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A NOTE ON COMPLETE INTERSECTIONS

BY

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ABSTRACT. Let R be a regular local ring and let $R[T]$ be a polynomial algebra in one variable over R. In this paper the author proves that every maximal ideal of $R[T]$ is complete intersection in each of the following cases: (1) R is a local ring of an affine algebra over an infinite perfect field, (2) R is a power series ring over a field.

Introduction. Let R be a regular local ring. Let $R[T]$ be a polynomial algebra in one variable over R. In **[D-G]** the following question has been asked.

Question. Is every maximal ideal of $R[T]$ complete intersection?

In this paper we prove that the answer to the above question is affirmative in each of the following cases:

(1) R is a local ring of an affine algebra over an infinite perfect field.

(2) R is a power series ring over a field.

This paper is divided into three sections. In §1 we fix notations and state a theorem without proof which is used in §§2 and 3. In §2 we prove some lemmas and propositions which are used in proving the result when R is a local ring of an affine algebra. §3 deals with the power series case.

1. Throughout this paper we consider commutative noetherian rings with 1. For a ring R, dim R denotes its Krull dimension which we always assume to be finite. If R is a local ring then $\mathfrak{M}(R)$ will always denote its unique maximal ideal. If M is a finitely generated R-module then $\mu(M)$ will denote the minimal number of generators of M. For an ideal I of R $\mathrm{ht}(I)$ denotes the height of I.

DEFINITION. Let I be an unmixed ideal of R of height r. Then I is said to be complete intersection in R if $I = \sum_{i=1}^{r} Ra_i$, where $a_1, a_2, \ldots, a_r$ is a regular R-sequence.

REMARK. If R is Cohen-Macaulay then I is complete intersection if and only if $\mu(I) = \mathrm{ht}(I)$.

Let R and S be two local rings.

DEFINITION. R is said to be a local extension of S if S is a subring of R and $\mathfrak{M}(S) = \mathfrak{M}(R) \cap S$. R is said to be unramified over S if $\mathfrak{M}(S)R = \mathfrak{M}(R)$ and $R/\mathfrak{M}(R)$ is separable over $S/\mathfrak{M}(S)$.

Let L/K be a finite separable extension of K. Then L is a simple extension of K. By a minimal polynomial of L over K we always mean an irreducible monic polynomial over K satisfied by a generator of L over K.

Now we state a theorem which has been proved in **[D-G**, Theorem 3].

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THEOREM. *Let R be a regular ring. Let $A = R[X, Y]$ be a polynomial algebra in two variables over R. Then every maximal ideal of A is complete intersection.*

In subsequent sections this theorem will always be referred to as the D-G theorem.

2. In this section we prove the following theorem.

THEOREM 2.1. *Let k be an infinite perfect field. Let C be an affine k-algebra. Let $\mathfrak{I}$ be a prime ideal of C such that $C_{\mathfrak{I}} = R$ is regular. Let M be a maximal ideal of $R[T]$. Then M is complete intersection.*

For the proof of this theorem we need some lemmas and propositions.

LEMMA 2.2. *Let A be an affine domain of* $\dim 1$ *over a field K. Let $\overline{\mathfrak{M}}$ be a nonregular maximal ideal of A such that $A/\overline{\mathfrak{M}}$ is a finite separable (therefore simple) extension of K. Then there exist $y_1, y_2, \ldots, y_r \in A$ such that*

(1) *A is integral over $K[y_1]$,*

(2) *the inclusion map $K[y_1]/\overline{\mathfrak{M}} \cap K[y_1] \to A/\overline{\mathfrak{M}}$ is an isomorphism,*

(3) *$\overline{\mathfrak{M}} = (f(y_1), y_2, \ldots, y_r)$ where $r = \mu(\overline{\mathfrak{M}}/\overline{\mathfrak{M}}^2)$ and f is a minimal polynomial of $A/\overline{\mathfrak{M}}$ over K.*

PROOF. Since A is one dimensional and $\overline{\mathfrak{M}}$ nonregular we have $\mu(\overline{\mathfrak{M}}/\overline{\mathfrak{M}}^2) = r \geqslant 2 = \dim A + 1$. Therefore by [**Mo**, Corollary 3] it follows that $\mu(\overline{\mathfrak{M}}) = \mu(\overline{\mathfrak{M}}/\overline{\mathfrak{M}}^2)$.

