ON FINITE GENERATION OF KERNELS OF LOCALLY NILPOTENT R-DERIVATIONS OF $R[X,Y,Z]$

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Abstract. Let R be a noetherian domain containing the field of rationals. We show that if R is Dedekind then the kernel of any locally nilpotent R-derivation of $R[X,Y,Z]$ is a finitely generated R-algebra. Conversely, we show that if R is neither a field nor a Dedekind domain then there exists a locally nilpotent R-derivation of $R[X,Y,Z]$ whose kernel is not finitely generated over R.

1. Introduction

One of the main results of this paper is the following generalization of [DF01, Cor. 1.2]:

Theorem 1. *Let $D : \mathbf{k}[X_1,X_2,X_3,X_4] \to \mathbf{k}[X_1,X_2,X_3,X_4]$ be a locally nilpotent derivation where $\mathbf{k}$ is a field of characteristic zero. If $D(f)=0$ for some variable f of $\mathbf{k}[X_1,X_2,X_3,X_4]$, then $\ker(D)$ is a finitely generated $\mathbf{k}$-algebra.*

In the above statement, by a *variable* of $\mathbf{k}[X_1,X_2,X_3,X_4]$ we mean an element f satisfying $\mathbf{k}[X_1,X_2,X_3,X_4] = \mathbf{k}[f,f_2,f_3,f_4]$ for some f_2,f_3,f_4. It is not known whether the theorem remains valid without the assumption that D annihilates a variable.

Theorem 1 is an immediate consequence of the following fact, which is proved in this paper and which generalizes [DF01, Th. 1.1]:

Proposition. *Let R be a Dedekind domain containing $\mathbb{Q}$. For any locally nilpotent R-derivation $D : R[X,Y,Z] \to R[X,Y,Z]$, $\ker(D)$ is a finitely generated R-algebra.*

In view of this proposition one is led to consider the following more general question. Let n be a positive integer and R a domain of characteristic zero. We say that R has the property $\mathrm{FG}(n)$ if for every locally nilpotent R-derivation D of $R^{[n]}$, $\ker(D)$ is finitely generated as an R-algebra (where $R^{[n]}$ denotes the polynomial algebra in n variables over R). We will write $R \in \mathrm{FG}(n)$ to indicate that R has property $\mathrm{FG}(n)$. It is interesting to ask which rings have property $\mathrm{FG}(n)$, for each n. This paper gives some partial results in that direction, and the above Proposition is one of them.

It is clear that all domains of characteristic zero have the property $\mathrm{FG}(1)$. On the other hand, it is known (cf. [DF99], Theorem 3.3) that if $\mathbf{k}$ is a field of characteristic zero and $n \geq 5$ then there exists a locally nilpotent derivation of $\mathbf{k}^{[n]}$ whose kernel is not finitely generated over $\mathbf{k}$. In view of part (1) of Lemma 2.1, it follows:

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Corollary. *If $n \geq 5$ then no domain of characteristic zero has property* $\mathrm{FG}(n)$.

Consequently the problem of determining which rings have property $\mathrm{FG}(n)$ remains open only for $n = 2, 3, 4$. In Section 4 we will prove:

Theorem 2. *Let R be a noetherian domain containing $\mathbb{Q}$.*

1. *$R \in \mathrm{FG}(3) \Leftrightarrow R$ is a Dedekind domain or a field.*
2. *If R is not a field then $R \notin \mathrm{FG}(4)$.*

This gives a satisfactory solution to the case $n = 3$ of the problem. The question whether fields of characteristic zero have property $\mathrm{FG}(4)$ is still open.

In this paper we will almost always assume that the base ring R is noetherian and contains $\mathbb{Q}$. We will first show that if $R \in \mathrm{FG}(n)$ for some $n > 1$ then R is normal, so to tackle the question one may also assume that R is normal. Under these assumptions, the question for $n = 2$ has been already investigated by Bhatwadekar and Dutta and a partial answer has been obtained, viz. (i) if the group $\mathrm{Cl}(R)/\operatorname{Pic}(R)$ is torsion then $R \in \mathrm{FG}(2)$; (ii) if $\dim(R) = 2$, then $R \in \mathrm{FG}(2)$ implies that $\mathrm{Cl}(R)/\operatorname{Pic}(R)$ is torsion (see [BD97], Corollary 3.7, Remark 3.10). A complete solution for $n = 2$ seems to be elusive at present.

Conventions. PID means *principal ideal domain* and DVR means *discrete valuation ring*. We write $\mathfrak{c}_{R'/R} = \{ x \in R \mid xR' \subseteq R \}$ for the conductor of a ring extension $R \subseteq R'$.

