A NOTE ON COMPLETE INTERSECTIONS

BY

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Abstract. Let $R$ be a regular local ring and let $R[T]$ be a polynomial algebra in one variable over $R$. In this paper the author proves that every maximal ideal of $R[T]$ is complete intersection in each of the following cases: (1) $R$ is a local ring of an affine algebra over an infinite perfect field, (2) $R$ is a power series ring over a field.

Introduction. Let $R$ be a regular local ring. Let $R[T]$ be a polynomial algebra in one variable over $R$. In [D-G] the following question has been asked.

Question. Is every maximal ideal of $R[T]$ complete intersection?

In this paper we prove that the answer to the above question is affirmative in each of the following cases:

(1) $R$ is a local ring of an affine algebra over an infinite perfect field.

(2) $R$ is a power series ring over a field.

This paper is divided into three sections. In §1 we fix notations and state a theorem without proof which is used in §§2 and 3. In §2 we prove some lemmas and propositions which are used in proving the result when $R$ is a local ring of an affine algebra. §3 deals with the power series case.

1. Throughout this paper we consider commutative noetherian rings with 1. For a ring $R$, $\dim R$ denotes its Krull dimension which we always assume to be finite. If $R$ is a local ring then $\mathfrak{m}(R)$ will always denote its unique maximal ideal. If $M$ is a finitely generated $R$-module then $\mu(M)$ will denote the minimal number of generators of $M$. For an ideal $I$ of $R$ $\ht(I)$ denotes the height of $I$.

Definition. Let $I$ be an unmixed ideal of $R$ of height $r$. Then $I$ is said to be complete intersection in $R$ if $I = \Sigma_{i=1}^{r}Ra_i$, where $a_1, a_2, \ldots, a_r$ is a regular $R$-sequence.

Remark. If $R$ is Cohen-Macaulay then $I$ is complete intersection if and only if $\mu(I) = \ht(I)$.

Let $R$ and $S$ be two local rings.

Definition. $R$ is said to be a local extension of $S$ if $S$ is a subring of $R$ and $\mathfrak{m}(S) = \mathfrak{m}(R) \cap S$. $R$ is said to be unramified over $S$ if $\mathfrak{m}(S)R = \mathfrak{m}(R)$ and $R/\mathfrak{m}(R)$ is separable over $S/\mathfrak{m}(S)$.

Let $L/K$ be a finite separable extension of $K$. Then $L$ is a simple extension of $K$. By a minimal polynomial of $L$ over $K$ we always mean an irreducible monic polynomial over $K$ satisfied by a generator of $L$ over $K$.

Now we state a theorem which has been proved in [D-G, Theorem 3].

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Theorem. Let $R$ be a regular ring. Let $A = R[X, Y]$ be a polynomial algebra in two variables over $R$. Then every maximal ideal of $A$ is complete intersection.

In subsequent sections this theorem will always be referred to as the D-G theorem.

2. In this section we prove the following theorem.

**Theorem 2.1.** Let $k$ be an infinite perfect field. Let $C$ be an affine $k$-algebra. Let $\mathfrak{m}$ be a prime ideal of $C$ such that $C_{\mathfrak{m}} = R$ is regular. Let $M$ be a maximal ideal of $R[T]$. Then $M$ is complete intersection.

For the proof of this theorem we need some lemmas and propositions.

**Lemma 2.2.** Let $A$ be an affine domain of dim 1 over a field $K$. Let $\mathfrak{m}$ be a nonregular maximal ideal of $A$ such that $A/\mathfrak{m}$ is a finite separable (therefore simple) extension of $K$. Then there exist $y_1, y_2, \ldots, y_r \in A$ such that

1. $A$ is integral over $K[y_1]$,
2. the inclusion map $K[y_1]/\mathfrak{m} \cap K[y_1] \to A/\mathfrak{m}$ is an isomorphism,
3. $\mathfrak{m} = (f(y_1), y_2, \ldots, y_r)$ where $r = \mu(\mathfrak{m}/\mathfrak{m}^2)$ and $f$ is a minimal polynomial of $\mathfrak{m}$ over $K$.

