CHARACTERISTIC LINES OF A HYPERSURFACE V_n IMBEDDED IN A RIEMANNIAN V_{n+1}

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1. INTRODUCTION

UPON a surface of positive curvature there is a unique conjugate system for which the angle between the directions at any point is the minimum angle between conjugate directions at that point. This system of lines is called *characteristic lines* or *mean-conjugate net* (Eisenhart, 1909, 1947). The object of the present paper is to obtain the equation of characteristic lines of a hypersurface in a Riemannian V_{n+1} and to discuss some properties of these curves.

2. HYPERSURFACE IN V_{n+1}

Consider a hypersurface V_n of co-ordinates x^i, $i = 1, 2, \ldots n$ and metric

$$ds^2 = g_{ij} dx^i dx^j, \tag{2.1}$$

imbedded in a V_{n+1} of co-ordinates y^α, $\alpha = 1, 2, \ldots n+1$, and metric

$$ds^2 = a_{\alpha\beta} dy^\alpha dy^\beta. \tag{2.2}$$

We then have the relation

$$g_{ij} = a_{\alpha\beta}\, y^\alpha,_i\, y^\beta,_j \tag{2.3}$$

where (,) followed by an index indicates covariant derivative with respect to the x with that index. If N^α, $\alpha = 1, 2, \ldots n+1$, be the contravariant components of the unit normal to V_n,

$$a_{\alpha\beta}\, N^\alpha N^\beta = 1, \tag{2.4}$$

$$y^\alpha;_{ij} = \Omega_{ij} N^\alpha \tag{2.5}$$

where (;) followed by an index indicates tensor derivative with respect to the x with that index and Ω_{ij} are the components of a symmetric covariant tensor of the second order symmetric in the indices i and j.

3. CHARACTERISTIC LINES IN V_n

Let e_{p_1}, $p = 1, 2, \ldots n$ be the unit tangents to the lines of curvature in V_n. If $\mathbf{t}, \bar{\mathbf{t}}$ be any two directions in V_n, we may express $\mathbf{t}, \bar{\mathbf{t}}$ as

$$\mathbf{t} = \sum_p e_{p_1} \cos \theta_p, \tag{3.1}$$

$$\overline{\mathbf{t}} = \sum_p e_{p_1} \cos \overline{\theta}_p, \tag{3.2}$$

where θ_p, $\overline{\theta}_p$ are the angles which $\mathbf{t}$ and $\overline{\mathbf{t}}$ make with the direction e_{p1}.

The angle ϕ between $\mathbf{t}$ and $\overline{\mathbf{t}}$ is given by

$$\cos \phi = g_{ij} \sum_p e_{p_1}{}^i \cos \theta_p \sum_q e_{q_1}{}^j \cos \overline{\theta}_q$$

$$= \sum_p \cos \theta_p \cos \overline{\theta}_p, \tag{3.3}$$

since the congruences e_{p_1} $(p = 1, 2, \ldots . n)$ are mutually orthogonal.

Defining characteristic directions as a pair of conjugate (but not asymptotic) directions, for which ϕ is extremum (so that $\cos \phi$ must also be extremum), the characteristic directions are given by those values of θ_p for which $\cos \phi$ is extremum, subject to the relations

$$\sum_p \cos^2 \theta_p = 1, \tag{3.4}$$

$$\sum_p \cos^2 \overline{\theta}_p = 1, \tag{3.5}$$

and

$$\Omega_{ij} \sum_p e_{p_1}{}^i \cos \theta_p \sum e_{q_1}{}^j \cos \overline{\theta}_q = 0$$

or

$$\sum \Omega_{ij}\, e_{p_1}{}^i e_{p_1}{}^j \cos \theta_p \cos \overline{\theta}_p = 0$$

or

$$\sum k_p \cos \theta_p \cos \overline{\theta}_p = 0 \tag{3.6}$$

where k_p is the principal curvature in the direction e_{p_1}.