2. Preliminaries

2.1. **Lemma.** *Let R be a domain of characteristic zero and suppose that $R \in \mathrm{FG}(n)$.*

1. *If $S \subset R$ is a multiplicative set then $S^{-1}R \in \mathrm{FG}(n)$.*
2. *If $n > 1$ then $R \in \mathrm{FG}(n-1)$ and $R^{[1]} \in \mathrm{FG}(n-1)$.*

Proof. (1) Let $D : S^{-1}R[X_1, \dots, X_n] \to S^{-1}R[X_1, \dots, X_n]$ be a locally nilpotent $S^{-1}R$-derivation. For each i, there exists $s_i \in S$ such that $s_i D(X_i) \in R[X_1, \dots, X_n]$. Let $s = s_1 \cdots s_n$. As $s \in \ker D$, the $S^{-1}R$-derivation $sD : S^{-1}R[X_1, \dots, X_n] \to S^{-1}R[X_1, \dots, X_n]$ is locally nilpotent; moreover, sD maps $R[X_1, \dots, X_n]$ into itself. Let $d : R[X_1, \dots, X_n] \to R[X_1, \dots, X_n]$ be the restriction of sD, then $\ker d$ is a finitely generated R-algebra since $R \in \mathrm{FG}(n)$. As $S^{-1} \ker(d) = \ker(sD) = \ker D$, $\ker D$ is a finitely generated $S^{-1}R$-algebra. Assertion (2) is trivial. $\square$

It was noted in [BD97] that by combining results 2.14 and 2.20 of [Ono84] with 2.1 of [Gir81], one obtains:

2.2. **Lemma.** *Let $R \subseteq A \subseteq B$ be domains, where R is noetherian and B is finitely generated as an R-algebra. Then the following are equivalent:*

1. *A is finitely generated as an R-algebra;*
2. *for every maximal ideal $\mathfrak{m}$ of R, $A_{\mathfrak{m}}$ is finitely generated as an $R_{\mathfrak{m}}$-algebra.*

The next fact is Lemma 3.2 of [DF01], and easily follows from 2.2:

2.3. **Lemma.** *Let R be a noetherian domain containing $\mathbb{Q}$, B a finitely generated overdomain of R and $D : B \to B$ a locally nilpotent R-derivation. Then the following conditions are equivalent:*

(1) $\ker(D)$ *is finitely generated as an R-algebra;*
(2) *for every maximal ideal $\mathfrak{m}$ of R, $\ker(D_{\mathfrak{m}})$ is finitely generated as an $R_{\mathfrak{m}}$-algebra (where $D_{\mathfrak{m}} : B_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is the $R_{\mathfrak{m}}$-derivation obtained by localizing D).*

2.4. **Lemma.** *Let R be a noetherian domain containing $\mathbb{Q}$ and n a positive integer. Then the following are equivalent:*

(1) $R \in \mathrm{FG}(n)$
(2) $R_{\mathfrak{m}} \in \mathrm{FG}(n)$ *for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Implication $(1 \Rightarrow 2)$ follows from part (1) of 2.1, and the converse is an immediate consequence of 2.3. $\square$

We also mention the following useful (and trivial) fact:

2.5. **Lemma.** *Suppose that $R \to S$ is a faithfully flat homomorphism of rings and that A is an R-algebra. Then A is finitely generated as an R-algebra if and only if $S \otimes_R A$ is finitely generated as an S-algebra.*

Proof. Suppose that $S \otimes_R A$ is finitely generated as an S-algebra and consider a finite set $\{g_1, \dots, g_m\}$ of generators. There exists a finite subset $E = \{x_1, \dots, x_n\}$ of A with the property that each g_i is a finite sum, $g_i = \sum_j s_{ij} \otimes \alpha_{ij}$, with $s_{ij} \in S$ and $\alpha_{ij} \in E$. Now consider the polynomial ring $R[X_1, \dots, X_n]$ and the R-homomorphism $\varphi : R[X_1, \dots, X_n] \to A$ defined by $\varphi(X_i) = x_i$ for all i. Applying the functor $S \otimes_R (\_)$ to φ yields an S-homomorphism $\Phi : S \otimes_R R[X_1, \dots, X_n] \to S \otimes_R A$ whose image contains $\{1 \otimes x_1, \dots, 1 \otimes x_n\}$. Thus Φ is surjective and, by faithful flatness, it follows that φ is surjective and hence that A is finitely generated. The converse is trivial (and holds without assuming faithful flatness). $\square$

3. Normality

3.1. **Lemma.** *Let $(R, \mathfrak{m})$ be a local domain containing $\mathbb{Q}$ and such that $\mathfrak{m} \neq \mathfrak{m}^2$. If there exists an overdomain R' of R such that $\mathfrak{c}_{R'/R} = \mathfrak{m}$ then $R \notin \mathrm{FG}(2)$.*

Proof. Fix an element $t \in R' \setminus R$. Let $F = tX + Y \in R'[X,Y]$, $A' = R'[F]$ and $A = R[X,Y] \cap A'$. We will prove the following claims, where (1) and (2) suffice for proving the Lemma:

(1) *A is the kernel of a locally nilpotent R-derivation of $R[X,Y]$,*

(2) *A is not finitely generated as an R-algebra,*

(3) *A is not a noetherian ring.*

The assumption $\mathfrak{m} \neq \mathfrak{m}^2$ implies in particular $\mathfrak{m} \neq 0$, so we may pick $c \in \mathfrak{m} \setminus \{0\}$ and define the R'-derivation

$$D = c\big(\frac{\partial}{\partial X} - t\frac{\partial}{\partial Y}\big) : R'[X,Y] \to R'[X,Y].$$

Then D is locally nilpotent and $\ker(D) = A'$. Moreover, D maps $R[X,Y]$ into itself (because $cR' \subset R$). Let $d : R[X,Y] \to R[X,Y]$ be the restriction of D, then d is a locally nilpotent R-derivation and $\ker(d) = R[X,Y] \cap A' = A$, proving (1).