**Proof.** Since $A$ is one dimensional and $\mathfrak{m}$ nonregular we have $\mu(\mathfrak{m}/\mathfrak{m}^2) = r > 2 = \dim A + 1$. Therefore by [Mo, Corollary 3] it follows that $\mu(\mathfrak{m}) = \mu(\mathfrak{m}/\mathfrak{m}^2)$.

Let $A/\mathfrak{m} = K[a]$. Let $f(X)$ be the minimal polynomial of $a$ over $K$. Let $b \in A$ be such that $a = b \mod \mathfrak{m}$. Then $\alpha$ is separable over $K$ and $f(X)$ is its minimal polynomial imply that $f(b) \in \mathfrak{m}$ and $\partial f(b)/\partial X \not\in \mathfrak{m}$. If $f(b) \in \mathfrak{m}^2$ then replacing $b$ by $b + x$ for some $x \in \mathfrak{m} - \mathfrak{m}^2$ we get $f(b) \not\in \mathfrak{m}^2$. This in particular implies that $b$ is not algebraic over $K$.

Since $A$ is one dimensional affine, by the normalization theorem [Z-S, p. 200] there exists $y \in A$ such that $A$ is integral over $K[y]$. Let $\mathfrak{m} \cap K[y] = (h(y))$. Let $y_1 = b + h(y)^l$ where $l$ is a positive integer. Then by taking sufficiently large $l \geq 2$ one can see that $K[y, b] = K[y_1, y]$ is integral over $K[y_1]$. Moreover

$$f(y_1) = f(b) + (\partial f/\partial X)(b)h(y)^l + ch(y)^{2l}, \quad c \in K[y, b].$$

Since $f(b) \not\in \mathfrak{m}^2$, $h(y) \in \mathfrak{m}$ and $l \geq 2$ we get $f(y_1) \in \mathfrak{m} - \mathfrak{m}^2$. Since $A$ is integral over $K[y, b]$, $A$ is integral over $K[y_1]$ and $\mathfrak{m} \cap K[y_1] = (f(y_1))$. Therefore the inclusion map $K[y_1]/\mathfrak{m} \cap K[y_1] \to A/\mathfrak{m}$ is an isomorphism.

Let $A' = A/(f(y_1))$, $\mathfrak{m}' = \mathfrak{m}/(f(y_1))$. Then $A'$ is zero dimensional and $\mu(\mathfrak{m}'/\mathfrak{m}'^2) = \mu(\mathfrak{m}/\mathfrak{m}^2) - 1 = r - 1 > 1$. Therefore by [Mo, Corollary 3] there exist $y_2', y_3', \ldots, y_r' \in A'$ such that $\mathfrak{m}' = (y_2', y_3', \ldots, y_r')$. Let $y_i$ be a pull back of $y_i'$ in $A$ for $2 \leq i \leq r$. Then $\mathfrak{m} = (f(y_1), y_2, y_3, \ldots, y_r)$.

This completes the proof of Lemma 2.2. Now we state a lemma the proof of which is easy and can be found in [L, Lemma 2].

**Lemma 2.3.** Let $k$ be a perfect field. Let $C$ be an affine $k$-algebra. Let $\mathfrak{m}$ be a prime ideal of $C$ such that $C_{\mathfrak{m}} = R$ is regular. Then there exists a field extension $K/k$ and regular affine $K$-domain $B$ contained in $R$ such that

1. $R = B_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$ of $B$,
2. $B/\mathfrak{m} = R/\mathfrak{m}(R)$ is a finite separable extension of $K$.
The following two propositions are very crucial for the proof of Theorem 2.1.

