1}}\, e^{i\theta_r\xi}. \end{aligned}\right. \tag{26}$$

$$\begin{aligned} I_0 = |S_0|^2 &= \sum_1^3 A_r^2 + 2\,\Sigma\, A_r\, A_s \cos(\theta_r - \theta_s)\,\xi \\ &= (\sum_1^3 A_r)^2 - 2\,\Sigma\, A_r\, A_s\,(1 - \cos \overline{\theta_r - \theta_s\,\xi}) \\ &= 1 - 4\,\Sigma\, A_r\, A_s \sin^2\left(\frac{\theta_r - \theta_s}{2}\right)\xi. \end{aligned}$$

$$\left\{\begin{aligned} I_0 &= 1 - 4\sum_1^3 A_r\; A_s\; \sin^2\left(\frac{\theta_r - \theta_s}{2}\right)\xi. \\ I_1 &= -4\sum_1^3 \frac{A_r\, A_s}{(2\theta_r - 1 + a)(2\theta_s - 1 + a)} \sin^2\left(\frac{\theta_r - \theta_s}{2}\right)\xi \\ I_{-1} &= -4\sum_1^3 \frac{A_r\, A_s}{(2\theta_r - 1 - a)(2\theta_s - 1 - a)} \sin^2\left(\frac{\theta_r - \theta_s}{2}\right)\xi. \end{aligned}\right. \tag{27}$$

Appendix

Interpretation of expressions of the form $\sum_1^n C_n \sum_1^n C_n \sum_1^n C_n \cdots$

$$\sum_1^n C_r = S_n.$$

$$\begin{aligned} \sum_1^n C_n \left\{\sum_1^n C_n\right\} &= \sum_1^n C_n\, S_n \\ &= \sum_1^n {}^{r,\,s}\, C_r\, C_s \text{ (where like terms are to be written once only)} \end{aligned}$$

as, *e.g.*, $\left(\sum_1^n {}^{r,\,s}\, C_r\;\, C_s\right)$

for $n = 1,\quad C_1^2$

$n = 2,\quad C_1^2 + C_2^2 + C_1\, C_2$

$n = 3,\quad C_1^2 + C_2^2 + C_3^2 + C_1\, C_2 + C_2\, C_3 + C_3\, C_1 \cdots$

$$\begin{aligned} \sum_1^n C_n \sum_1^n C_n \sum_1^n C_n &= \sum_1^n C_n \left\{\sum_1^n C_n\, S_n\right\} \\ &= \sum_1^n C_n \left\{\sum_1^n C_r\, C_s\right\} \\ &= \sum_1^n {}^{r,\,s,\,t}\, C_r\, C_s\, C_t. \end{aligned}$$

$\sum_{1}^{n} r, s, t\, C_r C_s C_t$

$$\begin{cases} \text{for } n = 1, & C_1^3 \\ n = 2, & C_1^3 + C_2^3 + C_1^2 C_2 + C_2^2 C_1. \\ n = 3, & C_1^3 + C_2^3 + C_3^3 + C_1^2 C_2 + C_2^2 C_1 + C_1^2 C_3 + C_3^2 C_1 \\ & \quad + C_2^2 C_3 + C_3^2 C_2 + C_1 C_2 C_3. \end{cases}$$

$$\sum_{1}^{n} C_n \sum_{1}^{n} C_n \sum_{1}^{n} C_n \sum_{1}^{n} C_n$$

$$= \sum_{1}^{n} C_n \left\{ \sum_{1}^{n} C_n \left[\sum_{1}^{n} C_r C_s \right] \right\} = \sum_{1}^{n} C_r C_s C_t C_u$$

$$\sum_{1}^{n} C_n \sum_{1}^{n} C_n \sum_{1}^{n} C_n \sum_{1}^{n} C_n \sum_{1}^{n} C_n = \sum_{1}^{n} C_r C_s C_t C_u C_v.$$

Generally,

$$\underbrace{\sum_{1}^{n} C_n \sum_{1}^{n} C_n \sum_{1}^{n} C_n \sum_{1}^{n} C_n \cdots \sum_{1}^{n} C_n}_{r \text{ times}} = \sum_{1}^{n} \lambda_1, \lambda_2, \cdots \lambda_r\, C_{\lambda_1} C_{\lambda_2} \cdots C_{\lambda_r} \qquad (28)$$

where the terms under the sign of summation are to be counted once only.