Consider an element α of A. As $\alpha \in A'$, we have $\alpha = \sum_{n=0}^{d} a_n F^n$ where $a_n \in R'$ for all n. So:

$$\alpha = \sum_{n=0}^{d} a_n \sum_{i=0}^{n} \binom{n}{i} t^i X^i Y^{n-i} = \sum_{(i,j)\in E} \binom{i+j}{i} a_{i+j} t^i X^i Y^j,$$

where $E = \big\{\, (i,j) \in \mathbb{N}^2 \mid i+j \leq d \,\big\}$. As $\alpha \in R[X,Y]$ and $\mathbb{Q} \subseteq R$ we obtain $a_{i+j}t^i \in R$ for all $(i,j) \in E$; it follows that $a_n \in R$ for all $n = 0,\dots,d$ and that $a_n t \in R$ for all $n = 1,\dots,d$; so $a_0 \in R$ and $a_n \in \mathfrak{m}$ for $n \geq 1$ (a_n cannot be a unit of R because $a_n t \in R$ would imply that $t \in R$, which is not the case). Thus $\alpha \in R + \mathfrak{m}[F]$ and hence $A \subseteq R + \mathfrak{m}[F]$. The reverse inclusion being clear,

$$A = R + \mathfrak{m}[F].$$

So A is generated by $G = \big\{\, aF^n \mid a \in \mathfrak{m} \text{ and } n \in \mathbb{N} \,\big\}$ as an R-algebra. If A is finitely generated then $A = R[a_1F^{n_1},\dots,a_pF^{n_p}]$ for some $a_i \in \mathfrak{m}$ and $n_i \in \mathbb{N}$; pick $a_0 \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $n_0 > \max(n_1,\dots,n_p)$ then $a_0F^{n_0} \in R[a_1F^{n_1},\dots,a_pF^{n_p}]$, so $a_0Y^{n_0} \in R[a_1Y^{n_1},\dots,a_pY^{n_p}]$, which is impossible. This proves (2).

Before proving (3) we observe that $\mathfrak{m}R' = \mathfrak{m}$, so $A \cap \mathfrak{m}A' = \mathfrak{m}A' = (\mathfrak{m}R')[F] = \mathfrak{m}[F]$, so $A/(A \cap \mathfrak{m}A') = (R + \mathfrak{m}[F])/\mathfrak{m}[F] \cong R/\mathfrak{m}$. On the other hand, $A' = R'[F] = R'^{[1]}$ implies that $A'/\mathfrak{m}A' = (R'/\mathfrak{m}R')^{[1]}$, so

$$A'/\mathfrak{m}A' \text{ is transcendental over } A/(A \cap \mathfrak{m}A'). \tag{4}$$

Assume that A is noetherian. Pick $a \in \mathfrak{m} \setminus \{0\}$, then aA' is an ideal of A, hence a finitely generated A-module. As A' and aA' are isomorphic A-modules, A' is finite over A and consequently $A'/\mathfrak{m}A'$ is finite over $A/(A \cap \mathfrak{m}A')$. This contradicts (4), so (3) is proved. □

3.2. **Proposition.** *Let R be a noetherian domain containing $\mathbb{Q}$. If $R \in \mathrm{FG}(n)$ for some $n > 1$, then R is normal.*

Proof. Assume that R is not normal. Then there exists a ring R' such that $R \subset R' \subset \operatorname{Frac} R$, $R' \neq R$ and R' is finite over R. Then $0 \neq \mathfrak{c}_{R'/R} \neq R$. Let $\mathfrak{p} \in \operatorname{Spec} R$ be a minimal prime over ideal of $\mathfrak{c}_{R'/R}$ and consider the rings $R_\mathfrak{p} \subset R'_\mathfrak{p}$. Let us denote the local ring $R_\mathfrak{p}$ by $(A,\mathfrak{m})$ and let $B = R'_\mathfrak{p}$. Then the radical of the ideal $\mathfrak{c}_{B/A}$ of A is $\mathfrak{m}$, so we may consider an integer $\ell \geq 1$ satisfying $\mathfrak{m}^\ell \subseteq \mathfrak{c}_{B/A}$ and $\mathfrak{m}^{\ell-1} \not\subseteq \mathfrak{c}_{B/A}$. Define $B' = A + \mathfrak{m}^{\ell-1}B$, then $\mathfrak{c}_{B'/A} = \mathfrak{m}$. As $\mathfrak{m}^2 \neq \mathfrak{m}$, 3.1 implies that $A \notin \mathrm{FG}(2)$. By 2.1, we conclude that $R \notin \mathrm{FG}(2)$ and hence that $R \notin \mathrm{FG}(n)$ for all $n > 1$. □

4. Simple sequences and the property FG(3)

The material in 4.1–4.5 is taken from [DF01].

4.1. **Setup.** We consider triples (R,t,S) satisfying:

(1) R is a PID containing $\mathbb{Q}$ and t a prime element of R;
(2) S is an overdomain of R such that t is prime in S.

Given a triple (R,t,S) as above, we use the following notations:

- $\bar{S} = S/tS$ (an integral domain);
- given $s \in S$, write $\bar{s} = s + tS \in \bar{S}$;
- given a ring A such that $R \subseteq A \subseteq S$, (i) let $\bar{A}$ be the image of the composite $A \hookrightarrow S \to \bar{S}$; (ii) let $A^+ = A[S \cap \frac{1}{t}A]$;
- the field R/tR is denoted κ (note that $\bar{R} = \kappa$).