Let $A/\overline{\mathfrak{M}} = K[\alpha]$. Let $f(X)$ be the minimal polynomial of α over K. Let $b \in A$ be such that $\alpha = b \bmod \overline{\mathfrak{M}}$. Then α is separable over K and $f(X)$ is its minimal polynomial imply that $f(b) \in \overline{\mathfrak{M}}$ and $\partial f(b)/\partial X \notin \overline{\mathfrak{M}}$. If $f(b) \in \overline{\mathfrak{M}}^2$ then replacing b by $b + x$ for some $x \in \overline{\mathfrak{M}} - \overline{\mathfrak{M}}^2$ we get $f(b) \notin \overline{\mathfrak{M}}^2$. This in particular implies that b is not algebraic over K.

Since A is one dimensional affine, by the normalization theorem [**Z-S**, p. 200] there exists $y \in A$ such that A is integral over $K[y]$. Let $\overline{\mathfrak{M}} \cap K[y] = (h(y))$. Let $y_1 = b + h(y)^l$ where l is a positive integer. Then by taking sufficiently large $l \geqslant 2$ one can see that $K[y, b] = K[y_1, y]$ is integral over $K[y_1]$. Moreover

$$f(y_1) = f(b) + (\partial f/\partial X)(b)h(y)^l + ch(y)^{2l}, \qquad c \in K[y, b].$$

Since $f(b) \notin \overline{\mathfrak{M}}^2$, $h(y) \in \overline{\mathfrak{M}}$ and $l \geqslant 2$ we get $f(y_1) \in \overline{\mathfrak{M}} - \overline{\mathfrak{M}}^2$. Since A is integral over $K[y, b]$, A is integral over $K[y_1]$ and $\overline{\mathfrak{M}} \cap K[y_1] = (f(y_1))$. Therefore the inclusion map $K[y_1]/\overline{\mathfrak{M}} \cap K[y_1] \to A/\overline{\mathfrak{M}}$ is an isomorphism.

Let $A' = A/(f(y_1))$, $\mathfrak{M}' = \overline{\mathfrak{M}}/(f(y_1))$. Then A' is zero dimensional and $\mu(\mathfrak{M}'/\mathfrak{M}'^2) = \mu(\overline{\mathfrak{M}}/\overline{\mathfrak{M}}^2) - 1 = r - 1 \geqslant 1$. Therefore by [**Mo**, Corollary 3] there exist $y_2', y_3', \ldots, y_r' \in A'$ such that $\mathfrak{M}' = (y_2', y_3', \ldots, y_r')$. Let y_i be a pull back of y_i' in A for $2 \leqslant i \leqslant r$. Then $\overline{\mathfrak{M}} = (f(y_1), y_2, y_3, \ldots, y_r)$.

This completes the proof of Lemma 2.2. Now we state a lemma the proof of which is easy and can be found in [**L**, Lemma 2].

LEMMA 2.3. *Let k be a perfect field. Let C be an affine k-algebra. Let $\mathfrak{I}$ be a prime ideal of C such that $C_{\mathfrak{I}} = R$ is regular. Then there exists a field extension K/k and regular affine K-domain B contained in R such that*

(1) *$R = B_{\mathfrak{M}}$ for some maximal ideal $\mathfrak{M}$ of B,*

(2) *$B/\mathfrak{M} = R/\mathfrak{M}(R)$ is a finite separable extension of K.*

The following two propositions are very crucial for the proof of Theorem 2.1.

PROPOSITION 2.4. *Let k, C, $\mathfrak{J}$, R, K, B, $\mathfrak{M}$ be as in Lemma 2.3. Let p be a prime ideal of R such that R/p is one dimensional and nonregular. Then R contains a local domain S such that*

(1) *S is a localization of a polynomial algebra C' over K at some maximal ideal η of C',*

(2) *there exists $h \in p \cap S$ such that the inclusion of S in R gives rise to an inclusion of S/hS in R/hR which is an isomorphism, i.e. $S/hS = R/hR$.*

PROOF. Since $R = B_{\mathfrak{M}}$ there exists a prime ideal q of B such that $qB_{\mathfrak{M}} = p$. Then B/q is one dimensional and $\mathfrak{M}/q$ is a nonregular maximal ideal of B/q.

