DEVIATIONS FROM PARALLELISM AND EQUIDISTANCE IN FINSLER SPACE

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ABSTRACT

In the present paper the deviations from parallelism and equidistance are studied for Finsler spaces in Cartan's sense and for Finsler space which is locally Minkowskian. By considering an orthogonal ennuple of hypersurfaces in F_n properties of associate curvature or angular spread vector have been studied.

INTRODUCTION

PARALLELISM and equidistance in classical differential geometry were studied by Graustein (1932); the same notions were studied and extended for Riemannian spaces by R. M. Peters (1935, 1937).

In the present paper the deviations from parallelism and equidistance are studied for Finsler spaces. In first section the study is based on Finsler space in the sense of E. Cartan (1934) and in second section the Finsler space under consideration is as given by A. Kawaguchi (1953, 1956), H. Rund (1954, 1955) and other writers.

SECTION I

1. In Riemannian geometry the associate curvature or angular spread is defined as the measure of deviation of vectors λ^i from parallelism, and is given by

$$\frac{1}{r} = (g_{ij}\mu^i\mu^j)^{\frac{1}{2}} \tag{1.1}$$

where

$$\mu^i = \lambda^i{}_{,k}\frac{dx^k}{ds} \tag{1.2}$$

are perpendicular to unit vector λ having λ^i as contravariant component.

The latin indices followed by a vertical bar denote a particular vector.

If we take Finsler space equipped with Euclidean connection, then the associate curvature (angular spread) vector μ^i given by the relation

$$\frac{1}{r^2} = g_{ij}(x, x')\,\mu^i\mu^j \tag{1.3}$$

is perpendicular to unit vector λ, as the co-variant derivative of fundamental tensor $g_{ij}(x, x')$ vanishes. $1/r$ is the measure of the deviation of vector λ^i at a point $\bar{P}(\bar{x}, d\bar{x})$ of a curve C and the vector which has been transported parallelly from $P(x, dx)$ to $\bar{P}$.

If n linearly independent congruences of curves C_k $(k = 1, 2, \ldots, n)$ with tangent vectors $\lambda^i_{k|} = dx^i/ds_k$ are considered, then the associate curvature vectors of the vectors $\lambda_{k|}{}^i$ with reference to the curves

$$C_l\;(l = 1, 2, \ldots, n, k \neq l)$$

are given by

$$\mu^j{}_{kl|} = \lambda^i{}_{l|}\,\lambda^j{}_{k|,i}. \tag{1.3 a}$$

This is called the associate curvature vector of the curves C_k with respect to curves C_l and its length

$$\frac{1}{r^2{}_{kl|}} = g_{ij}(x, x')\,\mu^i{}_{kl|}\mu^j{}_{kl|}$$

is called the associate curvature of the curves C_k with respect to C_l. Since

$$g_{ij}(x, x')\,\frac{dx^i}{ds_k}\cdot\frac{dx^j}{ds_k} = 1 \qquad (k = 1, \ldots n)$$

vectors $\mu^j{}_{k'|}$ are perpendicular to $\lambda^j{}_{k|}$ $(k \neq l)$.

We take n independent families of hypersurfaces S_l defined locally by the equations $f_l(x^1, x^2, \ldots, x^n, x^{1'}, \ldots, x^{n'}) = C_l$ $(l = 1, 2, \ldots, n')$ such that the distance between any two neighbouring points x^i, $x^i + dx^i$ is given as (E. Cartan, 1934)

$$ds = £(x, dx) \tag{2.1}$$

$$s_l = \int £_l(x, dx). \tag{2.2}$$

These hypersurfaces intersect in n linearly independent congruences of curves C_l whose unit tangent vectors are $\lambda^i{}_{l|} = dx^i/ds_l$. The family of hypersurfaces S_l have $(n-1)$ independent congruences of curves C_k $(k \neq l)$ on them and the family of curves C_l are their transversals.[1]

If d_l be the distance measured along a transversal between two hypersurfaces $S_l : f_l = f_l'$ and $f_l = f_l^1 + f_l^0$ (where f_l^0 is of the order Δf_l^1) and logarithmic directional derivative of d_l be taken in the direction of a curve C_k on f_l then this derivative as $\mathrm{L}t.\, f_l^0 \to 0$ is called the distantial spread of the hypersurfaces S_l with respect to the curves C_l measured along C_k and it is denoted by $1/d_{kl}$.

Thus

$$\frac{1}{d_{kl}} = \mathrm{L}t._{\,f_l^0 \to 0} \frac{\partial \log dl}{\partial S_k} \qquad (k \neq l). \tag{2.3}$$

If S_l is the directed arc of an individual curve C_l as defined by the integral (2.2) then

$$df_l = \frac{\partial f_l}{\partial S_l} dS_l$$

where $\partial f_l / \partial S_l$ is the directional derivative of f_l in the positive direction of C_l.

Therefore

$$\frac{1}{d_{kl}} = -\frac{\partial \log \left(\dfrac{\partial f_l}{\partial S_l} \right)}{\partial S_k} \qquad (k \neq l).$$

If $1/d_{kl} = 0$ then $\log(\partial f_l / \partial S_l)$ is constant and $s_l = \mathrm{F}(f_l)$ is the common directed arc measured from a fixed hypersurface S_l with congruence of transversals C_l.

Conversely when $1/d_{kl}$ the distantial spread of hypersurfaces S_l with respect to curves C_l in the directions of the curves C_k $(k \neq l)$ vanishes for all points then each pair of hypersurfaces cut equal segments on transversals.

