Generalized epimorphism theorem

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Abstract. Let $R[X, Y]$ be a polynomial ring in two variables over a commutative ring $R$ and let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ (a polynomial ring in one variable). In this set-up we prove that $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$ if either $R$ contains a field of characteristic zero or $R$ is a seminormal domain of characteristic zero.

Keywords. Epimorphism theorem; polynomial ring; seminormal domain; characteristic zero.

1. Introduction

Let $k$ be a field of characteristic zero. Let $k[X, Y]$ be a polynomial ring in two variables over $k$ and $F \in k[X, Y]$ such that $k[X, Y]/(F) = k[Z]$ (a polynomial ring in one variable). In this set-up the famous epimorphism theorem of Abhyankar and Moh ([2], Theorem 1.2) says that $k[X, Y] = k[F, G]$ for some $G \in k[X, Y]$. Russell and Sathaye had obtained the following analog of the epimorphism theorem ([6], Theorem 2.6.2): If $R$ is a locally factorial Krull domain of characteristic zero and $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$, then $R[X, Y] = R[F, G]$. Therefore one asks the following natural question:

Is the foregoing result valid for an arbitrary commutative domain $R$ of characteristic zero?

In this paper we answer this question affirmatively under the assumption that $R$ is seminormal. We prove:

Theorem A. Let $R$ be a seminormal commutative domain of characteristic zero. Let $I$ be an ideal of $R[X, Y]$ such that $R[X, Y]/I = R[Z]$. Then $I$ is a principal ideal say generated by $F$ and $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$.

Moreover we give an example (Example 3.8) to show that $I$ need not be principal if $R$ is not seminormal. When $R$ contains a field of characteristic zero we prove the following (weaker) epimorphism theorem:

Theorem B. Let $R$ be a commutative ring containing a field of characteristic zero. Let $F \in R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$. Then $R[X, Y] = R[F, G]$ for some $G \in R[X, Y]$.
2. Preliminaries

Throughout this paper all rings will be commutative.

In this section we set up notations and state some results for later use.

$R$ will denote a commutative ring.

$R^{(n)}$, polynomial ring in $n$ variables over $R$.

$R^n$: free $R$-module of rank $n$.

For a finitely generated $R$-algebra $A$,

$\Omega_{A/R}$: universal module of $R$-differentials of $A$.

For a prime ideal $\mathfrak{p}$ of $R$,

$k(\mathfrak{p})$: $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

DEFINITION

A reduced ring $R$ is said to be seminormal if it satisfies the condition: for $b, c \in R$ with $b^3 = c^2$, there is an $a \in R$ with $a^2 = b, a^3 = c$.

Lemma (2.1). Let $R$ be a noetherian ring and let $s \in R$ be a non-zero divisor. Let $M$ be a finitely generated $R$-module. If $M_s$ is a projective $R_s$-module of rank $d$ and $M/sM$ is $R/sR$-projective of rank $d$ then $M$ is $R$-projective of rank $d$.

Proof. Without loss of generality we can assume that $R$ is local.

Since $M/sM$ is $R/s$-projective and $R$ is local there exists a surjective $R$-linear map $\beta: R^d \to M$ ($d = \text{rank } M/sM$). Let $N = \ker \beta$. Since $M_s$ is $R_s$-projective of rank $d$ and $\beta$ is surjective we get $N_s = 0$. But $s$ is a non-zero-divisor of $R$ and $N \subseteq R^d$, therefore $N_s = 0 = N = 0$ and $\beta$ is an isomorphism.

Lemma (2.2). Let $R$ be a noetherian ring and $I$ be an ideal of $R^{(n)}$ such that $R^{(n)}/I \cong R^{(n-1)}$ as $R$-algebras. Then for an ideal $\mathfrak{G}$ of $R$, $I \cap \mathfrak{G}R^{(n)} = \mathfrak{G}I$. Moreover if $I$ is a principal ideal of $R^{(n)}$ say generated by $F$, then

(i) $F$ is a non-zero-divisor of $R^{(n)}$.

(ii) $F$ is a algebraically independent over $R$, i.e. $R[F] \cong R^{(1)}$.