Applying the Lagrange's method of undetermined multipliers, we have

$$\sum_p (\sin \theta_p \cos \overline{\theta}_p d\theta_p + \cos \theta_p \sin \overline{\theta}_p d\overline{\theta}_p)$$

$$+ \lambda \sum_p \sin \theta_p \cos \theta_p d\theta_p + \overline{\lambda} \sum \sin \overline{\theta}_p \cos \overline{\theta}_p d\overline{\theta}_p$$

$$+ \mu \sum_p (k_p \sin \theta_p \cos \overline{\theta}_p d\theta_p + k_p \cos \theta_p \sin \overline{\theta}_p d\overline{\theta}_p) = 0 \tag{3.7}$$

where λ, $\overline{\lambda}$ and μ are the Lagrange's multipliers.

Equating to zero the coefficients of the differentials $d\bar{\theta}_p$, $d\theta_p$ we have the following set of equations for the determination of λ, $\bar{\lambda}$, μ and the θ's:

$$\left.\begin{aligned} \sin\theta_p\,(\cos\bar{\theta}_p + \lambda\cos\theta_p + \mu k_p\cos\bar{\theta}_p) &= 0 \\ \sin\bar{\theta}_p\,(\cos\theta_p + \bar{\lambda}\cos\bar{\theta}_p + \mu k_p\cos\theta_p) &= 0 \end{aligned}\right\}, \qquad p = 1, 2, \ldots n. \tag{3.8}$$

But $\sin\theta_p \neq 0$ for any value of p. For, if $\sin\theta_p = 0$ for a fixed p, we have $\cos^2\theta_q = 0$, $q \neq p$, $q = 1, 2, \ldots n$ in which case the direction t must be a principal direction. Similarly $\sin\bar{\theta}_p \neq 0$ for any value of p.

Therefore the equations (3.8) reduce to

$$\left.\begin{aligned} \cos\bar{\theta}_p + \lambda\cos\theta_p + \mu k_p\cos\bar{\theta}_p &= 0, \\ \cos\theta_p + \bar{\lambda}\cos\bar{\theta}_p + \mu k_p\cos\theta_p &= 0 \end{aligned}\right\}, \qquad p = 1, 2, \ldots n. \tag{3.9}$$

Multiplying the first of the equations (3.9) by $\cos\theta_p$ and the second by $\cos\bar{\theta}_p$ and summing up on p we have,

$$\left.\begin{aligned} \cos\phi + \lambda &= 0 \\ \cos\phi + \bar{\lambda} &= 0 \end{aligned}\right\}, \tag{3.10}$$

by virtue of the equations (3.3)—(3.6).

From the equations (3.10) we have

$$\lambda = \bar{\lambda} \tag{3.11}$$

Using (3.11), we may write (3.9) as

$$\begin{aligned} (1 + \mu k_p)\cos\bar{\theta}_p + \lambda\cos\theta_p &= 0 \\ (1 + \mu k_p)\cos\theta_p + \lambda\cos\bar{\theta}_p &= 0 \end{aligned} \tag{3.12}$$

For a fixed p, say $p = r$, the solutions of (3.12) are either $\cos\theta_r = 0$, $\cos\bar{\theta}_r = 0$ or else

$$(1 + \mu k_r)^2 = \lambda^2 \tag{3.13}$$

Similarly if for $p = s$, $\cos\theta_p \neq 0$, $\cos\bar{\theta}_p \neq 0$,

$$(1 + \mu k_s)^2 = \lambda^2 \tag{3.14}$$

There must exist at least two values of p, say r and s, such that $\cos\theta_p$, $\cos\overline{\theta}_p$ are different from zero; for otherwise, the directions $\mathbf{t}$ and $\overline{\mathbf{t}}$ reduce to principal directions.

Two different cases arise according as the principal curvatures are all unequal or there are sets of equal values of the principal curvatures. We shall discuss the two cases separately.

(i) *When all the curvatures are unequal.*—From (3.13) and (3.14) it is found that there cannot be more than two values of p for which $\cos\theta_p$, $\cos\overline{\theta}_p$ may be different from zero. For, we cannot have more than two consistent equations of the form (3.13) for determining λ and μ.