4.2. **Definition.** Let (R,t,S) be as in 4.1. Let $f = (f_0, \dots, f_r)$ be a finite sequence in S and write $A_n = R[f_0, \dots, f_n]$ and $K_n = \mathrm{Frac}(\bar{A}_n)$ for $0 \le n \le r$. We say that f is a *simple sequence* of (R,t,S) if the following hold:

(1) $r \ge 1$;
(2) $\bar{f}_0 \in \bar{S}$ is transcendental over κ;
(3) for each n such that $0 < n < r$, $\bar{f}_n$ is algebraic over K_{n-1} and there exists a monic polynomial $\varphi_n \in A_{n-1}[T]$ satisfying: (i) $\bar{\varphi}_n \in \bar{A}_{n-1}[T]$ is the minimal polynomial of $\bar{f}_n$ over K_{n-1}; and (ii) $f_{n+1} = \varphi_n(f_n)/t$.

We distinguish three types of simple sequences:

(i) f is *transcendental* if $\bar{f}_r$ is transcendental over K_{r-1};
(ii) f is *extendable* if $\bar{f}_r$ is algebraic over K_{r-1} and its minimal polynomial is in $\bar{A}_{r-1}[T]$;
(iii) f is *obstructed* if it is neither transcendental nor extendable, i.e., if $\bar{f}_r$ is algebraic over K_{r-1} but its minimal polynomial fails to have all its coefficients in $\bar{A}_{r-1}$.

Remark. It is easy to see that a simple sequence $f = (f_0, \dots, f_r)$ of (R,t,S) is extendable if and only if $\exists f_{r+1} \in S$ such that $(f_0, \dots, f_r, f_{r+1})$ is a simple sequence of (R,t,S). Also note that if $S = R^{[1]}$ (as in the next definition) then no simple sequence of (R,t,S) is transcendental, because $\bar{S} = \kappa^{[1]}$ is algebraic over $\kappa[\bar{f}_0]$.

4.3. **Definition.** Let R be a PID containing $\mathbb{Q}$ and t a prime element of R. We call (R,t) a *simple pair* if no simple sequence of $(R,t,R^{[1]})$ is obstructed (i.e., if every simple sequence of $(R,t,R^{[1]})$ is extendable).

4.4. **Definition.** Let (R,t) and (R',t') be pairs satisfying the condition (1) of 4.1. We use the notation $(R,t) \prec (R',t')$ to indicate that the following conditions hold:

(1) R' is a DVR with maximal ideal $t'R'$;
(2) $R \subseteq R'$ and $R \cap t'R' = tR$;
(3) $R'/t'R'$ is an algebraic extension of R/tR.

4.5. **Lemma** (Cf. Lemma 2.10 of [DF01])**.** *Let (R,t) and (R',t') be pairs satisfying the condition* (1) *of* 4.1. *If $(R,t) \prec (R',t')$ and (R',t') is simple, then (R,t) is simple.*

4.6. **Lemma.** *Let $\mathbf{k}$ be a field of characteristic zero and t an indeterminate over $\mathbf{k}$. Then $(\mathbf{k}[[t]],t)$ is a simple pair.*

Proof. Let $\bar{\mathbf{k}}$ be the algebraic closure of $\mathbf{k}$, then $(\mathbf{k}[[t]],t) \prec (\bar{\mathbf{k}}[[t]],t)$. By Corollary 2.8 of [DF01],[1] $(\bar{\mathbf{k}}[[t]],t)$ is a simple pair. So $(\mathbf{k}[[t]],t)$ is a simple pair by Lemma 4.5. □

4.7. **Lemma.** *Let R be a DVR containing $\mathbb{Q}$ and t a uniformizing parameter of R. Then (R,t) is a simple pair.*

Proof. Let $\hat{R}$ be the completion of R with respect to tR. Then $\hat{R} \cong \kappa[[t]]$ where $\kappa = R/tR$ is a field of characteristic zero. So $(R,t) \prec (\kappa[[t]],t)$, where $(\kappa[[t]],t)$ is a simple pair by 4.6. We are done by Lemma 4.5. □

4.8. **Lemma.** *Let R be a PID containing $\mathbb{Q}$ and t a prime element of R. Then (R,t) is a simple pair.*

Proof. Let R' be the localization of R at the maximal ideal tR. Then $(R,t) \prec (R',t)$ where (R',t) is a simple pair by 4.7. We are done by Lemma 4.5. □

Remark. A posteriori we find that Definition 4.3 is a bit misleading: by Lemma 4.8, the simple pairs are *precisely* the pairs (R,t) where R is a PID containing $\mathbb{Q}$ and t a prime element of R.

4.9. **Lemma.** *Let R,t,S,U be such that each of the triples (R,t,S) and (R,t,U) satisfies conditions* (1) *and* (2) *of* 4.1. *Suppose:*

(∗) *Given any $g \in S$ such that $\bar{g} \in \bar{S}$ is transcendental over κ, there exists an R-homomorphism $\varepsilon : S \to U$ such that $\overline{\varepsilon(g)} \in \bar{U}$ is transcendental over κ.*

If no simple sequence of (R,t,U) is obstructed, then no simple sequence of (R,t,S) is obstructed.