(iii) $R[F] \cap \mathfrak{G}R^{(n)} = \mathfrak{G}R[F]$ for any ideal $\mathfrak{G}$ of $R$.

Proof. Since for any non-negative integer $l$, $R^{(l)}$ is a free $R$-module, the exact sequence

$0 \to I \to R^{(n-l)} \to R^{(n-l)} \to 0$; $x: R$-algebra homomorphism of $R$-modules gives rise to the exact sequence

$0 \to I \otimes_R \mathfrak{G} \to R^{(n)} \otimes_R \mathfrak{G} \otimes_{R^{(l)}} R^{(n-l)} \otimes_R \mathfrak{G} \to 0$

proving that the canonical map $I/\mathfrak{G}I \to I + \mathfrak{G}R^{(n-l)}/\mathfrak{G}R^{(n)}$ of $R/\mathfrak{G}R^{(n)}$-modules is an isomorphism. Hence $I \cap \mathfrak{G}R^{(n)} = \mathfrak{G}I$.

Now we assume that $I = (F)$.

(i) It is easy to see that $F \notin sR^{(n)}$ for any maximal ideal $s$ of $R$. This shows that $F$ is a non-zero-divisor of $R^{(n)}$.

(ii) Suppose $c_0 + c_1 F + \cdots + c_r F^r = 0$ where $c_i \in R \forall i$, $0 \leq i \leq r$. Then $0 = \alpha(c_0 + c_1 F + \cdots + c_r F^r) = c_0 \text{ i.e. } F(c_1 + c_2 F + \cdots + c_r F^{r-1}) = 0$. Therefore, as by (i) $F$ is a non-zero-
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divisor, \( c_1 + c_2 F + \cdots + c_r F^{r-1} = 0 \) showing that \( c_1 = 0 \). Repeating this argument we see that \( c_1 = 0 \forall i, 0 \leq i \leq r \).

(iii) Let \( \overline{F} \) be the image of \( F \) in \( R/\mathfrak{m}[F] \), then obviously \( R/\mathfrak{m}[F] \approx R/\mathfrak{m}^{r-1} \). Therefore by (ii) \( \overline{F} \) is algebraically independent over \( R/\mathfrak{m} \) and hence \( R[F] \cap \mathfrak{m}[F] = \mathfrak{m}[F] \).

Lemma (2.3). Let \( R \) be a noetherian ring and let \( \mathfrak{m} \) be the nilradical of \( R \). Let \( I \) be an ideal of \( R[\mathfrak{m}] \) such that \( R[\mathfrak{m}]/I \approx R^{[n-1]} \) as \( R \)-algebras. If \( I/\mathfrak{m}I \) is a projective \( R[\mathfrak{m}]-\)module of (constant) rank \( 1 \) then \( I \) is a projective \( R[\mathfrak{m}]-\)module of (constant) rank \( 1 \).

Proof. Since \( \mathfrak{m} \) is nilpotent, the canonical map \( \text{Pic}(R[\mathfrak{m}]) \to \text{Pic}(R/\mathfrak{m}[\mathfrak{m}]) \) is an isomorphism. Therefore there exists a projective \( R[\mathfrak{m}]-\)module \( L \) of constant rank \( 1 \) such that \( L/\mathfrak{m}L \approx I/\mathfrak{m}I \). Hence there exists a \( R[\mathfrak{m}]-\)linear map \( \psi: L \to I \) such that the induced map \( \tilde{\psi}: L/\mathfrak{m}L \to I/\mathfrak{m}I \) is an isomorphism.

We claim that \( \psi \) is an isomorphism.

Surjectivity of \( \psi \): Since \( \tilde{\psi} \) is an isomorphism, we have \( I = \psi(L) + \mathfrak{m}I \). But \( \mathfrak{m} \) is nilpotent and hence \( I = \psi(L) \).