**Proposition 2.4.** Let $k, C, \mathfrak{N}, R, K, B, \mathfrak{M}$ be as in Lemma 2.3. Let $p$ be a prime ideal of $R$ such that $R/p$ is one dimensional and nonregular. Then $R$ contains a local domain $S$ such that

(1) $S$ is a localization of a polynomial algebra $C'$ over $K$ at some maximal ideal $\eta$ of $C'$,

(2) there exists $h \in p \cap S$ such that the inclusion of $S$ in $R$ gives rise to an inclusion of $S/hS$ in $R/hR$ which is an isomorphism, i.e. $S/hS = R/hR$.

**Proof.** Since $R = B_{\mathfrak{M}}$, there exists a prime ideal $q$ of $B$ such that $qB_{\mathfrak{M}} = p$. Then $B/q$ is one dimensional and $\mathfrak{M}/q$ is a nonregular maximal ideal of $B/q$.

Let $A = B/q$, $\mathfrak{N} = \mathfrak{M}/q$. Then by Lemma 2.2 there exist $y_1, y_2, \ldots, y_r \in A$ satisfying properties 1, 2 and 3 of Lemma 2.2. Let $\phi: B \rightarrow A$ (= $B/q$) be the canonical map. Let $x_i \in B$ be such that $\phi(x_i) = y_i$ for $1 \leq i \leq r$. Then $q + (f(x_1), x_2, \ldots, x_r) = \mathfrak{N}$ and $f(x_1), x_2, \ldots, x_r$ generate $\mathfrak{M}/q$ modulo $\mathfrak{N}/q + q$ where $r = \dim_{B/\mathfrak{M}}(\mathfrak{M}/\mathfrak{N}^2 + q)$. Let $\dim_{B/\mathfrak{M}}(\mathfrak{M}/\mathfrak{N}^2) = \mu(B/\mathfrak{M}) = \dim R = n$. Then

we get $\dim B/\mathfrak{M}(q/q \cap \mathfrak{N}^2) = n - r$. Let $x_{r+1}, x_{r+2}, \ldots, x_n \in q$ be such that $(x_{r+1}, x_{r+2}, \ldots, x_n) + q \cap \mathfrak{N}^2 = q$. Then it is easy to see that $(f(x_1), \ldots, x_r, x_{r+1}, \ldots, x_n) + \mathfrak{M} = \mathfrak{N}$.

Since $R = B_{\mathfrak{M}}$ is regular of dim $n$ it follows that $(f(x_1), x_2, \ldots, x_n)R = \mathfrak{M}(R)$ and $f(x_1), x_2, \ldots, x_n$ are algebraically independent over $K$. Therefore $x_1, x_2, \ldots, x_n$ are also algebraically independent over $K$ and hence $C' = K[x_1, x_2, \ldots, x_n]$ is a polynomial algebra over $K$ contained in $B$.

Let $\eta = C' \cap \mathfrak{N}$. Then $\eta = (f(x_1), x_2, \ldots, x_n)$ is a maximal ideal of $C'$ and the inclusion map $C'/\eta \rightarrow B/\mathfrak{M}$ is an isomorphism. Moreover $A (= B/q)$ is integral over $C'/\eta_1$, where $\eta_1 = q \cap C'$ and $\mathfrak{N}$ is the only maximal ideal of $A$ lying over the maximal ideal $\eta/\eta_1$ of $C'/\eta_1$.

Let $L = \text{quotient field of } B$, $L' = \text{quotient field of } C'$. Then since $B$ and $C'$ are affine $K$-domains of dim $n$, $L$ is a finite algebraic extension of $L'$. Let $B'$ be the integral closure of $C'$ in $L$. Then $B'$ is a finitely generated $C'$-module contained in $B$.