Proof. Let $f = (f_0,\dots,f_r)$ be a simple sequence of (R,t,S). Assuming that f is not transcendental, we show that it is extendable.

Note that $\bar{f}_0 \in \bar{S}$ is transcendental over κ; by assumption (∗), we may choose an R-homomorphism $\varepsilon : S \to U$ such that $\overline{\varepsilon(f_0)} \in \bar{U}$ is transcendental over κ. Define a sequence $e = (e_0,\dots,e_r)$ in U by $e_n = \varepsilon(f_n)$ and note that $\bar{e}_0 \in \bar{U}$ is transcendental over κ. We will show that e is a simple sequence of (R,t,U) and is extendable; then we will deduce that f is extendable.

For $n = 0,\dots,r$, define $A_n = R[f_1,\dots,f_n] \subseteq S$, $\bar{A}_n = \kappa[\bar{f}_1,\dots,\bar{f}_n] \subseteq \bar{S}$, $E_n = R[e_1,\dots,e_n] \subseteq U$ and $\bar{E}_n = \kappa[\bar{e}_1,\dots,\bar{e}_n] \subseteq \bar{U}$. Also, let C be the algebraic closure of $\bar{A}_0 = \kappa[\bar{f}_0]$ in $\bar{S}$; as we assumed that the simple sequence f is not transcendental,

[1] The proof of [DF01, Cor. 2.8] makes use of a result of Sathaye in the theory of generalized Newton-Puiseux expansions. Sathaye's result is quoted in [DF01] as Theorem 1.3 but there is a misprint in the statement: it should be "$f_0(0)$ does not belong to $\mathbf{k}$" in place of "$f_1(0)$ does not…"

$\bar f = (\bar f_0, \dots, \bar f_r)$ is a sequence in C. Also note that there is a unique R-homomorphism $\bar\varepsilon : \bar S \to \bar U$ such that $\bar\varepsilon \circ \pi = \pi' \circ \varepsilon$, where $S \xrightarrow{\pi} \bar S$ and $U \xrightarrow{\pi'} \bar U$ are the canonical epimorphisms. So for each $n = 0, \dots, r$, we have the commutative diagram:

$$\begin{tikzcd}
 & S \arrow[rr, two heads, "\pi"] \arrow[dl, "\varepsilon"'] & & \bar S \arrow[dl, "\bar\varepsilon"'] \\
U \arrow[rr, two heads, "\pi'"] & & \bar U & C \arrow[u, hook] \\
 & A_n \arrow[uu, hook] \arrow[rr, two heads] \arrow[dl] & & \bar A_n \arrow[u, hook] \arrow[dl] \\
E_n \arrow[uu, hook] \arrow[rr, two heads] & & \bar E_n \arrow[uu, hook] &
\end{tikzcd}$$

By commutativity of the top square, $\bar\varepsilon(\bar f_n) = \bar e_n$ for all n. We claim:

(5) $$\text{the composite } C \hookrightarrow \bar S \xrightarrow{\bar\varepsilon} \bar U \text{ is injective.}$$

Indeed, suppose that $0 \neq x \in C$ is an element of the kernel of this homomorphism. Let $h : \kappa[\bar f_0, x] \to \bar U$ be the composite $\kappa[\bar f_0, x] \hookrightarrow \bar S \xrightarrow{\bar\varepsilon} \bar U$; then $h(x) = 0$. As $\kappa[\bar f_0, x]$ has transcendence degree 1 over κ, its Krull dimension is 1 and $\ker h$ is a maximal ideal of $\kappa[\bar f_0, x]$. Consequently, the image of h is a finite extension of κ. This is impossible because $h(\bar f_0) = \bar e_0$ is transcendental over κ. So (5) is true.

Define $C' = \bar\varepsilon(C)$; by (5), $\bar\varepsilon$ restricts to an isomorphism $\gamma : C \to C'$ of R-algebras. Clearly, $\gamma(\bar f_n) = \bar e_n$ for all $n = 0, \dots r$.

Let n be such that $0 < n < r$. Let $\varphi_n \in A_{n-1}[T]$ be a monic polynomial satisfying: (i) $\bar\varphi_n \in \bar A_{n-1}[T]$ is the minimal polynomial of $\bar f_n$ over $\operatorname{Frac} \bar A_{n-1}$; and (ii) $f_{n+1} = \varphi_n(f_n)/t$. Define[2] $\psi_n = \varphi_n^{(\varepsilon)} \in E_{n-1}[T]$; then ψ_n is monic and

$$\psi_n(e_n) = \varphi_n^{(\varepsilon)}(\varepsilon(f_n)) = \varepsilon(\varphi_n(f_n)) = \varepsilon(t f_{n+1}) = t\varepsilon(f_{n+1}) = t e_{n+1}.$$

Moreover, $\bar\psi_n = \bar\varphi_n^{(\bar\varepsilon)}$. Since $\gamma : C \to C'$ is a restriction of $\bar\varepsilon$ and $\bar\varphi_n \in C[T]$, we may also write $\bar\psi_n = \bar\varphi_n^{(\gamma)}$. Since the isomorphism γ maps $\bar f_n$ on $\bar e_n$ and $\bar A_{n-1}$ on $\bar E_{n-1}$, we obtain that $\bar\psi_n$ is the minimal polynomial of $\bar e_n$ over $\operatorname{Frac} E_{n-1}$. Thus e is a simple sequence.