Injectivity of \( \psi \): Let \( M = \ker \psi \). Then we get the following exact sequence of \( R[\mathfrak{m}]-\)modules:

\[ 0 \to M \to L \to I \to 0. \]

As in Lemma 2.2 we see that \( I \) is a projective \( R \)-module. Therefore the above exact sequence gives rise to the following exact sequence:

\[ 0 \to M/\mathfrak{m}M \to L/\mathfrak{m}L \to I/\mathfrak{m}I \to 0. \]

But \( \tilde{\psi} \) is an isomorphism. Therefore \( M/\mathfrak{m}M = 0 \) i.e. \( M = \mathfrak{m}M \). The nilpotency of \( \mathfrak{m} \) shows that \( M = 0 \).

Thus \( \psi \) is an isomorphism.

3. Main theorems

In this section we prove Theorem A and Theorem B which are quoted in the introduction. For the proof of these theorems we need some lemmas and a proposition. Lemma 3.1 is well known but for the lack of a proper reference we give a proof.

Lemma 3.1. Let \( R \) be a noetherian ring and \( S \) be a noetherian \( R \)-algebra. Let \( \pi \in R \) be such that \( S_\pi \) is a flat \( R_\pi \)-algebra and \( S/\pi S \) is a flat \( R/\pi R \)-algebra. Moreover assume that \( \text{Tor}_1(S, R/\pi R) = 0 \). Then \( S \) is a flat \( R \)-algebra.

Proof. Let \( M \) and \( N \) be finitely generated \( R \)-modules and let \( f: M \to N \) be a \( R \)-linear injective map. Then we want to show that the map \( f \otimes 1_S: M \otimes_R S \to N \otimes_R S \) is injective. Let \( K = \ker( f \otimes 1_S) \).
Since $S_x$ is $R_x$-flat we have $K_x = 0$. Let $T = 1 + \pi R$ and $T' = 1 + \pi S$. Then since $\text{Tor}_1^R(S_T, R_T/\pi R_T) = \text{Tor}_1^R(S, R/\pi R) \otimes_R S_T = 0$ and $R_T/\pi R_T = R/\pi R$, $S_T/\pi S_T = S/\pi S$, by ([1], Theorem 3.2, p. 91) $S_T$ is flat over $R_T$ and hence $K_T = 0$.

Thus $K_T = 0$, $K_x = 0$. Therefore $K = 0$ showing that $S$ is a flat $R$-algebra.

Lemma 3.2. Let $R$ be a noetherian ring of finite Krull dimension (denoted by $\dim R$). Let $F$ be an element of $R[X, Y]$ such that $R[X, Y]/(F) = R[Z]$ as $R$-algebras. Then $R[X, Y]$ is a flat $R[F]$-algebra.

Proof. Let $6$ be the nilradical of $R$. Since $R[F]$ ($F$ being algebraically independent over $R$) and $R[X, Y]$ are flat over $R$, for every module $M$ over $R$ we have $\text{Tor}_1^R(R[X, Y], M \otimes_R R[F]) = 0$. In particular for every ideal $J$ of $R$ we have $\text{Tor}_1^R(JR[F], R[F]/JR[F]) = 0$. Therefore, since $6$ is nilpotent, by ([1], Theorem 3.2, p. 91) $R[X, Y]$ is flat over $R[F]$ if $R[X, Y]/\pi R[X, Y] (= R/\pi R[X, Y])$ is flat over $R[F]/\pi R[F]$. So it is enough to prove the result when $R$ is a reduced ring.

We prove the result by induction on $\dim R$. Without loss of generality we can assume that $R$ is local.

If $\dim R = 0$ then $R$ is a field, $R[F]$ is a principal ideal domain and $R[X, Y]$ is a domain. Therefore $R[X, Y]$ is a flat $R[F]$-algebra.

Now we assume that $\dim R > 0$. Let $\pi R$ be a nonunit non-zero-divisor of $R$. Let $\bar{F}$ denote the image of $F$ in $R/\pi R[X, Y]$. Then $R/(\pi \bar{F}) \cong R[F]/\pi R[F]$. Since $\dim R/(\pi \bar{F}) < \dim R$ and $\dim R_x < \dim R$, by the induction hypothesis $R_x[X, Y]$ is flat over $R_\pi [F]$ and $R[X, Y]/\pi R[X, Y]$ is flat over $R[F]/\pi R[F]$. Moreover $\text{Tor}_1^R(R[X, Y], R[F]/\pi R[F]) = 0$. Therefore by Lemma 3.1 $R[X, Y]$ is a flat $R[F]$-algebra.