Now

$$f_l \left(x^1, x^2, \ldots, x^n, \frac{dx^1}{dS}, \ldots, \frac{dx^n}{dS} \right) = C_l$$

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[1] The word transversal does not stand here for the generalisation of orthogonality, it merely indicates the curves which interesct the remaining $(n-1)$ families of curves C_k. $(k = 1, 2, \ldots n$ $k \neq l)$.

and

$$\frac{\partial f_l}{\partial S_l} = \frac{\partial f_l}{\partial x^i}\frac{dx^i}{dS_l} + \frac{\partial f_l}{\partial x'^i}\frac{dx'^i}{dS_l}$$

or

$$\frac{\partial f_l}{\partial S_l} = f_{l,i}\lambda_{l_1}{}^i + f_{l,i'}\lambda_{l_1}{}^{i'}. \tag{2.4}$$

As s_l is the arc parameter for the transversals C_l, $\lambda_{l_1}{}^i \equiv dx^i/dS_l$ denotes the unit tangent vector to the curve. The dashed suffixes throughout denote the partial derivative with reference to x'^i and $\lambda_{l_1}{}^{i'} = dx'^i/dS_l$.

Now

$$\frac{\partial \log \frac{\partial f_l}{\partial S_l}}{\partial S_k} = \frac{1}{\frac{\partial f_l}{\partial S_l}}[f_{l,ij}\lambda_{l_1}{}^i\lambda_{k_1}{}^j + f_{l,i}\lambda^i{}_{l_1,j}\lambda_{k_1}{}^j + f_{l,i'j}\lambda_{l_1}{}^i\lambda_{k_1}{}^j$$

$$+ f_{l,i'}\lambda^i{}_{l_1,j}\lambda_{k_1}{}^j]. \tag{2.5}$$

Also

$$f_{l_1,ij}\lambda_{l_1}{}^i\lambda_{k_1}{}^j = \frac{\partial f_{l,j}\lambda_{k_1}{}^j}{\partial S_l} - f_{l,j}\lambda^j{}_{k_1,i}\lambda_{l_1}{}^i. \tag{2.6}$$

Substituting this value in (2.5) and simplifying it we have

$$\frac{\partial \log \frac{\partial f_l}{\partial S_l}}{\partial S_k} = \frac{1}{\frac{\partial f_l}{\partial S_l}}\Big[f_{l,i}(\lambda^j{}_{l_1,j}\lambda_{k_1}{}^j - \lambda^i{}_{k_1,j}\lambda_{l_1}{}^j) + \frac{\partial f_{l,j}\lambda_{k_1}{}^j}{\partial S_l}$$

$$+ f_{l,i'j}\lambda_{l_1}{}^i\lambda_{k_1}{}^j + f_{l,i'}\lambda^i{}_{l_1,j}\lambda_{k_1}{}^j\Big] \tag{2.7}$$

or from (1.3)

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}}\Big[f_{l,i}(\mu^i{}_{lk_1} - \mu^i{}_{kl_1}) + \frac{\partial f_{l,j}\lambda_{kl}{}^j}{\partial S_l} + f_{l,i'j}\lambda_{l_1}{}^i\lambda_{k_1}{}^j$$

$$+ f_{l,i'}\lambda^{i'}{}_{l_1,j}\lambda_{k_1}{}^j\Big]. \tag{2.8}$$

Thus we conclude that if the difference between the associated curvature vectors $\mu^i{}_{lk_1}$, $\mu^i{}_{kl_1}$ is zero and the 2nd, 3rd and 4th terms vanish the hypersurfaces are equidistant.

In particular if the vector field $\lambda_{l_1}{}^{i'}$ is null the last two terms vanish. In this case the hypersurfaces will be equidistant if the sum $f_{l,j}\lambda_{k_1}{}^{j}$ is constant and the vector difference $(\mu^{i}{}_{lk_1} - \mu^{i}{}_{kl_1})$ is zero.

Equation (2.8) reduces locally to equations in Riemannian space as given by R. M. Peters (1935).

From above considerations, we have a necessary and sufficient condition that the hypersurfaces of each family be equidistant with respect to the congruence of curves not contained in them is that the vector $\mu^{i}{}_{kl_1}$ is identical to $\mu^{i}{}_{lk_1}$, $f_{l,j}\lambda_{k_1}{}^{j}$ is invariant with respect to differentiation along their transversals and vectors $\lambda_{l_1}{}^{i}$ are zero ($l = 1, 2, \ldots, n$).

3. E. Cartan (1934) has shown that a curve in Finsler space with Cartesian basis $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$ can be chosen so that at any point P we have

$$\vec{e}_i\vec{e}_j = g_{ij}.$$

We also assume that the independent hypersurfaces form an orthogonal ennuple thus

$$\Sigma\, \lambda_{h_1}{}^{i}\lambda_{h_1}{}^{j} = g^{ij}, \quad \lambda_{h_1}{}^{i}\lambda_{k_1 i} = 0 \quad (h \neq k)$$

and

$$\lambda_{h_1}{}^{i}\lambda_{h_1 j} = \delta_j^{\,i}.$$

We define the projection of the derived vector of λ_{h_1} in the direction of λ_{l_1} on λ_{k_1} as

$$\lambda_{h_1 i,j}\lambda_{k_1}{}^{i}\lambda_{l_1}{}^{j} = \gamma_{hkl} \tag{3.2}$$

which are known as Ricci's coefficients of rotation.