Let $\mathfrak{M}' = \mathfrak{M} \cap B'$, $B'_\mathfrak{M} = R'$, $C'_\mathfrak{M} = S$. Then we get a tower of local extensions $S \rightarrow R' \rightarrow R$. Since $S/\mathfrak{M}(S) = C'/\eta \rightarrow B/\mathfrak{M} = R/\mathfrak{M}(R)$ and $R$ is unramified over $S$, $R$ is also unramified over $R'$ and $R'/\mathfrak{M}(R') \rightarrow R/\mathfrak{M}(R)$. But since $R'$ and $R$ have the same quotient field $L$ and $R'$ is normal, by Zariski’s main theorem [BI, p. 93] we have $R' = R$.

Let $q' = q \cap B'$. Then we get a tower of integral extensions $C'/\eta_1 \rightarrow B'/q' \rightarrow B/q (= A)$. Since $\mathfrak{M}' (= \mathfrak{M}/q)$ is the only maximal ideal of $A$ lying over $\eta/\eta_1$, $\mathfrak{M}'/q'$ will be the only maximal ideal of $B'/q'$ lying over $\eta/\eta_1$. Therefore $\eta B' + q'$ is $\mathfrak{M}'$-primary. Since $B'_\mathfrak{M} = R' = R$ and $\eta R = \mathfrak{M}(R)$ we have $\eta B' + \mathfrak{M}'^2 = \mathfrak{M}'$. But this implies that $\eta B' + \mathfrak{M}'^l = \mathfrak{M}'$ for every positive integer $l$. Since $\eta B' + q'$ is $\mathfrak{M}'$-primary, there exists a positive integer, say $l_0$, such that $\mathfrak{M}'^{l_0} \subset \eta B' + q'$. Therefore $\eta B' + q' = \mathfrak{M}'$. Moreover $\eta B' + \mathfrak{M}'^2 = \mathfrak{M}'$ implies that $\mathfrak{M}'/\eta B'$ is an
idempotent and therefore principal ideal of $B'/\eta B'$. Hence there exists $t \in q'$ such that $tB' + \eta B' = \mathfrak{m}'$.

Let $B'' = C[t], \mathfrak{m}'' = \mathfrak{m}' \cap B'', q'' = q' \cap B''$. It is obvious that $\mathfrak{m}'' B' = \mathfrak{m}'$ and $B''/\mathfrak{m}'' \cong B'/\mathfrak{m}'$. Since $B'$ is a finitely generated $B''$-module we have $B''_{\mathfrak{m}''} = B'_{\mathfrak{m}'} = R$ and $q'' R = p$.

Since $B''$ is a simple integral extension of $C'$ and $C'$ is a unique factorization domain we get $B'' = C'[T]/(g(T))$ where $g(T)$ is a monic irreducible polynomial in $T$.

Let $\psi: C'[T] \rightarrow B''$ be the canonical map. Let $M = \psi^{-1}(q'')$. Since $\psi(T) = t \in \mathfrak{m}''$ we have $T \in M$. Also $\mathfrak{m}'' \cap C' = \eta$ implies $M \cap C' = \eta$. Therefore $M = TC'[T] + \eta C'[T]$.

Let $g(T) = t^t + a_{t-1} t^{t-1} + \cdots + a_0$. Then $g(t) = 0$ and $t \in q''$ implies $a_0 \in q_1 = q'' \cap C'$. Since $B''_{\mathfrak{m}''} = R$ and $\eta R = \mathfrak{m}(R)$ it follows that $a_1 \not\in \eta$, and therefore $tR = hR$ where $h = a_0$. Therefore the map $S/hS \rightarrow R/hR$ is an isomorphism. Thus the proof of Proposition 2.4 is complete.

**Remark.** Under the assumptions of Proposition 2.4 Lindel [L, Proposition 2] also has shown the existence of $S$ and $h$. Our proof is a variation of his proof because of the requirement that $h$ should belong to $p$.