Thus, the characteristic directions are given by

$$\cos\theta_p = 0,\ \cos\overline{\theta}_p = 0,\ p = 1, 2, \ldots n,\quad p \neq r, s \tag{3.15}$$

$$\left.\begin{aligned} 1 + \mu k_r &= \lambda, \\ 1 + \mu k_s &= -\lambda \end{aligned}\right\} \tag{3.16}$$

since $k_r \neq k_s$.

Using (3.16) we have from (3.12)

$$\left.\begin{aligned} \cos\theta_r + \cos\overline{\theta}_r &= 0 \\ \cos\theta_s - \cos\overline{\theta}_s &= 0 \end{aligned}\right\} \tag{3.17}$$

Also using (3.15) we have from (3.4), (3.5) and (3.6)

$$\left.\begin{aligned} \cos^2\theta_r + \cos^2\theta_s &= 1 \\ \cos^2\overline{\theta}_r + \cos^2\overline{\theta}_s &= 1 \\ k_r\cos^2\theta_r - k_s\cos^2\theta_s &= 0 \end{aligned}\right\} \tag{3.18}$$

From the equations (3.18) we have

$$\left.\begin{aligned} \tan^2\theta_r = \tan^2\overline{\theta}_r &= k_r/k_s \\ \tan^2\theta_s = \tan^2\overline{\theta}_s &= k_s/k_r \end{aligned}\right\} \tag{3.19}$$

(3.15) and (3.19) define a pair of real directions provided k_r, k_s, be both of the same sign, showing that corresponding to every pair of principal curvatures having the same sign we have a real pair of characteristic lines.

The angle between the characteristic lines as defined above is given by

$$\begin{aligned}\cos\phi &= \Sigma \cos\theta_p \cos\bar{\theta}_p \\ &= \cos^2\theta_s - \cos^2\theta_r \\ &= 2\cos^2\theta_s - 1 \\ &= \cos 2\theta_s = \frac{k_s - k_r}{k_s + k_r}\end{aligned} \tag{3.20}$$

The normal curvature in the characteristic direction **t** is

$$\begin{aligned}k &= k_r \cos^2\theta_r + k_s \cos^2\theta_s \\ &= \frac{2k_r k_s}{k_r + k_s}\end{aligned} \tag{3.21}$$

Similarly the normal curvature in the characteristic direction $\bar{\mathbf{t}}$ is

$$\begin{aligned}\bar{k} &= k_r \cos^2\bar{\theta}_r + k_s \cos^2\bar{\theta}_s \\ &= \frac{2k_r k_s}{k_r + k_s}\end{aligned} \tag{3.22}$$

Thus *the normal curvatures in the characteristic directions lying in the pencil determined by the principal directions* e_{r_1} *and* e_{s_1} *are equal, each being equal to the harmonic mean between the principal curvatures* k_r *and* k_s.

(ii) *When there are sets of equal values of the curvature at the point.*

Let $\quad k_1 = k_2 = \ldots. = k_r \neq k_s = k_{s+1} = \ldots. = k_{s+q-1}$, etc.

Following the same reasoning as in case (i) we find that corresponding to two distinct values of the principal curvatures, say k_r and k_s, there exist values of p such that

$$(1 + \mu k_p)^2 = \lambda^2 \tag{3.23}$$

$$\cos\theta_p = 0, \cos\bar{\theta}_p = 0, \tag{3.24}$$

$$p = r+1, r+2, \ldots. s-1, s+q, s+q+1, \ldots. n.$$

In this case, (3.4), (3.5) and (3.6) give

$$\left.\begin{aligned}&(\cos^2\theta_1 + \cos^2\theta_2 + \ldots. + \cos^2\theta_r) \\ &\qquad + (\cos^2\theta_s + \ldots. + \cos^2\theta_{s+q-1}) = \\ &(\cos^2\bar{\theta}_1 + \cos^2\bar{\theta}_2 + \ldots. + \cos^2\bar{\theta}_r) \\ &\qquad + (\cos^2\bar{\theta}_s + \ldots. + \cos^2\bar{\theta}_{s+q-1}) = 1 \\ &k_r(\cos\theta_1\cos\bar{\theta}_1 + \ldots. + \cos\theta_r\cos\bar{\theta}_r) \\ &\qquad + k_s(\cos\theta_s\cos\bar{\theta}_s + \ldots. \\ &\qquad + \cos\theta_{s+q-1}\cos\bar{\theta}_{s+q-1}) = 0.\end{aligned}\right\} \tag{3.25}$$