Now

$$\mu^{i}{}_{lk_1}\lambda_{l_1 i} = 0 \tag{3.3}$$

and directions of the curves C_l coincide with the system of normals to the surface thus $f_{l,i}\mu^{i}{}_{lk_1} = 0$ and (2.9) reduces to

$$\frac{1}{d_{kl}} = \frac{1}{\dfrac{\partial f_l}{dS_l}}\left\{\frac{\partial f_{l,j}\lambda_{k_1}{}^{j}}{dS_l} - f_{l,i}\mu^{i}{}_{kl_1} + f_{l,i'j}\lambda_{l_1}{}^{i'}\lambda_{k_1}{}^{j} + f_{l,i'}\lambda^{i}{}_{l_1,j}\lambda_{k_1}{}^{j}\right\}. \tag{3.4}$$

Further

$$\mu^i{}_{kl_1} = \Sigma\, \gamma_{krl} \lambda_{r_1}{}^i$$

and

$$\mu^i{}_{lk_1} = \Sigma\, \gamma_{lrk} \lambda_{r_1}{}^i. \tag{3.5}$$

Thus equation (2.8) can be put in terms of coefficients of rotation.

If we take the derivative of $\lambda_{h_1}{}^i \lambda_{k_1 i} = 0$ with respect to x^j and then multiply by $\lambda_{l_1}{}^j$ we have the relation

$$\gamma_{hkl_1} = -\gamma_{khl_1}. \tag{3.6}$$

Putting $h = k$ we get (3.7) $\gamma_{kkl_1} = 0$ in conformity with (3.3).

With the help of (3.5) we put (3.4) as

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}} \left\{ \frac{\partial f_{l,j} \lambda_{k_1}{}^j}{\partial S_l} + f_{l,i} \Sigma\, \gamma_{rkl} \lambda_{r_1}{}^i + f_{l,i';j} \lambda_{l_1}{}^i \lambda_{k_1}{}^j \right.$$

$$\left. + f_{l,i'} \lambda^i{}_{l_1,j} \lambda_{k_1}{}^j \right\}.$$

Thus if $\gamma_{rkl} = 0$ $(r = 1, 2, \ldots, n)$, $\lambda_{l_1}{}^i = 0$ and $f_{l,i} \lambda_{k_1}{}^i$ is invariant with respect to differentiation $\partial/\partial S_l$ the orthogonal ennuple of hypersurfaces is equidistant each hypersurface being so with respect to the congruence of curves not contained in it.

4. Distantial spread of one congruence of curves with respect to another congruence:—

We consider the n independent congruences of curves as obtained by the intersection of n independent families of hypersurfaces S_l,

$$f_l\left(x^1, x^2, \ldots, x^n, \frac{dx^1}{dS}, \frac{dx^2}{dS}, \ldots, \frac{dx^n}{dS}\right) = C_l$$

the curves C_l being the transversals of the hypersurfaces S_l.

Upon a two-dimensional surface $f_i = C_i$ $(i = 1, 2, \ldots, n;\ i \neq k, l)$ there are ∞^1 curves C_l defined by the equations $f_k = C_k$ and ∞^1 curves C_k defined by the equations $f_l = C_l$.

Upon an arbitrary surface we consider these two sets of curves and take the logarithmic directional derivative of the distance measured along a curve of congruence C_l between the two neighbouring curves given by

$f_l = f_l^1, f_l = f_l^1 + f_l^0$ (where f_l^0 is of the order $4f_l^1$) of the congruence C_k. The limit of this derivative as $f_l^0 \to 0$ is the distantial spread of the curves C_k with respect to the curves C_l.

From (2.12) the distantial spread of the congruences of curves to one another can be put down as

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}} \left[\frac{\partial f_{l,j} \lambda_{k_l}{}^j}{\partial S_l} + f_{l,i} (\mu^i{}_{lk_l} - \mu^i{}_{kl_l}) + f_{l,i'j} \lambda_{l_l}{}^i \lambda_{k_l}{}^j \right.$$

$$\left. + f_{l,i'} \lambda^i{}_{l_l,j} \lambda_{k_l}{}^j \right]$$

and

$$\frac{1}{d_{lk}} = \frac{1}{\frac{\partial f_k}{\partial S_k}} \left[\frac{\partial f_{k,j} \lambda_{l_l}{}^j}{\partial S_k} + f_{k,i} (\mu^i{}_{kl_l} - \mu^i{}_{lk_l}) + f_{k,i'j} \lambda_{k_l}{}^i \lambda_{l_l}{}^j \right.$$

$$\left. + f_{k,i'} \lambda_{k_l}{}^{i'} \lambda_{l_l}{}^j \right]. \tag{4.1}$$

Thus

If the curves of each of two congruences are parallel with respect to one another, $\lambda_{l_l}{}^i$ and $\lambda_{k_l}{}^{i'}$ are null and the terms

$$\frac{\partial f_{l,j} \lambda_{k_l}{}^j}{\partial S_l}, \quad \frac{\partial f_{k,j} \lambda_{l_l}{}^j}{\partial S_k}$$

vanish then the curves of each congruence are equidistant with respect to the curves of the other.

Conversely if the first, third and fourth terms of both the brackets vanish and the curves of each congruence are equidistant with respect to the curves of another then their associate curvatures with respect to one another are identical.

Finally if the first, third and fourth terms of (4.1) vanish and the curves of each of the two congruences are equidistant with respect to another and the curves of one of the congruences are known to be parallel with respect to another then the curves of latter congruence are also parallel with respect to the curves of first.