Thus the proof of Lemma 3.2 is complete.

We state a definition before stating the next lemma.

DEFINITION
An element $F$ of $R[X, Y]$ is called a residual variable if for every prime ideal $\mathfrak{y}$ of $R$, $k(\mathfrak{y}) [X, Y] = k(\mathfrak{y}) [\bar{F}]^\dagger$ where $\bar{F}$ denotes the image of $F$ in $k(\mathfrak{y}) [X, Y]$.

Lemma 3.3. Let $R$ be a ring and $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as $R$-algebras. Assume that $F$ is a residual variable. Then for every prime ideal $\mathfrak{y}$ of $R[F]$, $k(\mathfrak{y}) \otimes_{R[F]} R[X, Y] = k(\mathfrak{y})^\dagger$.

Proof. Let $\mathfrak{y} \cap R = \mathfrak{y}$. Then $R[R[F]] \subset R[\mathfrak{y}]$ and by (2.2) $R[F] = R[X, Y] \cap R[F]$. Since $F$ is a residual variable, we have $k(\mathfrak{y}) [X, Y] = k(\mathfrak{y})[\bar{F}]^\dagger$ where $\bar{F}$ denotes the image of $F$ in $k(\mathfrak{y}) [X, Y]$. Moreover $k(\mathfrak{y}) [\bar{F}] \cong k(\mathfrak{y}) \otimes_{R[F]} R[F]$ and there exists a $R[F]$-algebra homomorphism $k(\mathfrak{y})[\bar{F}] \rightarrow k(\mathfrak{y})$. Therefore

$$k(\mathfrak{y}) \otimes_{R[F]} R[X, Y] = k(\mathfrak{y}) \otimes_{k(\mathfrak{y})[\bar{F}]} k(\mathfrak{y}) \otimes_{R[F]} R[X, Y] = k(\mathfrak{y})^\dagger.$$
Proof. We have the following right exact sequence of $R[X, Y]$-modules

$$\Omega_{R,F/R} \otimes_{R[F]} R[X, Y] \rightarrow \Omega_{R[X,Y]/R} \rightarrow \Omega_{R[X,Y]/R[F]} \rightarrow 0.$$ 

Since $\Omega_{R[X,Y]/R}$ is a free $R[X, Y]$-module of rank two with a basis $dX, dY$ and $\text{Im}(\theta) = N$ is the cyclic submodule generated by $F_X dX + F_Y dY$ where $F_X = \partial F/\partial X$ and $F_Y = \partial F/\partial Y$, it is enough to show that the ideal $(F_X, F_Y) = R[X, Y]$.

Suppose $\mathfrak{M}$ is a maximal ideal of $R[X, Y]$ such that $(F_X, F_Y) \subset \mathfrak{M}$. Let $\mathfrak{M} \cap R = \mathfrak{m}$. Then replacing $R$ by $R_{\mathfrak{m}}$ and $\mathfrak{M}$ by $\mathfrak{M}_{\mathfrak{m}}$ we can assume that $R$ is a local ring with the maximal ideal $\mathfrak{m}$, $\mathfrak{M}$ is a maximal ideal of $R[X, Y]$ with $\mathfrak{M} \cap R = \mathfrak{m}$ and $(F_X, F_Y) \subset \mathfrak{M}$. But then, since $F$ is a residual variable, we have $R[X, Y] = (F_X, F_Y) + \mathfrak{M} R[X, Y] \subset \mathfrak{M}$ which is absurd. Hence $(F_X, F_Y) = R[X, Y]$.

Lemma 3.5. Let $R$ be a noetherian ring such that no prime integer is a zero-divisor in $R$. Let $F \in R[X, Y]$ be such that $R[X, Y]/(F) = R[Z]$ as $R$-algebras. Then $F$ is a residual variable.

Proof. Let $\mathfrak{M}$ be a prime ideal of $R$ and let $\tilde{F}$ denote the image of $F$ in $k(\mathfrak{M})[X, Y]$. Then $k(\mathfrak{M})[X, Y]/(\tilde{F}) = k(\mathfrak{M})[Z]$.