From (3·23) we have

$$\left.\begin{aligned} 1 + \mu k_r &= \lambda, \\ 1 + \mu k_s &= -\lambda \end{aligned}\right\}, \tag{3·26}$$

giving

$$\mu = -\frac{2}{k_r + k_s}, \quad \lambda = -\frac{k_r - k_s}{k_r + k_s}. \tag{3·27}$$

(3·9) now yields

$$\cos\theta_u + \cos\overline{\theta}_u = 0, \qquad u = 1, 2, \ldots r \tag{3·28}$$

$$\cos\theta_v - \cos\overline{\theta}_v = 0, \qquad v = s, s+1, \ldots, s+q-1. \tag{3·29}$$

Using (3·28) and (3·29), the last of the equations (3·25) becomes

$$k_r(\cos^2\theta_1 + \cos^2\theta_2 + \ldots + \cos^2\theta_r) - k_s(\cos^2\theta_s + \ldots + \cos^2\theta_{s+q-1}) = 0,$$

which together with the first of the equations (3·25) gives

$$\cos^2\theta_1 + \cos^2\theta_2 + \ldots + \cos^2\theta_r = \cos^2\overline{\theta}_1 + \cos^2\overline{\theta}_2 + \ldots + \cos^2\overline{\theta}_r = k_s/k_r + k_s, \tag{3·30}$$

$$\cos^2\theta_s + \cos^2\theta_{s+1} + \ldots + \cos^2\theta_{s+q-1} = \cos^2\overline{\theta}_s + \cos^2\overline{\theta}_{s+1} + \ldots + \cos^2\overline{\theta}_{s+q-1} = k_r/k_r + k_s.$$

(3·24) and (3·30) show that the directions **t** and $\overline{\mathbf{t}}$ are linear combinations of the directions $\frac{k_r + k_s}{k_s} \sum_{u=1}^{r} e_{u_1} \cos\theta_u$, and $\frac{k_r + k_s}{k_r} \sum_{v=s}^{s+q-1} e_{v_1} \cos\theta_v$, $\frac{k_r + k_s}{k_s} \sum_{u=1}^{r} e_{u_1} \cos\overline{\theta}_u$ and $\frac{k_r + k_s}{k_r} \sum_{v=s}^{s+q-1} e_{v_1} \cos\overline{\theta}_v$ respectively, which being linear combinations of the principal directions corresponding to equal values of curvature, are themselves principal directions.

Hence we find that *the characteristic directions are linear combinations of the principal directions corresponding to any two distinct values of the principal curvatures.*

The angle between the characteristic directions **t** and $\overline{\mathbf{t}}$ is given by

$$\cos\phi = -\lambda = \frac{k_r - k_s}{k_r + k_s}, \tag{3·31}$$

by virtue of (3·27).

The normal curvatures in the characteristic directions $\mathbf{t}$ and $\overline{\mathbf{t}}$ are

$$
\begin{aligned}
k = {} & k_r(\cos^2\theta_1 + \ldots + \cos^2\theta_r) \\
& + k_s(\cos^2\theta_s + \ldots + \cos^2\theta_{s+q-1}) \\
= {} & 2k_r k_s/k_r + k_s.
\end{aligned} \tag{3.32}
$$

and

$$
\begin{aligned}
\bar{k} = {} & k_r(\cos^2\bar{\theta}_1 + \ldots + \cos^2\bar{\theta}_r) \\
& + k_s(\cos^2\bar{\theta}_s + \ldots + \cos^2\bar{\theta}_{s+q-1}) \\
= {} & 2k_r k_s/k_r + k_s,
\end{aligned} \tag{3.33}
$$

by virtue of (3.30)

Relations (3.31)–(3.33) are the same as (3.20)–(3.22).