The isomorphism $\gamma : C \to C'$ maps $\bar A_0$ onto $\bar E_0$; as C is algebraic over $\bar A_0$, it follows that C' is algebraic over $\bar E_0$. In particular $\bar e_r$ is algebraic over $\bar E_0$, so the simple sequence e is not transcendental. As no simple sequence of (R, t, U) is obstructed, it follows that e is extendable. Consequently, the minimal polynomial of $\bar e_r$ over $\operatorname{Frac} \bar E_{r-1}$ has all its coefficients in $\bar E_{r-1}$; via γ, this implies that the minimal polynomial of $\bar f_r$ over $\operatorname{Frac} \bar A_{r-1}$ has all its coefficients in $\bar A_{r-1}$. In other words, f is extendable. $\square$

4.10. **Corollary.** *Let (R, t) be a simple pair and m a positive integer. Then no simple sequence of $(R, t, R^{[m]})$ is obstructed.*

[2]If $P = \sum_i a_i T^i \in A[T]$ is a polynomial ($a_i \in A$) and $h : A \to B$ is a ring homomorphism then define the polynomial $P^{(h)} \in B[T]$ by $P^{(h)} = \sum_i h(a_i) T^i$.

Proof. Let $S = R^{[m]}$ and $U = R^{[1]}$. By definition of simple pair, no simple sequence of (R, t, U) is obstructed. To prove the Corollary, it suffices to verify that (R, t, S) and (R, t, U) satisfy the condition $(*)$ of 4.9.

Let $g \in S$ be such that $\bar{g} \in \bar{S} = \kappa^{[m]}$ is transcendental over κ. Then we may choose $X, Y_1, \dots, Y_{m-1} \in S$ satisfying $S = R[X, Y_1, \dots, Y_{m-1}]$ and such that, if we regard g as a polynomial in X with coefficients in $R[Y_1, \dots, Y_{m-1}]$, then:

$$\text{The leading term of } g \text{ is } aX^N \text{ for some } N > 0 \text{ and } a \in R \setminus tR.$$

Write $U = R[X]$ and let $\varepsilon : S \to U$ be the R-homomorphism defined by $f(X, Y) \mapsto f(X, 0)$. Then $\overline{\varepsilon(g)} \in \bar{U} = \kappa[X]$ is transcendental over κ. □

4.11. **Lemma.** *Let (R, t, S) be a triple satisfying conditions (1) and (2) of 4.1 and let A be a ring such that $R \subseteq A \subseteq S$, $A \cap tS = tA$ and $\bigcap_{n=0}^{\infty} t^n A = \{0\}$. Then*

$$\operatorname{trdeg}_{\bar{A}}(\bar{S}) \le \operatorname{trdeg}_A(S).$$

Proof. Consider a family $(z_i)_{i \in I}$ of elements of S and the corresponding family $(\bar{z}_i)_{i \in I}$ of elements of $\bar{S}$. It suffices to show that if $(\bar{z}_i)_{i \in I}$ is algebraically independent over $\bar{A}$, then $(z_i)_{i \in I}$ is algebraically independent over A.

Actually we prove the contrapositive. If $(z_i)_{i \in I}$ is algebraically dependent over A then there exists a nonempty finite subset $\{i_1, \dots, i_n\}$ of I and a nonzero polynomial $P(T_1, \dots, T_n)$ with coefficients in A such that $P(z_{i_1}, \dots, z_{i_n}) = 0$. Since $\bigcap_{n=0}^{\infty} t^n A = \{0\}$, we may consider the largest $n \in \mathbb{N}$ such that t^n divides all coefficients of P. Replacing P by P/t^n, we arrange that some coefficient a of P is not in tA; then the element $\bar{a}$ of $\bar{A}$ is nonzero, because $A \cap tS = tA$. Thus $\bar{P} \in \bar{A}[T_1, \dots, T_n]$ is nonzero and satisfies $\bar{P}(\bar{z}_{i_1}, \dots, \bar{z}_{i_n}) = 0$, so $(\bar{z}_i)_{i \in I}$ is algebraically dependent over $\bar{A}$. □

The following is Lemma 2.3 of [DF01]. We defined the notation A^+ in 4.1.

4.12. **Lemma.** *Let (R, t, S) be as in 4.1 and let $f = (f_0, \dots, f_r)$ be a simple sequence of (R, t, S). Then the rings $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r$ (where $A_n = R[f_0, \dots, f_n]$) have the following properties:*

(1) *$A_0^+ = A_0$;*
(2) *for all n such that $0 < n < r$, $A_n^+ = A_{n+1}$;*
(3) *if f is of transcendental type, then $A_r^+ = A_r$.*

4.13. **Proposition.** *If R is a Dedekind domain containing $\mathbb{Q}$ then $R \in \mathrm{FG}(3)$.*

Proof. By Proposition 2.4, we may assume that R is a DVR containing $\mathbb{Q}$. Let t be a uniformizing parameter of R then, by 4.7, (R, t) is a simple pair. Let $B = R^{[3]}$ and $D : B \to B$ a locally nilpotent R-derivation. We have to show that $\ker D$ is finitely generated as an R-algebra. We may assume that $D \neq 0$. Consider the localization $D_t : B_t \to B_t$ of D at $\{1, t, t^2, \dots\}$. As $B_t = R_t^{[3]}$ where R_t is a field of characteristic zero, $\ker D_t = R_t[F, G] = R_t^{[2]}$ for some $F, G \in B_t$ (by [Miy85]). In fact we may arrange that $F, G \in B$ and that the element $\bar{F}$ of $\bar{B} = \kappa^{[3]}$ is transcendental over κ (where