**Proposition 2.5.** Let $K$ be an infinite field. Let $D = K[X_1, X_2, \ldots, X_n]$ be a polynomial algebra over $K$. Let $\mathfrak{m} = (f(X_1), X_2, \ldots, X_n)$ be a maximal ideal of $D$. Let $p$ be a prime ideal of dim 1 contained in $\mathfrak{m}$. If $n \geq 3$ then $D$ contains a $K$-algebra $D'$ of dim $n - 1$ such that

1. $D = D'[Y]$,
2. $p + \mathfrak{m}' D$ is $\mathfrak{m}$-primary where $\mathfrak{m}' = \mathfrak{m} \cap D'$.

**Proof.** If $p$ contains one of the generators $f(X_1), X_2, \ldots, X_n$, say $f(X_1)$, then $p + (X_2, \ldots, X_n) = \mathfrak{m}$. Therefore by taking $D' = K[X_2, \ldots, X_n]$ we get the required result.

Now we assume that $X_i \not\in p$ for $2 \leq i \leq n$ and $f(X_1) \not\in p$. Then $p + (X_2, \ldots, X_n) = \mathfrak{m}$. Therefore by taking $D' = K[X_2, \ldots, X_n]$ we get the required result.

For every $i$, $2 \leq i \leq t$, let $V_i$ denote a subspace of $K^n$ consisting of $n$-tuples $(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 f(X_1) + \lambda_2 X_2 + \cdots + \lambda_n X_n \in \mathfrak{m}_i$. Then $V_i \neq K^n$ for $2 \leq i \leq t$. Since $K$ is infinite we have $\bigcup_{2 \leq i \leq t} V_i = K^n$. Let $(\beta_1, \beta_2, \ldots, \beta_n) \not\in V_i$ for every $i, 2 \leq i \leq t$. Let $Z = \beta_1 f(X_1) + \beta_2 X_2 + \cdots + \beta_n X_n$. Since $X_i \in \mathfrak{m}_i$ for every $i, 2 \leq i \leq t$, we have $\beta_i \neq 0$ for some $l, 1 \leq l \leq n - 1$.

If $\beta_2 = 0$ then taking $D' = D[X_1, X_3, \ldots, X_n]$ we get $X_n, Z \in \mathfrak{m}' = \mathfrak{m} \cap D'$, and the ideal $p + (X_n, Z)D$ is $\mathfrak{m}$-primary. Therefore $p + \mathfrak{m}' D$ is $\mathfrak{m}$-primary. Since $D = D'[X_2]$ we get the required result.

If $\beta_2 \neq 0$ then obviously $D = K[X_1, Z, X_3, \ldots, X_n]$. Taking $D' = K[Z, X_3, \ldots, X_n]$ we get $X_n, Z \in \mathfrak{m}' = \mathfrak{m} \cap D'$. Therefore as before we see that $\mathfrak{m}' D + p$ is $\mathfrak{m}$-primary. Since $D = D'[X_1]$ we get the required result.
Proof of Theorem 2.1. Let $p = M \cap R$. Then $\dim R/p \leq 1$. If $R/p$ is regular then since $\operatorname{ht}(M/pR[T]) = 1$, $M/pR[T]$ is a principal ideal of $R/p[T]$. Therefore

$$\mu(M) \leq 1 + \mu(pR[T]) = 1 + \mu(p) = 1 + \operatorname{ht}(p) = \operatorname{ht}(M).$$

Since we always have $\operatorname{ht}(M) \leq \mu(M)$ we get the equality $\mu(M) = \operatorname{ht}(M)$ which shows that $M$ is complete intersection.

Now we suppose that $R/p$ is not regular. Then $\dim R/p = 1$, $\operatorname{ht}(M) = \operatorname{ht}(p) + 1 = \dim R$ and $\dim R \geq 2$.