In the course of the work we require the sums of $\sum_{1}^{n} r^p$ for various values of p ($p = 1, 2, 3, 4, \cdots 8$). For this purpose the well-known formula derived from Euler's summation formula has been employed.

$$\begin{aligned} \sum_{1}^{n} r^p = {} & \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \frac{p}{12} n^{p-1} - \frac{p(p-1)(p-2)}{720} n^{p-3} \\ & + \frac{p(p-1)(p-2)(p-3)(p-4)}{30240} n^{p-5} \\ & - \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)}{\lfloor 8 \cdot 30} n^{p-7} \\ & + \frac{5p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)}{66 \cdot \lfloor 10} n^{p-9} \\ & - \frac{691 \cdot p(p-1)(p-2)(p-3)(p-4)(p-5)(p-6)(p-7)(p-8)(p-9)(p-10)}{2730 \cdot \lfloor 12} \\ & \times n^{p-11} + \cdots\cdots \end{aligned} \qquad (29)$$

$$\sum_{1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{1}^{n} r^3 = \left\{ \frac{n(n+1)}{2} \right\}^2.$$

$$\sum_{1}^{n} r^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

$$\sum_1^n r^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

$$\sum_1^n r^6 = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42} \tag{30}$$

$$\sum_1^n r^7 = \frac{n^2(n+1)^2(3n^4+6n^3+n^2-4n+2)}{24}$$

$$\sum_1^n r^8 = \frac{n^9}{9} + \frac{n^8}{2} + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{n}{30}.$$

$$= \frac{n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)}{90}$$

$$C_r = i\rho(r^2 + ar)$$

$$\boxed{\sum_1^n C_r = i\rho\left\{\frac{n(n+1)(2n+1)}{6} + \frac{a}{2}n(n+1)\right\}} \tag{31}$$

$$\sum_1^n C_r^2 = -\rho^2\sum_1^n(r^2+ar)^2 = -\rho^2\sum_1^n(r^4+2ar^3+a^2r^2)$$

$$\boxed{\sum_1^n C_r^2 = -\rho^2\left\{\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + \frac{a}{2}n^2(n+1)^2 + a^2\frac{n(n+1)(2n+1)}{6}\right\}} \tag{32}$$

$$\sum_1^n C_r^3 = -i\rho^3\sum_1^n(r^2+ar)^3$$

$$= -i\rho^3\left\{\sum_1^n r^6 + 3a\sum_1^n r^5 + 3a^2\sum_1^n r^4 + a^3\sum_1^n r^3\right\}$$

$$\sum_1^n C_r^3 = -i\rho^3\left\{\frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}\right.$$

$$+ \frac{a}{4}n^2(n+1)^2(2n^2+2n-1) + \frac{a^2}{10}n(n+1)(2n+1)(3n^2+3n-1)$$

$$\left. + \frac{a^3}{4}n^2(n+1)^2\right\} \tag{33}$$

$$\sum_1^n C_r^4 = \rho^4\sum_1^n(r^2+ar)^4$$

$$= \rho^4\sum_1^n(r^8+4ar^7+6a^2r^6+4a^3r^5+a^4r^4)$$

$$\left\{\begin{aligned}\sum_1^n C_r^4 = \rho^4 \Big\{&\frac{\alpha^4}{30} n(n+1)(2n+1)(3n^2+3n-1) + \frac{\alpha^3}{3} n^2(n+1)^2(2n^2+2n-1) \\ &+ \frac{\alpha^2}{7} n(n+1)(2n+1)(3n^4+6n^3-3n+1) \\ &+ \frac{\alpha}{6} n^2(n+1)^2(3n^4+6n^3-n^2-4n+2) \\ &+ \frac{n^9}{9} + \frac{n^8}{2} + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{n}{30}\Big\}\end{aligned}\right. \tag{34}$$

$$\sum_1^n {}^{r,s}\, C_r C_s = \tfrac{1}{2}\Big\{\sum_1^n C_r^2 + \big(\sum_1^n C_r\big)^2\Big\}$$

$$= -\frac{\rho^2}{2}\Big\{\Big[\frac{n(n+1)(2n+1)}{6} + \frac{\alpha}{2}n(n+1)\Big]^2$$

$$+\Big[\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + \frac{\alpha}{2}n^2(n+1)^2$$

$$+\alpha^2\frac{n(n+1)(2n+1)}{6}\Big]\Big\}$$

$$= -\frac{\rho^2}{2}\Big\{\alpha^2\frac{n(n+1)}{12}\Big[3n(n+1)+2(2n+1)\Big]$$

$$+\frac{\alpha}{2}n^2(n+1)^2\Big[1+\frac{(2n+1)}{3}\Big] + \frac{n(n+1)(2n+1)}{180}\Big[5n(2n^2+3n+1)$$