$\kappa = R/tR$ as usual). Then we set $f_0 = F$ and $f_1 = G$ and we note that (f_0, f_1) is a simple sequence of (R, t, B). Moreover,

$$R[f_0, f_1] \subseteq \ker D \quad \text{and} \quad R[f_0, f_1]_t = (\ker D)_t.$$

Let E be the set of simple sequences $f' = (f'_0, \dots, f'_r)$ of (R, t, B) satisfying $f'_0 = f_0$ and $f'_1 = f_1$. Note that E is nonempty, since $(f_0, f_1) \in E$.

Consider $(f_0, \dots, f_r) \in E$ and set $A_n = R[f_0, \dots, f_n]$ for $n = 0, \dots, r$. By Lemma 4.12, $A_n^+ = A_{n+1}$ for all n such that $0 < n < r$; moreover, we have $A_1 \subseteq \ker D$ and it is clear that $A_n \subseteq \ker D$ implies $A_n^+ \subseteq \ker D$; hence $A_r \subseteq \ker D$. To summarize,

$$\text{if } (f_0, \dots, f_r) \in E \text{ then } R[f_0, \dots, f_r] \subseteq \ker D \subseteq R[f_0, \dots, f_r]_t. \tag{6}$$

We claim that some element of E is a transcendental simple sequence. Indeed, suppose the contrary; then, by 4.10, every element of E is extendable. By the remark following 4.2, there exists an infinite sequence $(f_0, f_1, f_2, \dots)$ such that $(f_0, \dots, f_n) \in E$ for each $n \geq 1$. Let $A_i = R[f_0, \dots, f_i]$ and $A = \bigcup_{i=1}^{\infty} A_i$. Then $A \subseteq \ker D$ by (6). If a is any element of $\ker D$ then $a \in R[f_0, f_1]_t$, so $t^n a \in R[f_0, f_1]$ for some n; now if $t^m a \in A_i$ where $i \geq 1$ and $m > 0$ then $t^{m-1} a \in B \cap t^{-1} A_i \subseteq A_i^+ = A_{i+1}$ (where we used Lemma 4.12) and by induction we get $a \in A$. Thus $A = \ker D$. In particular A is factorially closed in B so $A \cap tB = tA$ and $\bigcap_{n=0}^{\infty} t^n A = \{0\}$ (the last claim follows from $\bigcap_{n=0}^{\infty} t^n B = \{0\}$, which follows from $B = R^{[3]}$ and $\bigcap_{n=0}^{\infty} t^n R = \{0\}$). Then 4.11 implies that $\operatorname{trdeg}_{\bar{A}}(\bar{B}) \leq \operatorname{trdeg}_A(B) = 1$, so $\operatorname{trdeg}_\kappa(\bar{A}) > 1$. This is absurd because $\bar{A} = \kappa[\bar{f}_0, \bar{f}_1, \bar{f}_2, \dots]$ and $\bar{f}_i$ is algebraic over $\kappa[\bar{f}_0]$ for each $i \geq 1$.

This contradiction shows that some element of E is a transcendental simple sequence of (R, t, B). Let $f = (f_0, \dots, f_r) \in E$ be such an element. Let $A = R[f_0, \dots, f_r]$, then $A \subseteq \ker D \subseteq A_t$ by (6). Since f is transcendental, we have $A^+ = A$ by 4.12, so $A \cap tB = tA$. We conclude that $\ker D = A$, so $\ker D$ is a finitely generated R-algebra. □

In order to obtain a converse of 4.13, we show:

4.14. **Lemma.** *Let R be a noetherian domain containing $\mathbb{Q}$. If $\dim(R) > 1$ then $R \notin \mathrm{FG}(3)$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R of height 2. To prove the result it is enough to show that $R_\mathfrak{p} \notin \mathrm{FG}(3)$ (cf. 2.1). So we assume that $\dim(R) = 2$ and that R is local. By 3.2, we also assume that R is normal. So R is a Cohen-Macaulay ring.

Let $\mathfrak{m}$ be the maximal ideal of R and let $\kappa = R/\mathfrak{m}$ be the residue field of R. Let $a, b \in \mathfrak{m}$ such that the ideal (a, b) has height 2. Let $D : R[X,Y,Z] \to R[X,Y,Z]$ be the locally nilpotent R-derivation given by

$$D(Z) = Y, \quad D(Y) = aX + b, \quad D(X) = a^2. \tag{7}$$

We claim:

$\ker(D)$ is not finitely generated as an R-algebra.

To prove this, consider the completion $\hat{R}$ of R with respect to $\mathfrak{m}$. Note that $\hat{R}$ contains κ as a coefficient field. Since R is Cohen-Macaulay and $\mathrm{ht}(a,b) = 2$, a, b is a regular R-sequence and hence a regular $\hat{R}$-sequence as $\hat{R}$ is (faithfully) flat over R. Therefore, since $\dim(R) = \dim(\hat{R}) = 2$, $\hat{R}$ is a Cohen-Macaulay ring of depth 2 and a, b is a system of parameters of $\hat{R}$. It follows that a, b are analytically independent over κ ([Mat89], Theorem 14.5, p. 107). Hence the κ-subalgebra $\kappa[[a,b]]$ of $\hat{R}$ is a complete regular local ring of dimension 2.