If the congruences form an orthogonal system, equations (4.1) simplify to

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}} \left\{ \frac{\partial f_{l,j} \lambda_{k_1}{}^j}{\partial S_l} + f_{l,i} \Sigma \, \gamma_{rkl} \lambda_{r_1}{}^i + f_{l,i'j} \lambda_{l_1}{}^{i'} \lambda_{k_1}{}^j + f_{l,i'} \lambda^i{}_{l_1,j} \lambda_{k_1}{}^j \right\}$$

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_k}{\partial S_k}} \left\{ \frac{\partial f_{k,j} \lambda_{l_1}{}^j}{\partial S_k} + f_{k,i} \Sigma \, \gamma_{rlk} \lambda_{r_1}{}^i + f_{k,i'j} \lambda_{k_1}{}^{i'} \lambda_{l_1}{}^j + f_{l,i'} \lambda^{i'}{}_{k_1,j} \lambda_{l_1}{}^j \right\}. \tag{4.2}$$

Thus if

$$\frac{\partial f_{l,j} \lambda_{k_1}{}^j}{\partial S_l}, \quad \frac{\partial f_{k,j} \lambda_{l_1}{}^j}{\partial S_k}, \quad \lambda_{l_1}{}^{i'}, \quad \lambda_{k_1}{}^{i'}$$

are zero and the coefficients of rotation

$$\gamma_{rkl} = 0 \quad \text{for} \quad r = 1, 2, \ldots, n$$

then the curves of each of the congruences are equidistant with respect to another.

Conversely if

$$\frac{\partial f_{l,j} \lambda_{k_1}{}^j}{\partial S_l}, \quad \frac{\partial f_{k,j} \lambda_{l_1}{}^j}{\partial S_k}, \quad \lambda_{l_1}{}^i, \quad \lambda_{k_1}{}^i$$

are zero and the curves of each congruence are equidistant with respect to another, then the coefficients of rotation γ_{rkl} must be zero for all r.

Now

$$\lambda_{l_1}{}^{i'} = \frac{dx'^i}{dS_l} = x,_j{}'^i \lambda_{l_1}{}^j$$

$$= \lambda^i{}_{m_1,j} \lambda_{l_1}{}^j$$

where m may stand for any number $1, 2, \ldots, n$.

Therefore,

$$\lambda_{l_1}{}^i = \mu^i{}_{ml_1}.$$

and

$$\mu^i{}_{ml_1} = \Sigma \gamma_{mrl} \lambda_{r_1}{}^i$$
$$= - \Sigma \gamma_{rml} \lambda_{r_1}{}^i.$$

Putting this value of $\lambda_{l_1}{}^i$ and $\lambda_{k_1}{}^{i'}$ in (4.2) we have

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}} \left[\frac{\partial f_{l,j} \lambda_{k_1}{}^j}{\partial S_l} + f_{l,i} \sum_r \gamma_{rkl} \lambda_{r_1}{}^i - f_{l,i'j} \sum_r \gamma_{rml} \lambda_{r_1}{}^i \lambda_{k_1}{}^j \right.$$
$$\left. - f_{l,i'} \left(\sum_r \gamma_{rml} \lambda_{r_1}{}^i \right), {}_j \lambda_{k_1}{}^j \right].$$

Hence we have, if

$$\gamma_{rkl} = 0 \quad \text{for} \quad r = 1, 2, \ldots, n$$

and

$$\frac{\partial f_{l,j} \lambda_{k_1}{}^j}{\partial S_l}, \quad \frac{\partial f_{k,j} \lambda_{l_1}{}^j}{\partial S_k}$$

are zero the curves of the orthogonal congruences are equidistant with respect to each other.

Conversely if the curves of two congruences are equidistant with respect to each other and

$$\frac{\partial f_{l,j} \lambda_{k_1}{}^j}{\partial S_l}, \quad \frac{\partial f_{k,j} \lambda_{l_1}{}^j}{\partial S_k}$$

are zero the coefficients of rotation γ_{rkl} must be zero for all r.

SECTION II

5. In the first section our findings in Finsler spaces were based on element of support and, therefore, on Euclidean connection.

In the present section we make a general approach by considering the Finsler space which is locally Minkowskian.

The metric function of n-dimensional Finsler space F_n is denoted by $F(x^i, X^i)$ where X^i is the vector attached to the point $P(x)$ of F_n. The function F is restricted by following assumptions:

(i) F is analytic in its $2n$ arguments $(x^1, x^2, \ldots, x^n; X^1, X^2, \ldots, X^n)$.

(ii) F is positive unless all the X^i vanish simultaneously.

(iii) It is positively homogeneous of the first degree in X^i.

(iv) The quadratic form $g_{ij}(x, X)\ \xi^i \xi^j > 0$ for $\xi^i \neq 0$

where

$$g_{ij}(x, X) = \tfrac{1}{2}\frac{\partial^2 F^2(x, X)}{\partial X^i \partial X^j}.$$

From assumption (iii) and by Euler's theorem on Homogeneous functions it follows:

$$\frac{\partial g_{ij}(x, X)}{\partial X^k} X^k = \frac{\partial g_{ij}(x, X)}{\partial X^h} X^j = 0 \tag{5.1}$$

$$g_{ij,k}(x, X)\, X^k = g_{ij,k}(x, X)\, X^j = 0. \tag{5.1 a}$$

In particular if $X^i = x'^i$ is contravariant component of the tangent vector then (5.1) takes the form

$$\frac{\partial g_{ij}(x, x')}{\partial x'^k} x'^k = \frac{\partial g_{ij}(x, x')}{\partial x'^h} x'^j = 0. \tag{5.2}$$