Case 1. $\dim R = 2$. Then $\dim R/p = 1$ implies $\operatorname{ht}(p) = 1$. Therefore we have $\operatorname{ht}(M) = \operatorname{ht}(p) + 1 = 2$. Since $R[T]$ is regular, $M$ is locally generated by a regular sequence of length 2. Therefore $\operatorname{hd}_{R[T]} M = 1$ where $\operatorname{hd}_{R[T]} M$ denotes the homological dimension of the $R[T]$-module $M$. Since

$$\operatorname{Ext}^1_{R[T]}(M, R[T]) \cong \operatorname{Ext}^2_{R[T]}(R[T]/M, R[T]) \cong R[T]/M,$$

we get $\operatorname{Ext}^1_{R[T]}(M, R[T])$ to be a cyclic $R[T]$-module. Therefore by [S, p. 8] there is an exact sequence $0 \to R[T] \to P \to M \to 0$ with $P$ finitely generated projective $R[T]$-module of rank 2. But by [Mu, Theorem] $P$ is free. Therefore $\mu(P) = 2$. Since $M$ is an epimorphic image of $P$ we have

$$\mu(M) \leq \mu(P) = 2 = \operatorname{ht}(M) \leq \mu(M).$$

Hence $M$ is complete intersection.

Case 2. $\dim R = n > 3$. By Lemma 2.3 and Proposition 2.4 there exist a field extension $K/k$ and a local domain $S$ contained in $R$ such that

1. $S = K[X_1, \ldots, X_n]_\eta$ where $\eta$ is a maximal ideal of $K[X_1, \ldots, X_n]$ generated by $f(X_1), X_2, \ldots, X_n$ for some irreducible monic polynomial $f(X)$ over $K$.

2. There exists $h \in p \cap S$ such that $S/hS = R/hR$ and therefore $S[T]/hS[T] = R[T]/hR[T]$.

Let $\tilde{M} = M \cap S[T]$. Since $h \in \tilde{M}$, $\tilde{M}$ is a maximal ideal of $S[T]$. Moreover $\tilde{M}R[T] = M$ and $\operatorname{ht}(\tilde{M}) = \operatorname{ht}(M)$. Therefore it is enough to prove that $\tilde{M}$ is a complete intersection ideal of $S[T]$.

Let $q = S \cap p = \tilde{M} \cap S$. Then $h \in q$ and hence $S/q = R/p$. Therefore $\dim S/q = 1$. Let $D = K[X_1, \ldots, X_n]$, $\tilde{M}' = \tilde{M} \cap D[T]$, $q' = q \cap D = \tilde{M}' \cap D$. Then since $D_\eta = S$ we have $\tilde{M}'S[T] = \tilde{M}$, $\operatorname{ht}(\tilde{M}') = \operatorname{ht}(\tilde{M}) = n = \dim D$ and $\operatorname{ht}(q') = \operatorname{ht}(q) = n - 1$. Therefore $\dim q' = \dim D/q' = 1$.

Since $n > 3$ by Proposition 2.5 there exists a subalgebra $D'$ of $D$ of dim $n-1$ such that

1. $D = D'[Y]$,
2. $\eta'D + q'$ is $\eta$-primary where $\eta' = \eta \cap D'$.

Consider the following commutative diagram

$$
\begin{array}{ccc}
D' & \hookrightarrow & D'[Y] = D \\
\downarrow & & \downarrow \\
D[T] & \hookrightarrow & S[T]
\end{array}
$$

$\tilde{M}'$ is a prime ideal of $D[T]$ of height $n = \dim D[T] - 1$. Therefore every prime ideal of $D[T]$ which contains $\tilde{M}'$ properly is a maximal ideal of $D[T]$. Let $M_1$ be one
such maximal ideal. Then since \( D', D, D[T] \) all are affine rings, \( N_i = M_i \cap D' \) will be a maximal ideal of \( D' \). If \( \eta' = N_i \) then since \( M' \subset M_i \) we have \( \eta' D + q' \subset M_i \cap D \). But \( \eta' D + q' \) is \( \eta \)-primary and \( \eta \) is maximal; therefore \( \eta = M_i \cap D \). Since \( S = D_\eta, \eta = M_i \cap D \) implies that \( M_i S[T] \) is a prime ideal of \( S[T] \) which contains \( \tilde{M}'S[T] = \tilde{M} \) properly which contradicts the fact that \( \tilde{M} \) is maximal. Therefore \( N_i \neq \eta' \).