We denote by I the maximal ideal of $\kappa[[a,b]]$. Since $\dim(\hat{R}) = 2$ and $I\hat{R}$ is an ideal of $\hat{R}$ of height 2, we get that $\hat{R}/I\hat{R}$ is an artinian local ring and hence finite dimensional vector space over κ. We regard $\hat{R}$ as a module over $\kappa[[a,b]]$. It is obvious that $\hat{R}$ is separated in I-adic topology. Since $\kappa[[a,b]]$ is complete local (with respect to I-adic topology) and $\hat{R}/I\hat{R}$ is finite dimensional over $\kappa = \kappa[[a,b]]/I$, it follows that $\hat{R}$ is a finite $\kappa[[a,b]]$-module ([Mat80], Lemma, p. 212). Since $\kappa[[a,b]]$ is regular local and $\hat{R}$ is finite over $\kappa[[a,b]]$, by Auslander-Buchsbaum result (see [BH98], Theorem 1.3.3, p. 17),

$$\mathrm{projdim}(\hat{R}) + \mathrm{depth}(\hat{R}) = \mathrm{depth}(\kappa[[a,b]]) = 2.$$

Since a, b is a regular sequence for $\hat{R}$, depth of $\hat{R}$ as a $\kappa[[a,b]]$-module is 2. Therefore $\mathrm{projdim}(\hat{R}) = 0$. Thus $\hat{R}$ is a free $\kappa[[a,b]]$-module (of finite rank).

The locally nilpotent R-derivation D of $R[X,Y,Z]$ naturally extends to a locally nilpotent $\hat{R}$-derivation $\hat{D}$ of $\hat{R}[X,Y,Z]$, and $\hat{D}$ maps $\kappa[[a,b]][X,Y,Z]$ into itself as is clear from (7). The restriction $D_1 : \kappa[[a,b]][X,Y,Z] \to \kappa[[a,b]][X,Y,Z]$ of $\hat{D}$ is a locally nilpotent $\kappa[[a,b]]$-derivation. As $\hat{R}$ is faithfully flat over each of R and $\kappa[[a,b]]$, we have

$$\hat{R} \otimes_R \ker(D) = \ker(\hat{D}) = \hat{R} \otimes_{\kappa[[a,b]]} \ker(D_1)$$

and

(8) $\ker(D)$ *is finitely generated as an* R*-algebra* $\iff$ $\ker(D_1)$ *is finitely generated as a* $\kappa[[a,b]]$*-algebra.*

Let $B = \kappa[T_1, T_2]$ be a polynomial algebra in two variables over κ and let $d : B[X,Y,Z] \to B[X,Y,Z]$ be the locally nilpotent B-derivation given by

$$d(Z) = Y, \quad d(Y) = T_1 X + T_2, \quad d(X) = {T_1}^2. \tag{9}$$

Then, by Theorem 3.3 of [DF99], $\ker(d)$ is not finitely generated over B. However, it can be proved that $\ker(d)[1/T_i]$ is finitely generated over B for each $i = 1, 2$. These two facts together with 2.2 imply that the S-algebra $A' = \ker(d) \otimes_B S$ is not finitely generated over S, where $S = B_M$ is the local ring of B at the maximal ideal $M = (T_1, T_2)$. Note that A' is the kernel of the induced derivation $d' : S[X,Y,Z] \to S[X,Y,Z]$.

Let $\alpha : S \to \kappa[[a,b]]$ be the κ-algebra homomorphism given by $\alpha(T_1) = a, \alpha(T_2) = b$. Through α we regard $\kappa[[a,b]]$ as an S-algebra. Since a, b are analytically independent over κ, it is obvious that $\kappa[[a,b]]$ is faithfully flat over S. Hence, as A' is not finitely generated over S, $A'' = A' \otimes_S \kappa[[a,b]]$ is not finitely generated over $\kappa[[a,b]]$.

Since $A' = \ker(d')$ and $\kappa[[a, b]]$ is flat over S, A'' is the kernel of the extension $d'' : \kappa[[a, b]][X, Y, Z] \to \kappa[[a, b]][X, Y, Z]$ of d'. Comparing (7) and (9) we see that $d'' = D_1$, so the $\kappa[[a, b]]$-algebra $\ker(D_1)$ is not finitely generated. It follows from (8) that $\ker(D)$ is not finitely generated over R, so the proof is complete. □

4.15. **Theorem.** *For a noetherian domain R containing $\mathbb{Q}$,*

$$R \in \mathrm{FG}(3) \iff R \text{ is a Dedekind domain or a field.}$$

Proof. If $R \in \mathrm{FG}(3)$ then $\dim(R) \leq 1$ by 4.14 and R is normal by 3.2, so R is a Dedekind domain or a field. The converse is 4.13. □

4.16. **Corollary.** *Let R be a noetherian domain containing $\mathbb{Q}$. If R is not a field then $R \notin \mathrm{FG}(4)$.*

Proof. If R is not a field then $\dim(R^{[1]}) > 1$, so $R^{[1]} \notin \mathrm{FG}(3)$ by 4.14 or 4.15, so $R \notin \mathrm{FG}(4)$. □

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