4. Directions for which the Ratio of the Geodesic Torsion (or Second Curvature) and the Normal Curvature is Extremum

The geodesic torsion in the direction $\mathbf{t} = \sum_p e_{p_1} \cos\theta_p$ is given by (Manikarnikamma, 1952).

$$\tau_g^2 = \tfrac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \cos^2\theta_p \cos^2\theta_q (k_p - k_q)^2 \tag{4.1}$$

The normal curvature in the direction of $\mathbf{t}$ is

$$k_n = \sum_{p=1}^{n} k_p \cos^2\theta_p \tag{4.2}$$

The directions for which τ_g/k_n is extremum are the same as those for τ_g^2/k_n^2 is extremum, (for τ_g is zero only in a principal direction).
Now,

$$
\begin{aligned}
\tau_g^2/k_n^2 &= (\tau_g^2 + k_n^2)\, k_n^{-2} - 1 \\
&= \sum_p k_p^2 \cos^2\theta_p \Big(\sum_p k_p \cos^2\theta_p\Big)^{-2} - 1.
\end{aligned} \tag{4.3}
$$

The directions for which τ_g/k_n is extremum are therefore given by

$$
\left.
\begin{aligned}
& \sum_{p=1}^{n} \frac{\partial}{\partial\theta_p} \left[\frac{\sum_1^n k_r^2 \cos^2\theta_r}{\left(\sum_1^n k_r \cos^2\theta_r\right)^2} - 1 \right] = 0, \\
& \sum_{p=1}^{n} \cos^2\theta_p = 1, \\
\text{and} \quad & \\
& \sum_{p=1}^{n} \cos\theta_p \sin\theta_p \, d\theta_p = 0.
\end{aligned}
\right\} \tag{4.4}
$$

The first of the equations (4.4) may be written as

$$\sum_p [k_p^2 \cos\theta_p \sin\theta_p (\Sigma k_r \cos^2\theta_r)^2$$
$$-2(\Sigma k_r \cos^2\theta_r) k_p \cos\theta_p \sin\theta_p \Sigma k_r^2 \cos^2\theta_r] \, d\theta_p = 0 \quad (4.5)$$

Applying the Lagrange's method of undetermined multipliers we have,

$$\left(\sum_1^n k_r \cos^2\theta_r\right) k_p^2 \cos\theta_p \sin\theta_p - 2k_p \cos\theta_p \sin\theta_p \Sigma k_r^2 \cos^2\theta_r$$
$$+ \lambda \cos\theta_p \sin\theta_p = 0, \quad (4.6)$$
$$(p = 1, 2, \ldots n)$$

where λ is the Lagrange's multiplier.

Now $\sin\theta_p \neq 0$ for any p, for otherwise the direction **t** must be a principal direction. (4·6) therefore gives

$$\cos\theta_p \left[k_p^2 \sum_1^n k_r \cos^2\theta_r - 2k_p \sum_1^n k_r^2 \cos^2\theta_r + \lambda\right] = 0 \quad (4.7)$$
$$(p = 1, 2, \ldots n)$$

(4·7) gives either (i) Cos $\theta_p = 0$,

$$\text{or (ii) } k_p^2 \sum_1^n k_r \cos^2\theta_r - 2k_p \sum_1^n k_r^2 \cos^2\theta_r + \lambda = 0.$$

As in §3 two different cases arise:

CASE (i). *When all the principal curvatures are unequal.*

Since **t** is not a principal direction, there must exist at least two values of p, say r and s, such that cos $\theta_p = 0$, *i.e.*, for at least two values of p,

$$k_p^2 \sum_1^n k_p \cos^2\theta_p - 2k_p \sum_1^n k_p^2 \cos^2\theta_p + \lambda = 0 \quad (4.8)$$

Putting $p = r, s$ in (4.8) and subtracting we have

$$(k_r \div k_s) \sum_1^n k_p \cos^2\theta_p - 2\sum_1^n k_p^2 \cos^2\theta_p = 0 \quad (4.9)$$

since $$k_r \neq k_s.$$

Multiplying (4.8) throughout by $\cos^2\theta_p$ and summing up on p we have

$$\lambda = \sum_1^n k_p^2 \cos^2\theta_p \sum_1^n k_p \cos^2\theta_p \quad (4.10)$$