The above discussion shows that no prime ideal of \( D[T] \) which contains \( \tilde{M}' \) properly can lie over a prime ideal of \( D' \) contained in \( \eta' \). Therefore \( \tilde{M}'S[T] \) becomes a maximal of \( S'[T] \) of height = \( \text{ht}(\tilde{M}') \) where \( S' = D_\eta'[Y] \). Then by the D-G theorem \( \tilde{M}'S[T] \) is complete intersection. Now we have the following tower of rings:

\[
D'[Y,T] = D[T] \hookrightarrow S'[T] \hookrightarrow S[T].
\]

Since \( \tilde{M}'S[T] \) is complete intersection, \( \tilde{M}'S[T] = \tilde{M} \) and \( \text{ht}(\tilde{M}'S[T]) = \text{ht}(\tilde{M}') = \text{ht}(\tilde{M}) \), it follows that \( \tilde{M} \) is also complete intersection.

Thus the proof of Theorem 2.1 is complete.

3. We begin this section with the following theorem.

**Theorem 3.1.** Let \( k \) be a field. Let \( R = k[[X_1, X_2, \ldots, X_n]] \) be a power series ring in \( n \) variables over \( k \). Let \( M \) be a maximal ideal of \( R[T] \). Then \( M \) is complete intersection.

**Proof.** Let \( p = R \cap M \). If \( p = 0 \) then \( \text{ht}(M) = \text{ht}(p) + 1 = 1 \).

Since \( R[T] \) is a unique factorization domain, \( M \) will be a principal ideal and hence complete intersection.

If \( p \neq 0 \) then let \( f \) be a nonzero element of \( p \). It is easy to see that there exist \( Y_1, Y_2, \ldots, Y_n \in R \) such that \( R = k[[Y_1, Y_2, \ldots, Y_n]] \) and \( f \) as a power series in \( Y_1, Y_2, \ldots, Y_n \) is regular in \( Y_n \). Therefore without loss of generality we can assume that \( f = f(X_1, \ldots, X_n) \) is regular in \( X_n \). Then by the Weierstrass preparation theorem [Z-S, p. 139] there exists a unit \( u(X_1, \ldots, X_n) \) in \( R \) such that

\[
u(X_1, \ldots, X_n)f(X_1, \ldots, X_n) = f'(X_1, \ldots, X_n) = X_n^r + g_1X_n^{r-1} + \cdots + g_r,
\]

where \( g_i \in k[[X_1, \ldots, X_{n-1}]] \) and \( g_i(0, 0, \ldots, 0) = 0 \) for \( 1 \leq i \leq r \). Let \( S = k[[X_1, \ldots, X_{n-1}][[X_n]] \subset R \). Then it also follows from the above-mentioned theorem that \( S/f'S = R/f'R \). Therefore \( S[T]/f'S[T] = R[T]/f'R[T] \).

Let \( \tilde{M} = M \cap S[T] \). Then since \( f' \in p \cap S \subset \tilde{M} \) it follows that \( \tilde{M} \) is a maximal ideal of \( S[T] \), \( \tilde{M}R[T] = M \) and \( \text{ht}(\tilde{M}) = \text{ht}(M) \). Since \( S[T] = k[[X_1, \ldots, X_{n-1}][[X_n, T]] \) by the D-G theorem \( \tilde{M} \) is complete intersection. Hence \( M \) is also complete intersection.

This completes the proof of Theorem 3.1.