If $ht \mathfrak{M} = 0$, then since no prime integer is a zero-divisor in $R$, $k(\mathfrak{M})$ is a field of characteristic zero. Therefore by the Abhyankar-Moh epimorphism theorem ([2], Theorem 1.2) $k(\mathfrak{M})[X, Y] = k(\mathfrak{M})[\tilde{F}]^{[1]}$.

If $ht \mathfrak{M} > 0$ then there exists a discrete valuation ring $V$ of characteristic zero with the uniformizing parameter $\pi$ and a ring homomorphism $\alpha: R \rightarrow V$ such that $\alpha^{-1}(\mathfrak{m}) = \mathfrak{M}$ and the field extension $k(\mathfrak{M}) \rightarrow V/(\pi)$ (induced by $\alpha$) is algebraic.

Let $\tilde{F}$ denote the image (through $\alpha$) of $F$ in $V[X, Y]$. Then $V[X,Y]/(\tilde{F}) = V[Z]$. Therefore by ([6], Theorem 2.6.2) $V[X, Y] = V[\tilde{F}]^{[1]}$ and hence $V/(\pi)[X, Y] = V/(\pi)[\tilde{F}]^{[1]}$ where $\tilde{F}$ is the image of $F$ in $V/(\pi)[X, Y]$.

Since we have the following commutative diagram of rings

$$\begin{array}{ccc}
R & \rightarrow & V \\
\downarrow & & \downarrow \\
k(\mathfrak{M}) & \rightarrow & V/(\pi)
\end{array}$$

and $V/(\pi)$ is algebraic over $k(\mathfrak{M})$, by ([4], Proposition 1.16) $k(\mathfrak{M})[X, Y] = k(\mathfrak{M})[\tilde{F}]^{[1]}$.

Thus we prove that $F$ is a residual variable.

Proposition 3.6. Let $R$ be a ring and $I$ be an ideal of $R^{[s]}$ such that $R^{[s]}/I \approx R^{[s]-1}$ as $R$-algebras. Then $I$ is a projective $R^{[s]}$-module of (constant) rank 1. Moreover if there exists a projective $R$-module $L$ of rank 1 such that $L \otimes_R R^{[s]} = I$ as $R^{[s]}$-modules then $I$ is a free $R^{[s]}$-module of rank 1 i.e. $I$ is a principal ideal (necessarily generated by a non-zero-divisor of $R^{[s]}$).

Proof. It is easy to see that under the hypothesis of the proposition there exists a subring $R'$ of $R$ which is finitely generated over the ring of integers and an ideal $I'$ of $R^{[s]}$ such that $R^{[s]}/I' \approx R^{[s]-1}$ and $I = I' R^{[s]} \approx I' \otimes_R R = I' \otimes_R R^{[s]}$. Therefore for proving
the first part of the proposition we can assume without loss of generality that $R$ is
noetherian of finite Krull dimension.

We prove the result by induction on $\dim R$.

Let $\dim R = 0$. By Lemma 2.3 we can assume that $R$ is reduced. But then $R$ is a finite
product of fields and hence, since $R^{[0]/I} \cong R^{[n-1]}$, $I$ is a principal ideal (of height 1)
generated by a non-zero-divisor. Therefore $I$ is a free $R^{[0]}$-module of rank 1.

Now we assume that $\dim R > 0$. Again by Lemma 2.3 we can assume that $R$ is
reduced. Let $S$ be the set of non-zero-divisors of $R$. Then $R_S$ is a finite product of fields
and as before we conclude that $I_S$ is a free $R^{[0]}_S$-module of rank 1. Therefore there $s \in S$ such
that $I_s$ is a free $R^{[0]}_s$-module of rank 1. We may assume that $s$ is a nonunit of $R$.