Let $A = B/q$, $\overline{\mathfrak{M}} = \mathfrak{M}/q$. Then by Lemma 2.2 there exist $y_1, y_2, \ldots, y_r \in A$ satisfying properties 1, 2 and 3 of Lemma 2.2. Let ϕ: $B \to A$ $(= B/q)$ be the canonical map. Let $x_i \in B$ be such that $\phi(x_i) = y_i$ for $1 \leqslant i \leqslant r$. Then $q + (f(x_1), x_2, \ldots, x_r) = \mathfrak{M}$ and $f(x_1), x_2, \ldots, x_r$ generate $\mathfrak{M} \bmod \mathfrak{M}^2 + q$ where $r = \dim_{B/\mathfrak{M}}(\mathfrak{M}/\mathfrak{M}^2 + q)$. Let $\dim_{B/\mathfrak{M}}(\mathfrak{M}/\mathfrak{M}^2) = \mu(\mathfrak{M}B_{\mathfrak{M}}) = \dim R = n$. Then since we have the following exact sequence

$$0 \to q/q \cap \mathfrak{M}^2 \to \mathfrak{M}/\mathfrak{M}^2 \to \mathfrak{M}/\mathfrak{M}^2 + q \to 0$$

we get $\dim_{B/\mathfrak{M}}(q/q \cap \mathfrak{M}^2) = n - r$. Let $x_{r+1}, x_{r+2}, \ldots, x_n \in q$ be such that $(x_{r+1}, x_{r+2}, \ldots, x_n) + q \cap \mathfrak{M}^2 = q$. Then it is easy to see that $(f(x_1), \ldots, x_r, x_{r+1}, \ldots, x_n) + \mathfrak{M}^2 = \mathfrak{M}$. Since $R = B_{\mathfrak{M}}$ is regular of dim n it follows that $(f(x_1), x_2, \ldots, x_n)R = \mathfrak{M}(R)$ and $f(x_1), x_2, \ldots, x_n$ are algebraically independent over K. Therefore $x_1, x_2, \ldots, x_n$ are also algebraically independent over K and hence $C' = K[x_1, x_2, \ldots, x_n]$ is a polynomial algebra over K contained in B.

Let $\eta = C' \cap \mathfrak{M}$. Then $\eta = (f(x_1), x_2, \ldots, x_n)$ is a maximal ideal of C' and the inclusion map $C'/\eta \to B/\mathfrak{M}$ is an isomorphism. Moreover A $(= B/q)$ is integral over C'/q_1 where $q_1 = q \cap C'$ and $\overline{\mathfrak{M}}$ is the only maximal ideal of A lying over the maximal ideal η/q_1 of C'/q_1.

Let $L =$ quotient field of B, $L' =$ quotient field of C'. Then since B and C' are affine K-domains of dim n, L is a finite algebraic extension of L'. Let B' be the integral closure of C' in L. Then B' is a finitely generated C'-module contained in B.

Let $\mathfrak{M}' = \mathfrak{M} \cap B'$, $B'_{\mathfrak{M}'} = R'$, $C'_\eta = S$. Then we get a tower of local extensions $S \hookrightarrow R' \hookrightarrow R$. Since $S/\mathfrak{M}(S) = C'/\eta \xrightarrow{\sim} B/\mathfrak{M} = R/\mathfrak{M}(R)$ and R is unramified over S, R is also unramified over R' and $R'/\mathfrak{M}(R') \xrightarrow{\sim} R/\mathfrak{M}(R)$. But since R' and R have the same quotient field L and R' is normal, by Zariski's main theorem [**BI**, p. 93] we have $R' = R$.

Let $q' = q \cap B'$. Then we get a tower of integral extensions $C'/q_1 \hookrightarrow B'/q' \hookrightarrow B/q$ $(= A)$. Since $\overline{\mathfrak{M}}$ $(= \mathfrak{M}/q)$ is the only maximal ideal of A lying over η/q_1, $\mathfrak{M}'/q'$ will be the only maximal ideal of B'/q' lying over η/q_1. Therefore $\eta B' + q'$ is $\mathfrak{M}'$-primary. Since $B'_{\mathfrak{M}'} = R' = R$ and $\eta R = \mathfrak{M}(R)$ we have $\eta B' + \mathfrak{M}'^2 = \mathfrak{M}'$. But this implies that $\eta B' + \mathfrak{M}'^l = \mathfrak{M}'$ for every positive integer l. Since $\eta B' + q'$ is $\mathfrak{M}'$-primary, there exists a positive integer, say l_0, such that $\mathfrak{M}'^{l_0} \subset \eta B' + q'$. Therefore $\eta B' + q' = \mathfrak{M}'$. Moreover $\eta B' + \mathfrak{M}'^2 = \mathfrak{M}'$ implies that $\mathfrak{M}'/\eta B'$ is an idempotent and therefore principal ideal of $B'/\eta B'$. Hence there exists $t \in q'$ such that $tB' + \eta B' = \mathfrak{M}'$.