Let \( R \) be an equicharacteristic regular local ring. Let \( \hat{R} \) be the completion of \( R \) with respect to \( \mathfrak{m}(R) \)-adic topology. Then \( \hat{R} = k[[X_1, \ldots, X_n]] \) where \( k \) is the residue field of \( R \) and \( n = \dim R \).

Now we state a proposition which is a generalization of Theorem 3.1.

**Proposition 3.2.** Let \( R \) be an equicharacteristic regular local ring. Let \( \hat{R} \) be its completion with respect to \( \mathfrak{m}(R) \)-adic topology. Let \( M \) be a maximal ideal of \( R[T] \). Let \( I = M \hat{R}[T] \). Then \( \text{ht}(I) = \text{ht}(M) \) and \( I \) is complete intersection.
PROOF. Let \( \hat{R} = k[[X_1, \ldots, X_n]] \) where \( k = R/\mathfrak{m}(R) \). Since \( M \) is locally generated by a regular sequence of length \( = \text{ht}(M) \) and \( \hat{R}[T] \) is a faithfully flat extension of \( R[T] \) it follows that \( \text{ht}(M) = \text{ht}(I) \). If \( \text{ht}(M) = 1 \) then \( M \) itself is complete intersection and therefore \( I \) is also complete intersection. Now we assume that \( \text{ht}(M) \geq 2 \).

Let \( J = I \cap \hat{R} \). Then \( \text{ht}(I) = \text{ht}(M) \geq 2 \) implies that \( J \neq 0 \). Then as in Theorem 3.1 we can assume that \( J \) contains an element \( f \) such that \( f \in S, S/fS = R/fR \) where \( S = k[[X_1, \ldots, X_{n-1}]][X_n] \). Moreover we can assume that \( f \) is monic in \( X_n \).

Let \( I' = I \cap S[T] \). Since \( f \in I' \) we have \( \mu(I'/I'^2) = \mu(I/I^2) \) and \( I'R[T] = I \). But \( \hat{R}[T] \) is faithfully flat over \( R[T] \), \( M\hat{R}[T] = I \) and \( M \) is a maximal ideal of \( R[T] \). Therefore \( \mu(I/I^2) = \mu(M/M^2) = \text{ht}(M) = \text{ht}(I) \).

Since \( S[T] = k[[X_1, \ldots, X_{n-1}]][T][X_n] \) and \( f \in I' \), \( I' \) contains a monic polynomial in \( X_n \) with coefficients in \( k[[X_1, \ldots, X_{n-1}]][T] \). Since \( \mu(I'/I'^2) = \mu(I/I^2) = \text{ht}(I) \geq 2 \) and \( \dim S[T]/I' = \dim R[T]/I = 0 \) (this is easy to check) by [Mo, Theorem 5] there exists a finitely generated projective \( S[T] \)-module \( P \) of rank \( = \mu(I'/I'^2) \) and a surjective homomorphism \( \psi: P \to I' \). But by [L-L, Theorem 2] \( P \) is free and therefore \( \mu(P) = \text{rank}(P) = \mu(I'/I'^2) \). This implies that \( \mu(I') \leq \mu(I'/I'^2) = \mu(I/I^2) = \text{ht}(I) \). Since \( I'\hat{R}[T] = I \), we have \( \mu(I) \leq \mu(I') \leq \text{ht}(I) \leq \mu(I) \). Therefore \( I \) is complete intersection.

This completes the proof of Proposition 3.2.

REMARK. In view of known results regarding projective modules over \( R[T] \) when \( R \) is regular local, one can obtain the results of §§2 and 3 in one stroke if one can prove the following theorem.

**THEOREM.** Let \( R \) be a regular local ring. Let \( M \) be a maximal ideal of \( R[T] \). Then there exists a projective \( R[T] \)-module \( P \) of rank \( = \text{ht}(M) \) and a surjective homomorphism \( \psi: P \to M \).

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