Since $I \cap sR^{[0]} = Is$, $I/sI \cong I + sR^{[0]}/sR^{[0]}$ as $R/(s)^{[n]}$-modules. Therefore, since
$R^{[0]}/(I + sR^{[0]}) \cong R/(s)^{[n-1]}$ and $\dim R/(s) < \dim R$, by the induction hypothesis
$I/sI$ is a projective $R/(s)^{[n]}$-module. Since $s$ is a non-zero-divisor of $R$, $I \subset R^{[0]}$ and $I_s$
(resp. $I/sI$) is a projective $R^{[0]}_s$-module (resp. $R/(s)^{[n]}$-module) of rank 1, by Lemma 2.1
$I$ is a projective $R^{[0]}$-module of (constant) rank 1.

Now assume that there exists a projective $R$-module $L$ of rank 1 such that $L \otimes_R R^{[0]} \cong I$ as $R^{[0]}$-modules.

Since $R^{[0]}/I \cong R^{[n-1]}$ as $R$-algebras, we get the following right exact sequence of
$R^{[n-1]}$-modules:

$I/I^2 \to \Omega^{[n-1]}_R/R^{[n-1]} \to \Omega^{[n]}_R \to 0$.

Since, for non-negative integer $l$, $\Omega^{[n]}_R$ is a free $R^{[l]}$-module of rank 1 and $I/I^2$ is a
projective $R^{[n-1]}$-module (as $I$ is projective over $R^{[0]}$ of rank 1) we see that the
above sequence is also left exact and

$\Omega^{[n]}_R/R^{[n]} \cong \Omega^{[n-1]}_R \oplus I/I^2$.

Thus $I/I^2$ is a stably free $R^{[n-1]}$-module of rank 1 and therefore $I/I^2$ is free over $R^{[n-1]}$ of
rank 1.

Let $\delta: R^{[n-1]} \to R$ be a surjective $R$-algebra homomorphism. Then composite map

$R \to R^{[0]} \to R^{[0]}/I \cong R^{[n-1]} \xrightarrow{\delta} R$

is the identity automorphism of $R$.

Since $L \otimes_R R^{[0]} \cong I$, we get

$L = L \otimes_R R^{[0]} \otimes_R R^{[n-1]} \otimes_R R \cong I \otimes_R R^{[n-1]} \otimes_R R$

$= I/I^2 \otimes_R R$.

But $I/I^2$ is a free $R^{[n-1]}$-module of rank 1. Hence $L$ is a free $R$-module of rank 1 and
therefore $I$ is a free $R^{[0]}$-module of rank 1 i.e. $I$ is a principal ideal.

Thus the proof of Proposition 3.6 is complete.

Now we prove Theorem A.

**Theorem 3.7.** Let $R$ be a ring such that $R_{red}$ is seminormal and no prime integer is a zero-
diviisor in $R_{red}$. Let $I$ be an ideal of $R[X, Y]$ such that $R[X, Y]/I = R[Z]$ (as $R$-algebras).
Then $I$ is a principal ideal say generated by $F$ and $R[X, Y] = R[F]^{[1]}$. 
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Proof. Since \( R_{red} \) is seminormal by ([7], Theorem 6.1) \( \text{Pic}(R) = \text{Pic}(R^{[m]}) \) for every \( m \). Therefore by Proposition 3.6 \( I \) is a principal ideal say generated by \( F \).

Let \( \mathfrak{R} \) be the nilradical of \( R \) and let \( \bar{F} \) be the image of \( F \) in \( R/\mathfrak{R}[X, Y] \). If \( R/\mathfrak{R}[X, Y] = R/\mathfrak{R}[[X, Y]]^{[1]} \) then it is easy to see that \( R[X, Y] = R[[X, Y]]^{[1]} \). Therefore we can assume that \( R \) is reduced. It is also easy to see that there exists a subring \( S \) of \( R \) which is finitely generated over the ring of integers such that \( F \in S[X, Y] \) and \( S[X, Y]/(F) = S[Z] \) as \( S \)-algebras. Note that \( S \) is a noetherian ring of finite Krull dimension.