From (4.9) it follows that there cannot exist more than two values of p such that cos θ_p may be different from zero; for, if r, s, w be three such values, we have as in (4.9)

$$\left.\begin{aligned} (k_r + k_w) \sum_1^n k_p \cos^2 \theta_p - 2 \sum_1^n k_p^2 \cos^2 \theta_p = 0 \\ (k_s + k_w) \sum_1^n k_p \cos^2 \theta_p - 2 \sum_1^n k_p^2 \cos^2 \theta_p = 0 \end{aligned}\right\} \tag{4.11}$$

By subtracting the equations in (4.11) we find that either $k_r = k_s$ or else

$$\sum_1^n k_p \cos^2 \theta_p = 0,$$

i.e., direction of $\mathbf{t}$ must be asymptotic. Since none of these alternatives is possible, we find that

$$\cos \theta_p = 0, \;\; p = 1, 2, \ldots. n, p \neq r, s, \tag{4.12}$$

$$\left.\begin{aligned} k_r^2 \sum_1^n k_p \cos^2 \theta_p - 2k_r \sum_1^n k_p^2 \cos^2 \theta_p + \lambda = 0 \\ k_s^2 \sum_1^n k_p \cos^2 \theta_p - 2k_s \sum_1^n k_p^2 \cos^2 \theta_p + \lambda = 0 \end{aligned}\right\} \tag{4.13}$$

The second of the equations (4.4) now gives

$$\cos^2 \theta_r + \cos^2 \theta_s = 1 \tag{4.14}$$

Using (4.12), (4.9) gives

$$\left.\begin{aligned} \tan^2 \theta_r &= k_r/k_s, \\ \tan^2 \theta_s &= k_s/k_r. \end{aligned}\right\} \tag{4.15}$$

(4.12), (4.15) and (4.16) define a pair of real directions corresponding to every pair of distinct values of principal curvatures of the same sign.

CASE (ii). *When there are sets of equal values of principal curvatures at any point.*

Let $\quad k_1 = k_2 = \ldots. = k_r \neq k_s = k_{s+1} = \ldots. = k_{s+q-1}$, etc.

Proceeding as in § 3 above, we can show that the directions for which the ratio of geodesic torsion to normal curvature is extremum are given by

$$\cos \theta_p = 0, p \neq 1, 2, \ldots. r, s, s+1, \ldots. s+q-1, \tag{4.16}$$

$$\left.\begin{aligned} \cos^2 \theta_1 + \cos^2 \theta_2 + \ldots. + \cos^2 \theta_r &= \frac{k_r}{k_r + k_s} \\ \cos^2 \theta_s + \cos^2 \theta_{s+1} + \ldots. + \cos^2 \theta_{s+q-1} &= \frac{k_s}{k_r + k_s} \end{aligned}\right\} \tag{4.17}$$

Comparing (4.12), (4.15), (4.16) and (4.17) with (3.15), (3.19), (3.24) and (3.30) respectively we find that *the directions for which the ratio of the geodesic torsion and normal curvature is an extremum are characteristic directions,* *and are, as such, linear combinations of the principal directions corresponding to two distinct values of the principal curvatures.*

5. PARTICULAR CASE

If the enveloping space be an Euclidean 3-space, we find from §4 above the known result (Mishra, 1947), that the *characteristic lines on a surface are the directions for which the ratio of geodesic torsion and the normal curvature is an extremum.*

6. SUMMARY

Upon a surface of positive Gaussian curvature there exists a unique conjugate system for which the angle between the directions at any point is the minimum angle between the conjugate directions at that point. This system of lines is called characteristic lines. In the present paper characteristic lines of a hypersurface V_n imbedded in a Riemannian V_{n+1} have been studied. It has been proved that characteristic directions are linear combinations of the principal directions corresponding to any two distinct values of the principal curvatures. It has also been proved that the normal curvatures in the two characteristic directions lying in the pencil determined by the principal directions corresponding to two distinct values of the principal curvatures are equal, each being equal to the harmonic mean between the principal curvatures. The directions for which the ratio of the geodesic torsion and normal curvature is an extremum have also been studied and it has been shown that the directions for which the ratio of the geodesic torsion and the normal curvatures is an extremum are characteristic directions.

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