$$+6(3n^2+3n-1)\Big]\Big\}$$

$$= -\frac{\rho^2}{2}\Big\{\frac{\alpha^2}{12}n(n+1)(n+2)(3n+1) + \frac{\alpha}{3}n^2(n+1)^2(n+2)$$

$$+\frac{n(n+1)(n+2)(2n+1)(2n+3)(5n-1)}{180}.$$

$$\therefore \quad \boxed{\begin{aligned}\sum_1^n {}^{r,s}\, C_r C_s = -\rho^2\Big\{&\frac{\alpha^2}{24}n(n+1)(n+2)(3n+1) \\ &+\frac{\alpha}{6}n^2(n+1)^2(n+2) + \frac{n(n+1)(n+2)(2n+1)(2n+3)(5n-1)}{360}\Big\}\end{aligned}}$$

$$C_{-1} + C_{n+1} + (n+4)S_n$$

$$= i\rho\Big\{1-\alpha+(n+1)^2+(n+1)\alpha+(n+4)\Big[\frac{n(n+1)(2n+1)}{6}$$

$$+\frac{\alpha}{2}n(n+1)\Big]\Big\}$$

$$= i\rho\Big\{n^2+2n+2+\frac{n(n+1)(2n+1)(n+4)}{6} + \frac{\alpha}{2}n(n+1)(n+4) + n\alpha\Big\}$$

$$= i\rho \left\{ \frac{n(n+2)(n+3)}{2} a + \frac{n(n+1)(n+2)(2n+5)}{6} + n + 2 \right\}. \quad (36)$$

$$C_{-1} + \sum_1^{n+1} C_r = i\rho \left\{ 1 - a + \frac{(n+1)(n+2)(2n+3)}{6} + \frac{a}{2}(n+1)(n+2) \right\}$$

$$= i\rho \left\{ \frac{(n+1)(n+2)(2n+3)}{6} + 1 + \frac{a}{2} n(n+3). \right\} \quad (37)$$

To find the value of $\sum_1^n {}^{r,s,t} C_r C_s C_t$ we make use of the following relation :

$$6 \sum_1^n C_r C_s C_t = (\sum_1^n C_r)^3 + 2 \sum_1^n C_r^3 + 3 (\sum_1^n C_r)(\sum_1^n C_r^2.)$$

$$\therefore \quad 6 \sum_1^n C_r C_s C_t = -i\rho^3 \left\{ \left[\frac{n(n+1)(2n+1)}{6} + \frac{a}{2} n(n+1) \right]^3 \right.$$

$$+ 2 \left[\frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42} \right.$$

$$+ \frac{a}{4} n^2 (n+1)^2 (2n^2+2n-1) + \frac{a^2}{10} n(n+1)(2n+1)(3n^2+3n-1)$$

$$\left. + \frac{a^3}{4} n^2 (n+1)^2 \right]$$

$$+ 3 \left[\frac{n(n+1)(2n+1)}{6} + \frac{a}{2} n(n+1) \right] \left[\frac{n(n+1)(2n+1)(3n^2+3n-1)}{180} \right.$$

$$\left. \left. + \frac{a}{2} n^2 (n+1)^2 + a^2 \frac{n(n+1)(2n+1)}{6} \right] \right\} \quad (38)$$

Similarly to find, $\sum_1^n {}^{r,s,t,u} C_r C_s C_t C_u$, we use the following relation

$$24 \sum_1^n C_r C_s C_t C_u = (\sum_1^n C_r)^4 + 6 \sum_1^n C_r^4 + 8 \sum_1^n C_n^3 \cdot \sum_1^n C_r + 3 (\sum_1^n C_r^2)^2$$

$$+ 6 \sum_1^n C_r^2 \cdot (\sum_1^n C_r)^2. \quad (39)$$

All the quantities on the R.H.S. being known, the expression can be summed up.