Let $B'' = C'[t]$, $\mathfrak{M}'' = \mathfrak{M}' \cap B''$, $q'' = q' \cap B''$. It is obvious that $\mathfrak{M}''B' = \mathfrak{M}'$ and $B''/\mathfrak{M}'' \xrightarrow{\sim} B'/\mathfrak{M}'$. Since B' is a finitely generated B''-module we have $B''_{\mathfrak{M}''} = B'_{\mathfrak{M}'} = R$ and $q''R = p$.

Since B'' is a simple integral extension of C' and C' is a unique factorization domain we get $B'' = C'[T]/(g(T))$ where $g(T)$ is a monic irreducible polynomial in T.

Let ψ: $C'[T] \to B''$ $(= C'[t])$ be the canonical map. Let $M = \psi^{-1}(\mathfrak{M}'')$. Since $\psi(T) = t \in \mathfrak{M}''$ we have $T \in M$. Also $\mathfrak{M}'' \cap C' = \eta$ implies $M \cap C' = \eta$. Therefore $M = TC'[T] + \eta C'[T]$.

Let $g(T) = T^i + a_{i-1}T^{i-1} + \cdots + a_1 T + a_0$. Then $g(t) = 0$ and $t \in q''$ implies $a_0 \in q_1 = q'' \cap C'$. Since $B''_{\mathfrak{M}''} = R$ and $\eta R = \mathfrak{M}(R)$ it follows that $a_1 \notin \eta$, and therefore $tR = hR$ where $h = a_0$. Therefore the map $S/hS \to R/hR$ is an isomorphism. Thus the proof of Proposition 2.4 is complete.

Remark. Under the assumptions of Proposition 2.4 Lindel [**L**, Proposition 2] also has shown the existence of S and h. Our proof is a variation of his proof because of the requirement that h should belong to p.

Proposition 2.5. *Let K be an infinite field. Let $D = K[X_1, X_2, \ldots, X_n]$ be a polynomial algebra over K. Let $\mathfrak{M} = (f(X_1), X_2, \ldots, X_n)$ be a maximal ideal of D. Let p be a prime ideal of* dim 1 *contained in $\mathfrak{M}$. If $n \geq 3$ then D contains a K-algebra D' of* dim $n - 1$ *such that*

(1) $D = D'[Y]$,

(2) $p + \mathfrak{M}'D$ *is $\mathfrak{M}$-primary where $\mathfrak{M}' = \mathfrak{M} \cap D'$.*

Proof. If p contains one of the generators $f(X_1), X_2, \ldots, X_n$, say $f(X_1)$, then $p + (X_2, \ldots, X_n) = \mathfrak{M}$. Therefore by taking $D' = K[X_2, \ldots, X_n]$ we get the required result.

Now we assume that $X_i \notin p$ for $2 \leq i \leq n$ and $f(X_1) \notin p$. Then $p + (X_n) = I$ is a zero dimensional ideal of D and hence contained in only finitely many maximal ideals of D. Let $T = \{\mathfrak{M} = \mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_t\}$ be a finite set of maximal ideals of D containing I.

For every i, $2 \leq i \leq t$, let V_i denote a subspace of K^n consisting of n-tuples $(\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 f(X_1) + \lambda_2 X_2 + \cdots + \lambda_n X_n \in \mathfrak{M}_i$. Then $V_i \neq K^n$ for $2 \leq i \leq t$. Since K is infinite we have $\cup_{2 \leq i \leq t} V_i \neq K^n$. Let $(\beta_1, \beta_2, \ldots, \beta_n)$ be such that $(\beta_1, \beta_2, \ldots, \beta_n) \notin V_i$ for every i, $2 \leq i \leq t$. Let $Z = \beta_1 f(X_1) + \beta_2 X_2 + \cdots + \beta_n X_n$. Since $X_n \in \mathfrak{M}_i$ for every i, $2 \leq i \leq t$, we have $\beta_l \neq 0$ for some l, $1 \leq l \leq n-1$.

