A THEOREM ON NORMAL RECTILINEAR CONGRUENCES

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Abstract

1. Let a rectilinear congruence be defined by

$$\xi = x + tX, \eta = y + tY, \ \zeta = z + tZ,$$

where x, y, z, X, Y, Z are functions of two variables u and v.

Denoting Sannia's two quadratic forms[1]

$$E du^2 + 2\,F du dv + G dv^2 \text{ and } \delta du^2 + 2\,\delta' du dv + \delta'' dv^2,$$

by f and ϕ respectively, we know that the equations of (*i*) the surfaces whose spherical representations are minimal lines, (*ii*) developable surfaces, and (*iii*) surfaces of distribution,[2] are respectively

$$f = 0, \ \phi = 0, J\,(f, \phi) = 0.$$ [3]

The object of this paper is to obtain analytically a property of the spherical representations of the distributive ruled surfaces through a line of a normal rectilinear congruence.

2. When the congruence is normal, the focal planes are at right angles and hence the two quadratic forms f and ϕ are harmonic, so that

$$E\delta'' + G\delta - 2\,F\delta' = 0.$$

This relation can also be written in the form $f - f' = 0$ where f stands for $\Sigma X_1 x_2$ and f' for $\Sigma X_2 x_1$, the subscripts 1 and 2 denoting differentiation with regard to u and v respectively.

We have

$$\frac{1}{\sqrt{EG - F^2}}\left(\frac{\delta\gamma'}{\delta u} - \frac{\delta\gamma}{\delta v}\right) = \frac{2\,F\delta' - E\delta'' - G\delta}{EG - F^2}, \qquad (1)$$ [4]

where $\gamma \, (\equiv \Sigma X x_1)$ and $\gamma' \, (\equiv \Sigma X x_2)$ are given by

$$-\gamma \ \sqrt{EG - F^2} = \delta_2 - \delta_1' - \Gamma'\delta + (\Gamma - \Delta')\,\delta' + \Delta\,\delta'' \qquad (2)$$

and

$$\gamma' \ \sqrt{EG - F^2} = \delta_1'' - \delta_2' + \Gamma''\delta + (\Delta'' - \Gamma')\,\delta' - \Delta'\delta'' \qquad (3)$$

where the letters Γ, Γ', Γ''; Δ, Δ', Δ'' have their usual meanings.[5]

Let the curves on the sphere which represent the surfaces of distribution be taken as the parametric curves, then

$$J(f, \phi) \equiv \begin{vmatrix} E du + F dv & F du + G dv \\ \delta du + \delta' dv & \delta' du + \delta'' dv \end{vmatrix} = 0$$

must be the same as $du dv = 0$, and therefore we have

$$\delta' = 0, \; F = 0. \tag{4}$$

Also since the congruence is normal, we have

$$\frac{\delta\gamma'}{\delta u} - \frac{\delta\gamma}{\delta v} = 0; \; 2F\delta' - E\delta'' - G\delta = 0 \tag{5}$$

so that (1) is satisfied identically, and (2), (3), (5) become, making use of (4),

$$\delta_2 - \Gamma'\delta + \Delta\delta'' = 0 \tag{6}$$

$$\delta_1'' + \Gamma''\delta - \Delta'\delta'' = 0 \tag{7}$$

$$E\delta'' + G\delta = 0 \tag{8}$$

But $\Gamma' \equiv \dfrac{GE_2 - FG_1}{2(EG - F^2)} = \dfrac{E_2}{2E}$; $\Delta \equiv \dfrac{-FE_1 - EE_2 + 2EF_1}{2(EG - F^2)} = -\dfrac{E_2}{2G}$;

$\Gamma'' \equiv \dfrac{-FG_2 - GG_1 + 2GF_2}{2(EG - F^2)} = -\dfrac{G_1}{2E}$; $\Delta' \equiv \dfrac{EG_1 - FE_2}{2(EG - F^2)} = \dfrac{G_1}{2G}$;

$\therefore$ (6) reduces to $\quad \delta_2 - \left(\dfrac{\delta}{E} + \dfrac{\delta''}{G}\right) \cdot \dfrac{E_2}{2} = 0,$

and (7) reduces to $\quad \delta_1'' - \left(\dfrac{\delta}{E} + \dfrac{\delta''}{G}\right) \cdot \dfrac{G_1}{2} = 0.$

$\therefore \quad \delta_2 = 0$ and $\delta_1'' = 0$ from (8).

Hence δ is a function of u only and δ'' is a function of v only. But

$$\frac{\delta}{\delta''} = -\frac{E}{G} \text{ from (8).}$$

$$\therefore \quad \frac{E}{G} = \frac{\text{a function of } u \text{ only}}{\text{a function of } v \text{ only}}$$

$\therefore \quad \dfrac{\delta^2}{\delta u \delta v} \log\left(\dfrac{E}{G}\right) = 0$, which shows that the parametric curves are isometric.

Hence we get the theorem:

The curves on the sphere representing the distributive ruled surfaces through a line of a normal rectilinear congruence are isometric.

REFERENCES

1. Bianchi, *Lezioni di geometria differenziale*, **1**, 459.

2. These surfaces have been called 'surfaces of curvature' by Zindler, *Liniengeometrie*, **2**, 108; and 'mean ruled surfaces' by Eisenhart, *Diff. Geo.*, p. 422; Burgatti, *Atti dei Lincei*, 1899, p. 515; Cifarelli, *Annali di Matematica*, 1899, pp. 139–54.

3. K. Ogura, *Sc. Rep. Tohoku Imp. Univ.*, 1916, p. 114.

4. Bianchi, *Lezioni*, **1**, 497; Sannia, *Atti della Accademia di Torino*, **45**, 58.

5. Forsyth, *Diff. Geo.*, p. 45.