The co-variant derivative of a vector λ^i is given by

$$\lambda^i,_k = \frac{\partial \lambda^i}{\partial x^k} + P_{hk}{}^{*i}(x, X)\, \lambda^h \tag{5.3}$$

and the condition of parallelism is

$$\lambda^i,_k = 0.$$

In analogy to Riemannian space we define the associate curvature vector of λ^i with respect to X^k as its derived vector in the direction of X^k, that is

$$\lambda^i,_k X^k = \mu^i.$$

The measure of the associate curvature (angular spread) vector μ^i is given by the relation

$$\frac{1}{r^2} = g_{ij}(x, \mu)\, \mu^i \mu^j \tag{5.4}$$

and $1/r$ is called the associate curvature of λ^i with respect to curve C.

X^k are the contravariant components of the vector which is intrinsically attached to the point in question.

It is to be noted that in Riemannian space and locally Euclidean Finsler space the co-variant component μ_i of the associate curvature vector μ^i can be put as

$$\lambda_{i,k}\frac{dx^k}{dS} = \mu_i$$

whereas in this case on account of non-vanishing of $g_{ij,k}(x, X)$ it cannot be put in the above manner.

We call the vector

$$\lambda_{i,k}\frac{dx^k}{dS} = \mu_i$$

as the secondary associate curvature vector and the corresponding length given by

$$g^{ij}(x, \mu)\,\mu_i\mu_j = \frac{1}{r^2}$$

as the secondary associate curvature of the vector λ^i with respect to the curve C.

Obviously $g_{ij}(x, \mu)$ and $g^{ij}(x, \mu)$ are not the counterparts of each other. Corresponding to each associate curvature vector at a point of the curve we shall have the secondary curvature vector too.

As the contravariant components λ^i belong to a unit vector we have

$$g_{ij}(x, X)\,\lambda^i\lambda^j = 1$$

and the co-variant derivative with respect to x^k gives

$$g_{ij,k}(x, X)\,\lambda^i\lambda^j + 2g_{ij}(x, X)\,\lambda^i{}_{,k}\lambda^j = 0$$

or

$$g_{ij,k}(x, X)\,\lambda^i\lambda^j x'^k + 2g_{ij}(x, X)\,\lambda^i{}_{,k}\lambda^j x'^k = 0. \tag{5.5}$$

Thus we observe that unlike Finsler manifolds with Euclidean connection here the vector μ^i is not always perpendicular to λ^i; it is perpendicular only when first term of (5.5) vanishes, *i.e.*, when X^i are replaced by λ^i in $g_{ij}(x, X)$ or $X^i \equiv x'^i$.

If n linearly independent congruences of curves C_k ($k = 1, 2, \ldots, n$) with tangent vector

$$\lambda_{k|}{}^i = \frac{dx^i}{dS_k}$$

are considered then the associate curvature vectors of the vectors $\lambda_{k_l}{}^i$ with respect to curves C_l $(k, l = 1, 2, \ldots, n,\ k \neq l)$ are given by

$$\mu^j{}_{kl_l} = \lambda^j{}_{k_l, i}\lambda_{l_l}{}^i. \tag{5.6}$$

This is called the angular spread vector or the associate curvature vector of the curves C_k with respect to C_l and its length

$$\frac{1}{r_{kl}{}^2} = g_{ij}(x, \mu_k)\,\mu^i{}_{kl_l}\mu^j{}_{kl_l}$$

is called the associate curvature of the curves C_k with respect to C_l.

Here again we notice that $\mu^i{}_{kl_l}$ are not necessarily perpendicular to $\lambda_{k_l}{}^i$; they will be so only when

$$g_{ij,h}(x, X)\,\lambda_{k_l}{}^i\lambda_{k_l}{}^j\lambda_{l_l}{}^h = 0, \ \textit{i.e.},\ X^i = \lambda_{k_l}{}^i \ \text{or}\ X^i = \lambda_{l_l}{}^i. \tag{5.7}$$

If F_n is a subspace embedded in a Finsler space F_m, $(m > n)$ the vectors in F_m are specified by λ^i whose contravariant components in F_n are ξ^α thus

$$\lambda^i = X_\alpha{}^i\xi^\alpha \quad \left\{ \begin{array}{l} i, j, k \ = 1, 2, \ldots, n \\ \alpha, \beta, \gamma = 1, 2, \ldots, n \\ x^i = x^i(u^1, u^2, \ldots, u^n) \end{array} \right\}. \tag{5.8}$$

Its ordinary differentiation gives

$$\frac{d\lambda^i}{dS} = X_\alpha{}^i\frac{d\xi^\alpha}{dS} + \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\,\xi^\alpha\frac{du^\beta}{dS}. \tag{5.9}$$

The associate curvature vector of λ^i are denoted by μ then

$$\mu^i = \frac{dx^j}{dS}\lambda^i{}_{,j}. \tag{5.10}$$

On account of (5.8) and (5.9) it takes the following form

$$\mu^i = \frac{d\xi^\alpha}{dS}X_\alpha{}^i + \xi^\alpha\frac{du^\beta}{dS}\left(\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + P_{hk}{}^{*i}X_\alpha{}^h X_\beta{}^k\right). \tag{5.11}$$

Since

$$P^*{}_{\alpha\beta,\gamma}(u, u') = g_{ij}(x, x')\,X_\gamma{}^j\left(\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + P_{hk}{}^{*i}(x, x')\,X_\alpha{}^h X_\beta{}^k\right).$$

We have

$$g_{ij}(x, x')\,X_\gamma{}^i\mu^j = g_{\alpha\gamma}\xi^\alpha{}_{,\beta}\frac{du^\beta}{dS} = g_{\alpha\gamma}(u, u')\,\eta^\alpha \tag{5.12}$$

where we have put

$$\xi^{\alpha}{}_{,\beta}\frac{du^{\beta}}{dS}=\eta^{\alpha}$$

which is the associate curvature vector of ξ^{α} with respect to the same curve C.