If $\beta_2 = 0$ then taking $D' = D[X_1, X_3, \ldots, X_n]$ we get $X_n, Z \in \mathfrak{M}' = \mathfrak{M} \cap D'$, and the ideal $p + (X_n, Z)D$ is $\mathfrak{M}$-primary. Therefore $p + \mathfrak{M}'D$ is $\mathfrak{M}$-primary. Since $D = D'[X_2]$ we get the required result.

If $\beta_2 \neq 0$ then obviously $D = K[X_1, Z, X_3, \ldots, X_n]$. Taking $D' = K[Z, X_3, \ldots, X_n]$ we get X_n, $Z \in \mathfrak{M}' = \mathfrak{M} \cap D'$. Therefore as before we see that $\mathfrak{M}'D + p$ is $\mathfrak{M}$-primary. Since $D = D'[X_1]$ we get the required result.

PROOF OF THEOREM 2.1. Let $p = M \cap R$. Then $\dim R/p \leqslant 1$. If R/p is regular then since $\mathrm{ht}(M/pR[T]) = 1$, $M/pR[T]$ is a principal ideal of $R/p[T]$. Therefore

$$\mu(M) \leqslant 1 + \mu(pR[T]) = 1 + \mu(p) = 1 + \mathrm{ht}(p) = \mathrm{ht}(M).$$

Since we always have $\mathrm{ht}(M) \leqslant \mu(M)$ we get the equality $\mu(M) = \mathrm{ht}(M)$ which shows that M is complete intersection.

Now we suppose that R/p is not regular. Then $\dim R/p = 1$, $\mathrm{ht}(M) = \mathrm{ht}(p) + 1 = \dim R$ and $\dim R \geqslant 2$.

Case 1. $\dim R = 2$. Then $\dim R/p = 1$ implies $\mathrm{ht}(p) = 1$. Therefore we have $\mathrm{ht}(M) = \mathrm{ht}(p) + 1 = 2$. Since $R[T]$ is regular, M is locally generated by a regular sequence of length 2. Therefore $\mathrm{hd}_{R[T]}M = 1$ where $\mathrm{hd}_{R[T]}M$ denotes the homological dimension of the $R[T]$-module M. Since

$$\mathrm{Ext}^1_{R[T]}(M, R[T]) \simeq \mathrm{Ext}^2_{R[T]}(R[T]/M, R[T]) \simeq R[T]/M,$$

we get $\mathrm{Ext}^1_{R[T]}(M, R[T])$ to be a cyclic $R[T]$-module. Therefore by [**S**, p. 8] there is an exact sequence $0 \to R[T] \to P \to M \to 0$ with P finitely generated projective $R[T]$-module of rank 2. But by [**Mu**, Theorem] P is free. Therefore $\mu(P) = 2$. Since M is an epimorphic image of P we have

$$\mu(M) \leqslant \mu(P) = 2 = \mathrm{ht}(M) \leqslant \mu(M).$$

Hence M is complete intersection.

Case 2. $\dim R = n \geqslant 3$. By Lemma 2.3 and Proposition 2.4 there exist a field extension K/k and a local domain S contained in R such that

(1) $S = K[X_1,\dots,X_n]_\eta$ where η is a maximal ideal of $K[X_1,\dots,X_n]$ generated by $f(X_1), X_2,\dots,X_n$ for some irreducible monic polynomial $f(X)$ over K.

(2) There exists $h \in p \cap S$ such that $S/hS = R/hR$ and therefore $S[T]/hS[T] = R[T]/hR[T]$.

Let $\tilde{M} = M \cap S[T]$. Since $h \in \tilde{M}$, $\tilde{M}$ is a maximal ideal of $S[T]$. Moreover $\tilde{M}R[T] = M$ and $\mathrm{ht}(\tilde{M}) = \mathrm{ht}(M)$. Therefore it is enough to prove that $\tilde{M}$ is a complete intersection ideal of $S[T]$.

Let $q = S \cap p = \tilde{M} \cap S$. Then $h \in q$ and hence $S/q = R/p$. Therefore $\dim S/q = 1$. Let $D = K[X_1,\dots,X_n]$, $\tilde{M}' = \tilde{M} \cap D[T]$, $q' = q \cap D = \tilde{M}' \cap D$. Then since $D_\eta = S$ we have $\tilde{M}'S[T] = \tilde{M}$, $\mathrm{ht}(\tilde{M}') = \mathrm{ht}(\tilde{M}) = n = \dim D$ and $\mathrm{ht}(q') = \mathrm{ht}(q) = n - 1$. Therefore $\dim q' = \dim D/q' = 1$.