$\sum_1^n C_r^4$ for $n = 1, 2 \cdots$

$$\rho^4 \chi \left\{ \begin{array}{l} 1 + 4a + 6a^2 + 4a^3 + a^4 \\ 257 + 516a + 390a^2 + 132a^3 + 17a^4 \\ 6818 + 9264a + 4764a^2 + 1114a^3 + 98a^4 \\ 72354 + 74800a + 29340a^2 + 5200a^3 + 354a^4 \\ 462979 + 387292a + 82060a^2 + 17700a^3 + 979a^4) \end{array} \right\} \quad (40)$$

The values of $\sum_1^n C_r, \sum_1^n C_r^2, \cdots$ are tabulated below for $n = 1, 2, 3, 4, \cdots$

$\sum_1^n C_r$	$\sum_1^n C_r^2$	$\sum_1^n C_r^3$
$i\rho\ (1 + a)$	$-\rho^2(1 + a)^2$	$-i\rho^3 (1 + a)^3$
$i\rho\ (5 + 3a)$	$-\rho^2(5a^2 + 18a + 17)$	$-i\rho^3 (9a^3 + 51a^2 + 99a + 65)$
$i\rho\ (14 + 6a)$	$-\rho^2 (14a^2 + 72a + 98)$	$-i\rho^3 (36a^3 + 294a^2 + 828a + 794)$
$i\rho\ (30 + 10a)$	$-\rho^2 (30a^2 + 200a + 354)$	$-i\rho^3 (100a^3 + 1062a^2 + 3900a + 4890)$
$i\rho\ (55 + 15a)$	$-\rho^2(55a^2 + 450a + 979)$	$-i\rho^3 (225a^3 + 2937a^2 + 13275a + 20515)$
$i\rho\ (91 + 21a)$	$-\rho^2 (91a^2 + 882a + 2275)$	$-i\rho^3 (441a^3 + 6825a^2 + 36603a + 67171)$
$i\rho\ (140 + 28a)$	$-\rho^2 (140a^2 + 1568a + 4676)$	$-i\rho^3 (754a^3 + 13028a^2 + 81024a + 184820)$
$i\rho\ (204 + 36a)$	$-\rho^2(204a^2 + 2592a + 8772)$	$-i\rho^3 (1296a^3 + 26136a^2 + 185328a + 44694)$
$i\rho\ (285 + 45a)$	$-\rho^2(385a^2 + 6050a + 25333)$	
$i\rho\ (385 + 55a)$		

$\sum_1^n r,s\ C_r C_s$	$\sum_1^n r,s,t\ C_r C_s C_t$
$-\rho^2 (a + 1)^2$	$-i\rho^3 (1 + a)^3$
$-\rho^2 (7a^2 + 24a + 21)$	$-i\rho^3 (85 + 141a + 79a^2 + 15a^3)$
$-\rho^2 (25a^2 + 120a + 147)$	$-i\rho^3 (90a^3 + 664a^2 + 1662a + 1408)$
$-\rho^2 (65a^2 + 400a + 627)$	$-i\rho^3 (350a^3 + 3304a^2 + 10570a + 11440)$
$-\rho^2 (140a^2 + 1050a + 2002)$	$-i\rho^3 (1050a^3 + 12054a^2 + 46830a + 61490)$
$-\rho^2 (266a^2 + 2352a + 5278)$	$-i\rho^2 (2646a^3 + 35742a^2 + 163170a + 251498)$
$-\rho^2 (462a^2 + 4704a + 12138)$	$-i\rho^3 (5880a^3 + 91308a^2 + 472598a + 906200)$
$-\rho^2 (750a^2 + 8640a + 25194)$	
$-\rho^2 (1155a^2 + 14850a + 48279)$	
$-\rho^2 (1705a^2 + 24200a + 86779)$	

$C_{-1} + \sum_1^{n+1} C_r$	$C_{-1} + C_{n+1} + (n + 4) S_n$
$i\rho\ (6 + 2a)$	$i\rho\ (6a + 10)$
$i\rho\ (15 + 5a)$	$i\rho\ (20a + 40)$
$i\rho\ (31 + 9a)$	$i\rho\ (45a + 115)$
$i\rho\ (56 + 14a)$	$i\rho\ (84a + 266)$
$i\rho\ (92 + 20a)$	$i\rho\ (140a + 532)$
$i\rho\ (141 + 27a)$	$i\rho\ (216a + 960)$
$i\rho\ (205 + 35a)$	
$i\rho\ (286 + 44a)$	
$i\rho\ (386 + 54a)$	
$i\rho\ (507 + 65a)$	

In conclusion, it is my greatest pleasure to record my respectful thanks to Professor Sir C. V. Raman for suggesting the present investigation and for valuable guidance and criticism in the course of the work.

Summary

The three methods (Series, Bessel-function and Closed Expression) which have been used for dealing with the problem of the diffraction of light by supersonic waves have been worked out in detail and the amplitude expressions for the diffracted orders are written *in extenso*. In the case of the third method, the intensity expressions assume a simple form.

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