Thus the relation (5.12) shows that the associate curvature vector of vectors in subspace and the embedding space with respect to same curve is not necessarily the same.

The associate curvature of the vector λ^{i} with respect to curve C is essentially its measure of the deviation from parallelism with respect to curve C.

6. *Equidistance of families of hypersurfaces with respect to congruences of curves.*—Let n independent families of hypersurfaces S_l be defined in F_n by the equations $f_l(x^1, x^2, \ldots, x^n, x'^1, \ldots, x'^n) = C_l$ $(l = 1, 2, \ldots, n)$.

The n linearly independent congruences of curves which are the curves of intersection of these hypersurfaces are denoted by C_k $(k = 1, 2, \ldots, n)$ and their unit tangent vectors are

$$\lambda_{k|}{}^{i}=\frac{dx^{i}}{dS_k}.$$

The hypersurfaces S_l of the family $f_l = C_l$ contain $(n-1)$ congruences of curves C_k $(k \neq l)$ and the congruence C_l is their transversal.

We define the distantial spread of the hypersurfaces S_l with respect to the curves C_l measured along C_k in the same way as we did in Section I.

Thus

$$\frac{1}{d_{kl}}=\underset{f_l{}^0\to 0}{\mathrm{Lt.}}\ \frac{\partial \log d_l}{\partial S_k}\qquad (k\neq l)$$

or

$$\frac{1}{d_{kl}}=-\frac{\partial \log\left(\frac{\partial f_l}{\partial S_l}\right)}{\partial S_k}. \tag{6.1}$$

If $1/d_{kl}=0$ then $\log \partial f_l/\partial S_l$ is constant and $S_l = F(f_l)$ is the common directed arc measured from a fixed hypersurface S_k of all the curves C_l.

Conversely when $1/d_{kl}$ the distantial spread of hypersurfaces S_l with respect to curves C_l in the directions of the curves C_k $(k \neq l)$ is zero at all points then each pair of hypersurfaces cut equal segments on transversals.

Now

$$f_l\,(x^1, x^2, \ldots, x^n, x'^1, \ldots, x'^n) = C_l.$$

and

$$\frac{\partial f_l}{\partial S_l} = \frac{\partial f_l}{\partial x^i}\frac{dx^i}{dS_l} + \frac{\partial f_l}{\partial x'^i}\frac{dx'^i}{dS_l}$$
$$= f_{l,i}\lambda_{l_1}{}^{i} + f_{l,i'}\lambda_{l_1}{}^{i}$$

where we have put

$$\lambda_{l_1}{}^{i'} = \frac{dx'^i}{dS_l}. \tag{6·2}$$

Therefore

$$\frac{1}{d_{kl}} = -\frac{1}{\frac{\partial f_l}{\partial S_l}}(f_{l,i}\lambda_{l_1}{}^{i} + f_{l,i'}\lambda_{l_1}{}^{i})_{,j}\lambda_{k_1}{}^{j}$$
$$= -\frac{1}{\frac{\partial f_l}{\partial S_l}}(f_{l,ij}\lambda_{l_1}{}^{i} + f_{l,i}\lambda^{i}{}_{l_1,j} + f_{l,i'j}\lambda_{l_1}{}^{i} + f_{l,i'}\lambda^{i}{}_{l_1,j})\,\lambda_{k_1}{}^{j}. \tag{6.3}$$

Also

$$f_{l,ij}\lambda_{l_1}{}^{i}\lambda_{k_1}{}^{j} = \frac{\partial f_{l,j}\lambda_{k_1}{}^{j}}{\partial S_l} - f_{l,j}\lambda^{j}{}_{k_1,i}\lambda_{l_1}{}^{i}.$$

Substituting this value of $f_{l,ij}\lambda_{l_1}{}^{i}\lambda_{k_1}{}^{j}$ in (6.3) and taking into account (5.6) we have

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}}\Big[f_{l,i}(\mu^{i}{}_{kl_1} - \mu^{i}{}_{lk_1}) - \frac{\partial}{\partial S_l}(f_{l,j}\lambda_{k_1}{}^{j})$$
$$- f_{l,i'j}\lambda_{l_1}{}^{i'}\lambda_{k_1}{}^{j} - f_{l,i'}\lambda^{i}{}_{l_1,j}\lambda_{k_1}{}^{j}\Big]. \tag{6.4}$$

From (6.4) we obtain the following theorems:

THEOREM 1.—*The hypersurfaces are equidistantial with respect to the curves C_k, if the difference vector of the associate curvatures of curves C_l with respect to C_k and curves C_k with respect to C_l is null and the last three terms reduce to zero.*

In particular if the congruence of curves C_l are geodesics, $\lambda_{l_1}{}^{i'}$ will be zero.