Since $n \geqslant 3$ by Proposition 2.5 there exists a subalgebra D' of D of $\dim n - 1$ such that

(1) $D = D'[Y]$,

(2) $\eta' D + q'$ is η-primary where $\eta' = \eta \cap D'$.

Consider the following commutative diagram

$$\begin{array}{ccccc} D' & \hookrightarrow & D'[Y] = D & \hookrightarrow & D_\eta = S \\ & & \downarrow & & \downarrow \\ & & D[T] & \hookrightarrow & S[T] \end{array}$$

$\tilde{M}'$ is a prime ideal of $D[T]$ of height $n = \dim D[T] - 1$. Therefore every prime ideal of $D[T]$ which contains $\tilde{M}'$ properly is a maximal ideal of $D[T]$. Let M_1 be one such maximal ideal. Then since D', D, $D[T]$ all are affine rings, $N_1 = M_1 \cap D'$ will be a maximal ideal of D'. If $\eta' = N_1$ then since $\tilde{M}' \subset M_1$ we have $\eta' D + q' \subset M_1 \cap D$. But $\eta' D + q'$ is η-primary and η is maximal; therefore $\eta = M_1 \cap D$. Since $S = D_\eta$, $\eta = M_1 \cap D$ implies that $M_1 S[T]$ is a prime ideal of $S[T]$ which contains $\tilde{M}' S[T] = \tilde{M}$ properly which contradicts the fact that $\tilde{M}$ is maximal. Therefore $N_1 \neq \eta'$.

The above discussion shows that no prime ideal of $D[T]$ which contains $\tilde{M}'$ properly can lie over a prime ideal of D' contained in η'. Therefore $\tilde{M}' S'[T]$ becomes a maximal of $S'[T]$ of height $= \operatorname{ht}(\tilde{M}')$ where $S' = D'_{\eta'}[Y]$. Then by the D-G theorem $\tilde{M}' S'[T]$ is complete intersection. Now we have the following tower of rings:

$$D'[Y, T] = D[T] \hookrightarrow S'[T] \hookrightarrow S[T].$$

Since $\tilde{M}' S'[T]$ is complete intersection, $\tilde{M}' S[T] = \tilde{M}$ and $\operatorname{ht}(\tilde{M}' S'[T]) = \operatorname{ht}(\tilde{M}') = \operatorname{ht}(\tilde{M})$, it follows that $\tilde{M}$ is also complete intersection.

Thus the proof of Theorem 2.1 is complete.

3. We begin this section with the following theorem.

THEOREM 3.1. *Let k be a field. Let $R = k[[X_1, X_2, \ldots, X_n]]$ be a power series ring in n variables over k. Let M be a maximal ideal of $R[T]$. Then M is complete intersection.*

PROOF. Let $p = R \cap M$. If $p = 0$ then $\operatorname{ht}(M) = \operatorname{ht}(p) + 1 = 1$.

Since $R[T]$ is a unique factorization domain, M will be a principal ideal and hence complete intersection.

If $p \neq 0$ then let f be a nonzero element of p. It is easy to see that there exist $Y_1, Y_2, \ldots, Y_n \in R$ such that $R = k[[Y_1, Y_2, \ldots, Y_n]]$ and f as a power series in $Y_1, Y_2, \ldots, Y_n$ is regular in Y_n. Therefore without loss of generality we can assume that $f = f(X_1, \ldots, X_n)$ is regular in X_n. Then by the Weierstrass preparation theorem [**Z-S**, p. 139] there exists a unit $u(X_1, \ldots, X_n)$ in R such that

$$u(X_1, \ldots, X_n) f(X_1, \ldots, X_n) = f'(X_1, \ldots, X_n) = X_n^r + g_1 X_n^{r-1} + \cdots + g_r$$

where $g_i \in k[[X_1, \ldots, X_{n-1}]]$ and $g_i(0, 0, \ldots, 0) = 0$ for $1 \leqslant i \leqslant r$. Let $S = k[[X_1, \ldots, X_{n-1}]][X_n] \subset R$. Then it also follows from the above-mentioned theorem that $S/f'S = R/f'R$. Therefore $S[T]/f'S[T] = R[T]/f'R[T]$.