Since \( S \subset R \) and \( R \) is reduced, by the hypothesis of the theorem, no prime integer is a zero-divisor in \( S \). Therefore \( F \) is a residual variable in \( S[X, Y] \) by Lemma 3.5. Hence \( \Omega_{S[X, Y]/S[F]} \) is a free \( S[X, Y] \)-module of rank one by Lemma 3.4. Moreover by Lemma 3.3, for every prime ideal \( \mathfrak{p} \) of \( S[F] \), \( k(\mathfrak{p}) \otimes_{S[F]} S[X, Y] = k(\mathfrak{p})^{[1]} \). \( S[X, Y] \) is a (finitely generated) flat \( S[F] \)-algebra by Lemma 3.2. Therefore by ([3], Lemma 3.3) there exists a positive integer \( m \) such that \( S[X, Y]^{[m]} = S[F]^{[m+1]} \).

Now \( S[X, Y]^{[m]} = S[F]^{[m+1]} \) implies that \( R[X, Y]^{[m]} = R[F]^{[m+1]} \). Since \( R \) is seminormal (we have assumed \( R \) to be reduced) by ([5], Theorem 2.6) \( R[X, Y] = R[F]^{[1]} \).

Thus the proof of Theorem 3.7 is complete.

The following example shows that if \( R_{red} \) is not seminormal then \( R[X, Y]/I = R[Z] \) need not imply that \( I \) is principal.

Example 3.8. Let \( k \) be a field of characteristic zero and let \( \bar{R} = k[[r]] \); a power series in one variable over \( k \). Let \( R = k[[r^2, r^3]] \), considered as a subring of \( \bar{R} \). It is obvious that \( \bar{R} \) is the normalization of \( R \) and \( R \) is not seminormal.

Let \( \alpha: R[X, Y] \to R[Z] \) be the \( R \)-algebra homomorphism defined as: \( \alpha(X) = Z + r^2Z^2 \) and \( \alpha(Y) = r^2Z \). Let \( I = \text{ker} \, \alpha \). Then

1. \( \alpha \) is surjective
2. \( I \) is not a principal ideal of \( R[X, Y] \).

Proof. Since \( \alpha(X - r^2X^2 + r^2XY^2 + Y^2) = Z \), \( \alpha \) is surjective.

Let \( \tilde{\alpha}: \bar{R}[X, Y] \to \bar{R}[Z] \) be the \( \bar{R} \)-algebra homomorphism such that \( \tilde{\alpha}(X) = \alpha(X) = Z + r^2Z^2 \) and \( \tilde{\alpha}(Y) = \alpha(Y) = r^2Z \). Let \( \mathfrak{I} = \text{ker} \, \tilde{\alpha} \). Then \( \mathfrak{I} \) is a principal prime ideal of \( \bar{R}[X, Y] \) generated by \( F(X, Y) = r^2X - Y - rY^2 \). Moreover \( \mathfrak{I} = FR[X, Y]/(r \otimes_{\mathfrak{I}} \bar{R}) \).

If \( \mathfrak{I} \) is a principal ideal of \( R[X, Y] \) say generated by \( f \) then \( H = uF = ur^2X - Y - rY^2 \) where \( u \) is a unit in \( R \) and \( uR \in \mathfrak{I} \). \( \mathfrak{I} \) is a principal ideal of \( R[X, Y] \).

Thus we prove that \( \mathfrak{I} \) cannot be principal.

We conclude this section with the proof of Theorem B.

Theorem 3.9. Let \( R \) be a ring containing a field \( k \) of characteristic zero. Let \( F \in R[X, Y] \) such that \( R[X, Y]/(F) = R[Z] \) as \( R \)-algebras. Then \( R[X, Y] = R[F]^{[1]} \).

Proof. As in Theorem 3.7, we can assume that \( R \) is reduced and \( R \) contains a noetherian subring \( S \) of finite Krull dimension such that \( F \in S[X, Y] \) and \( S[X, Y]/(F) = S[Z] \) as \( S \)-algebras. Moreover we can assume that \( S \) contains \( k \). Repeating the same arguments we see that there exists a positive integer \( m \) such that \( S[X, Y]^{[m]} = S[Z]^{[m+1]} \). Now since \( S \) contains \( k \) (a field of characteristic zero) by ([5], Theorem 2.8), \( S[X, Y] = S[F]^{[1]} \). Hence \( R[X, Y] = R[F]^{[1]} \).