THEOREM 2.—*Similarly if the unit tangent vectors* $\lambda_{k|}{}^{i}$ *and* $\lambda_{l|}{}^{i}$ *are parallel with respect to the curves* C_l *and* C_k *respectively and the last three terms are zero the hypersurfaces* S_l *will be equidistant with respect to curves* C_k.

Now

$$\begin{aligned}\lambda_{l|}{}^{i} &= x'^{i}{}_{,j}\lambda_{l|}{}^{j}\\ &= \lambda^{i}{}_{m|,j}\lambda_{l|}{}^{j} = \mu^{i}{}_{ml|}.\end{aligned}$$

Thus $\lambda_{l|}{}^{i}$ is actually the associate curvature vector with one of the superscript suppressed meaning thereby that it stands for the associate curvature of all the curves of the congruences C_m $(m = 1, 2, \ldots, n)$.

Therefore

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}}\Big[f_{l,i}(\mu^{i}{}_{kl|} - \mu^{i}{}_{lk|}) - \frac{\partial}{\partial S_l}(f_{l,j}\lambda_{k|}{}^{j}) - f_{l,i'j}\mu^{i}{}_{ml|}\lambda_{k|}{}^{j} - f_{l,i'}\mu^{i}{}_{ml|,j}\lambda_{k|}{}^{j}\Big]. \tag{6.5}$$

Thus we have

THEOREM 3.—*The hypersurfaces* S_l *are equidistant with respect to the curves* C_k *if the associate curvature vectors of all the curves of the congruence with respect to the fixed congruence* C_l *are zero, the associate curvature vectors of* C_l *with respect to* C_k *are zero and the sum* $f_{l,j}\lambda_{k|}{}^{j}$ *is independent of* s_l.

7. We choose an orthogonal ennuple of hypersurfaces in F_n such that if

$$\lambda_{k|}{}^{i} = \frac{dx^i}{dS_l}, \quad \lambda_{l|}{}^{j} = \frac{dx^j}{dS_l}$$

be the contravariant components of the unit tangent vectors of the transversals C_k and C_l of the family of hypersurfaces $f_k = C_k$, $f_l = C_l$ then we have the relations

$$\begin{aligned} g_{ij}(x, \lambda_{k|})\,\lambda_{k|}{}^{i}\lambda_{l|}{}^{j} &= \delta_l{}^{k} = \lambda_{k|j}\lambda_{l|}{}^{j}\\ g_{ij}(x, \lambda_{l|})\,\lambda_{l|}{}^{i}\lambda_{k|}{}^{j} &= \delta_k{}^{l} = \lambda_{l|j}\lambda_{k|}{}^{j}. \end{aligned} \tag{7.1}$$

It is to be noted that $\delta_l{}^{k} \neq \delta_k{}^{l}$ in general, though of course $\delta_l{}^{k}$ and $\delta_k{}^{l}$ are both zero or one according as $k \neq l$ or $k = l$.

The projection of the derived vector of $\lambda_{h_|}$ in the direction of $\lambda_{l_|}$ on $\lambda_{k_|}$ is given by

$$\lambda^i{}_{h_|,j}\lambda_{k_|i}\lambda_{l_|}{}^j \tag{7.2}$$

we put it as γ_{hkl} and call it Ricci's coefficient of rotation.

Since

$$\lambda^i{}_{h_|,j}\lambda_{k_|i} \neq \lambda_{h_|i,j}\lambda_{k_|}{}^i$$

the coefficient of rotation so defined is not anti-symmetric in the first two indices.

We call the invariant defined by

$$\lambda_{h_|i,j}\lambda_{l_|}{}^j\lambda_{k_|}{}^i = \gamma_{hkl} \tag{7.3}$$

as the relative Ricci's coefficient of rotation.

Taking the co-variant derivative of the relation (7.1) with respect to x^m, multiplying it by $\lambda_{l_|}{}^m$ we have

$$(\lambda_{h_|j,m}\lambda_{k_|}{}^j + \lambda_{h_|j}\lambda^j{}_{k_|,m})\,\lambda_{l_|}{}^m = 0$$

$$(\lambda^j{}_{h_|,m}\lambda_{k_|j} + \lambda_{h_|}{}^j\lambda_{k_|j,m})\,\lambda_{l_|}{}^m = 0. \tag{7.4}$$

In view of (7.2) and (7.3) we have

$$\gamma_{hkl} + \gamma_{khl} = 0$$

$$\gamma_{hkl} + \gamma_{khl} = 0 \tag{7.5}$$

This shows that the relative coefficient of rotation and coefficient of rotation are connected by a reciprocal relation.

From (7.2) and (7.3) we have relations for associate and secondary associate curvature vector, thus,

$$\lambda^i{}_{h_|,j}\lambda_{l_|}{}^j = \sum_k \gamma_{hkl}\lambda_{k_|}{}^i = \mu^i{}_{hl_|}$$

$$\lambda_{h_|i,j}\lambda_{l_|}{}^j = \sum_k \gamma_{hkl}\lambda_{k_|i} = \mu_{hl_|i}. \tag{7.6}$$

We may remark here that if λ's are constant multiples of each other then in view of condition (iii), the distinction between two types of associate curvatures and Ricci's coefficient of rotation will vanish.