Let $\tilde{M} = M \cap S[T]$. Then since $f' \in p \cap S \subset \tilde{M}$ it follows that $\tilde{M}$ is a maximal ideal of $S[T]$, $\tilde{M} R[T] = M$ and $\operatorname{ht}(\tilde{M}) = \operatorname{ht}(M)$. Since $S[T] = k[[X_1, \ldots, X_{n-1}]][X_n, T]$ by the D-G theorem $\tilde{M}$ is complete intersection. Hence M is also complete intersection.

This completes the proof of Theorem 3.1.

Let R be an equicharacteristic regular local ring. Let $\hat{R}$ be the completion of R with respect to $\mathfrak{M}(R)$-adic topology. Then $\hat{R} = k[[X_1, \ldots, X_n]]$ where k is the residue field of R and $n = \dim R$.

Now we state a proposition which is a generalization of Theorem 3.1.

PROPOSITION 3.2. *Let R be an equicharacteristic regular local ring. Let $\hat{R}$ be its completion with respect to $\mathfrak{M}(R)$-adic topology. Let M be a maximal ideal of $R[T]$. Let $I = M\hat{R}[T]$. Then $\operatorname{ht}(I) = \operatorname{ht}(M)$ and I is complete intersection.*

PROOF. Let $\hat{R} = k[[X_1,\ldots,X_n]]$ where $k = R/\mathfrak{M}(R)$. Since M is locally generated by a regular sequence of length $= \mathrm{ht}(M)$ and $\hat{R}[T]$ is a faithfully flat extension of $R[T]$ it follows that $\mathrm{ht}(M) = \mathrm{ht}(I)$. If $\mathrm{ht}(M) = 1$ then M itself is complete intersection and therefore I is also complete intersection. Now we assume that $\mathrm{ht}(M) \geqslant 2$.

Let $J = I \cap \hat{R}$. Then $\mathrm{ht}(I) = \mathrm{ht}(M) \geqslant 2$ implies that $J \neq 0$. Then as in Theorem 3.1 we can assume that J contains an element f such that $f \in S$, $S/fS = R/fR$ where $S = k[[X_1,\ldots,X_{n-1}]][X_n]$. Moreover we can assume that f is monic in X_n.

Let $I' = I \cap S[T]$. Since $f \in I'$ we have $\mu(I'/I'^2) = \mu(I/I^2)$ and $I'\hat{R}[T] = I$. But $\hat{R}[T]$ is faithfully flat over $R[T]$, $M\hat{R}[T] = I$ and M is a maximal ideal of $R[T]$. Therefore $\mu(I/I^2) = \mu(M/M^2) = \mathrm{ht}(M) = \mathrm{ht}(I)$.

Since $S[T] = k[[X_1,\ldots,X_{n-1}]][T][X_n]$ and $f \in I'$, I' contains a monic polynomial in X_n with coefficients in $k[[X_1,\ldots,X_{n-1}]][T]$. Since $\mu(I'/I'^2) = \mu(I/I^2) = \mathrm{ht}(I) \geqslant 2$ and $\dim S[T]/I' = \dim R[T]/I = 0$ (this is easy to check) by [**Mo**, Theorem 5] there exists a finitely generated projective $S[T]$-module P of rank $= \mu(I'/I'^2)$ and a surjective homomorphism $\psi\colon P \to I'$. But by [**L-L**, Theorem 2] P is free and therefore $\mu(P) = \mathrm{rank}(P) = \mu(I'/I'^2)$. This implies that $\mu(I') \leqslant \mu(I'/I'^2) = \mu(I/I^2) = \mathrm{ht}(I)$. Since $I'\hat{R}[T] = I$, we have $\mu(I) \leqslant \mu(I') \leqslant \mathrm{ht}(I) \leqslant \mu(I)$. Therefore I is complete intersection.

This completes the proof of Proposition 3.2.

REMARK. In view of known results regarding projective modules over $R[T]$ when R is regular local, one can obtain the results of §§2 and 3 in one stroke if one can prove the following theorem.

THEOREM. *Let R be a regular local ring. Let M be a maximal ideal of $R[T]$. Then there exists a projective $R[T]$-module P of rank* $= \mathrm{ht}(M)$ *and a surjective homomorphism $\psi\colon P \to M$.*

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