The derived vector of $\lambda_{h_|}$ in its own direction is called the first curvature vector and is denoted by

$$\lambda^{i}{}_{h_1,j}\lambda_{h_1}{}^{j} = \sum_{k} \gamma_{hkh}\lambda_{k_1}{}^{i} = \mu_{h_1}{}^{i}$$

$$\lambda_{h_1 i,j}\lambda_{h_1}{}^{j} = \sum_{k} \gamma_{hkh}\lambda_{k_1 i} = \mu_{h_1 i}. \tag{7.7}$$

Applying (7.6) to (6.5) we have

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}}\Big[f_{l,i}\Big(\sum_{h} \gamma_{khl}\lambda_{h_1}{}^{i} - \sum_{h} \gamma_{lhk}\lambda_{h_1}{}^{i}\Big) - \frac{\partial}{\partial S_l}(f_{l,j}\lambda_{k_1}{}^{j})$$

$$- f_{l,i'j}\sum_{h} \gamma_{mhl}\lambda_{h_1}{}^{i}\lambda_{k_1}{}^{j} - f_{l,i'}\Big(\sum_{h} \gamma_{mhl}\lambda_{h_1}{}^{i}\Big)_{,j}\lambda_{k_1}{}^{j}\Big]. \tag{7.8}$$

Hence we have

THEOREM 4.—*If for an orthogonal ennuple of hypersurfaces the Ricci's coefficients of rotation* $\gamma_{hkl_1} = 0$ $(h, k, l = 1, 2, \ldots, n)$ and $\partial/S_l\,(f_{l,j}\lambda_{k_1}{}^{j})$ *is also zero then the family of hypersurfaces* $f_l = C_l$ *must be equidistant with respect to the family* $f_k = C_k$.

On account of (7.5) the relation (7.8) can also be put as

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}}\Big[f_{l,i}\Big(\sum_{h} \gamma_{hlk}\gamma_{h_1}{}^{i} - \sum_{h} \lambda_{hkl}\lambda_{h_1}{}^{i}\Big) - \frac{\partial}{\partial S_l}(f_{l,j}\lambda_{k_1}{}^{j})$$

$$+ f_{l,i'j}\sum_{h} \gamma_{hml}\lambda_{h_1}{}^{i}\lambda_{k_1}{}^{j} - f_{l,i'}\Big(\sum_{h} \gamma_{hml}\lambda_{h_1}{}^{i}\Big)_{,j}\lambda_{k_1}{}^{j}\Big].$$

Hence

THEOREM 5.—*The family of hypersurfaces* $f_l = C_l$ *will be equidistant with respect to* $f_k = C_k$ *if relative coefficients of rotation* γ_{hkl} *are zero and* $\partial/\partial S_l\,(f_{l,j}\lambda_{k_1}{}^{j})$ *is zero.*

8. We define the distantial spread of the curves of the congruences on the same lines as in Section I, thus the distantial spread of the curves of the congruence C_k with respect to the curves of the congruence C_l can be put as

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}}\Big[f_{l,i}(\mu^{i}{}_{lk_1} - \mu^{i}{}_{kl_1}) - \frac{\partial}{\partial S_l}(f_{l,j}\lambda_{k_1}{}^{j})$$

$$- f_{l,i'j}\mu^{i}{}_{ml_1}\lambda_{k_1}{}^{j} - f_{l,i'}\mu^{i}{}_{ml_1,j}\lambda_{k_1}{}^{j}\Big]. \tag{8.1}$$

Similarly we have

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_k}{\partial S_k}} \Big[f_{k,i} (\mu^i{}_{lk_1} - \mu^i{}_{kl_1}) - \frac{\partial}{\partial S_k} (f_{k,j} \lambda_{l_1}{}^j)$$

$$- f_{k,i'j} \mu^i{}_{mk_1} \lambda_{l_1}{}^j - f_{k,i'} \mu^i{}_{mk_1,j} \lambda_{k_1}{}^j \Big] . \qquad (8.2)$$

The distantial spread of congruence of curves C_k with respect to the congruences C_l will be zero if $1/d_{kl}$ is zero.

Thus we have

THEOREM 6.—*If the curves of all the congruences are parallel with respect to the curves of congruences C_l and curves of congruence C_l are parallel with respect to C_k and $f_{l,j} \lambda_{k_1}{}^j$ is invariant with respect to the differentiation $\partial/\partial S_l$ then the curves of congruence C_k are equidistant with respect to the curves of congruence C_l.*

As (8.1) is equivalent to the following equation

$$\frac{1}{d_{kl}} = \frac{1}{\frac{\partial f_l}{\partial S_l}} \Big[f_{l,i} (\mu^i{}_{kl_1} - \mu^i{}_{lk_1}) - \frac{\partial}{\partial S_l} (f_{l,j} \lambda_{k_1}{}^j)$$

$$- f_{l,i'j} \lambda_{l_1}{}^{i'} \lambda_{k_1}{}^j - f_{l,i'} \lambda^i{}_{l_1,j} \lambda_{k_1}{}^j \Big] \qquad (8.3)$$

We have

THEOREM 7.—*If the curves of two congruences C_k, C_l are parallel with respect to each other, $\partial/\partial S_l (f_{l,j} \lambda_{k_1}{}^j)$ is zero and the vector $\lambda_{l_1}{}^{i'}$ is null then the curves of congruence C_k are equidistant with respect to the curves of congruence C_l.*

Considering (8.1) and (8.2) together we have

THEOREM 8.—*If the curves of two congruences C_k and C_l are parallel with respect to each other $\partial/\partial S_l (f_{l,j} \lambda_{k_1}{}^j)$, $\partial/\partial S_k (f_{k,j} \lambda_{l_1}{}^j)$ is zero and the vectors $\lambda_{l_1}{}^{i'}$, $\lambda_{k_1}{}^{i'}$ are null then both the congruences are equidistant with respect to each other.*

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