TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 8, August 1997, Pages 3303–3319
S 0002-9947(97)01946-6

KERNEL OF LOCALLY NILPOTENT R-DERIVATIONS OF $R[X, Y]$

S. M. BHATWADEKAR AND AMARTYA K. DUTTA

Abstract. In this paper we study the kernel of a non-zero locally nilpotent R-derivation of the polynomial ring $R[X, Y]$ over a noetherian integral domain R containing a field of characteristic zero. We show that if R is normal then the kernel has a graded R-algebra structure isomorphic to the symbolic Rees algebra of an unmixed ideal of height one in R, and, conversely, the symbolic Rees algebra of any unmixed height one ideal in R can be embedded in $R[X, Y]$ as the kernel of a locally nilpotent R-derivation of $R[X, Y]$. We also give a necessary and sufficient criterion for the kernel to be a polynomial ring in general.

1. Introduction

Locally nilpotent R-derivations of the polynomial ring $R[X, Y]$, where R is a U.F.D. containing a field of characteristic zero, have been studied recently by Daigle-Freudenburg in ([D-F]). In this situation the kernel of a non-zero locally nilpotent R-derivation of $R[X, Y]$ is a polynomial ring in one variable over R ([D-F], 2.1). In our paper we first investigate locally nilpotent R-derivations of $R[X, Y]$ over a noetherian *normal* domain R containing a field of characteristic zero. We first describe the structure of the kernel of such derivations as a graded R-algebra. We prove :

Theorem 3.5. *Let R be a noetherian normal domain containing a field of characteristc zero and let D be a non-zero locally nilpotent R-derivation of the polynomial ring $R[X, Y]$ with kernel A. Then A has the structure of a graded ring $\bigoplus_{n\geq 0} A_n$ with $A_0 = R$ and A_n a finite reflexive R-module of rank one for every n. In fact, when R is not a field, there exists an ideal I in R of unmixed height one such that A is isomorphic to the symbolic Rees algebra $\bigoplus_{n\geq 0} I^{(n)}T^n$ as a graded R-algebra.*

Conversely, let R be as above and let I be any ideal of unmixed height one in R. Then there exists a non-zero locally nilpotent R-derivation of $R[X, Y]$ whose kernel is isomorphic to the symbolic Rees algebra $\bigoplus_{n\geq 0} I^{(n)}T^n$ as a graded R-algebra.

In fact we show (Proposition 3.3) that when R is a noetherian normal domain, any inert subring of $R[X, Y]$ of transcendence degree one over R has the graded

Received by the editors January 11, 1996.
1991 *Mathematics Subject Classification.* Primary 13B10; Secondary 13A30.
Key words and phrases. Locally nilpotent derivations, inert subrings, symbolic Rees algebra.

R-algebra structure described above. The crucial step in the proof of this result is a patching lemma (3.1).

The converse part of Theorem 3.5 would show (see 3.6) that the kernel of a locally nilpotent R-derivation of $R[X,Y]$ need not be finitely generated over R even when R is normal. We discuss the question of finite generation of the kernel and its connection with the class group of R (3.7 and 3.8). We also give an example (3.11) to show that Theorem 3.5 is not valid if R is not normal.

In ([Rn]), Rentschler had shown that over a field K of characteristic zero the kernel of a non-zero locally nilpotent K-derivation of $K[X,Y]$ is a polynomial ring $K[F]$ where F is a variable in $K[X,Y]$. As mentioned earlier, in ([D-F]), Daigle and Freudenburg have investigated locally nilpotent R-derivations of $R[X,Y]$, where R is a U.F.D. containing a field of characteristic zero. They proved that the kernel of a non-zero locally nilpotent R-derivation of $R[X,Y]$ is a polynomial ring $R[F]$ ([D-F], 2.1) and gave a necessary and sufficient condition for F to be a variable in $R[X,Y]$ ([D-F], 2.5). In general, when R is not a U.F.D., the kernel need not be a polynomial ring over R (in fact it need not be finitely generated over R), even when R is normal, as Theorem 3.5 (quoted above) shows.

In Section 4 of our paper we generalise the result ([D-F], 2.1) in another direction —we give a necessary and sufficient condition for the kernel to be a polynomial ring in one variable over R where R is *any* noetherian domain containing a field of characteristic zero. We prove :

Theorem 4.7. *Let R be a noetherian domain containing a field of characteristic zero and let D be a non-zero irreducible locally nilpotent R-derivation of the polynomial ring $R[X,Y]$. Then the kernel A of D is a polynomial ring in one variable over R if and only if DX and DY either form a regular $R[X,Y]$-sequence or are comaximal in $R[X,Y]$. Moreover if DX and DY are comaximal in $R[X,Y]$, then $R[X,Y]$ is a polynomial ring in one variable over A.*

A crucial step in the proof of Theorem 4.7 is Proposition 4.5, which gives a necessary and sufficient condition for a singly generated R-subalgebra of $R[X,Y]$ to be an inert subring of $R[X,Y]$.

We also show (Proposition 4.11) that in the situation of Theorem 3.5 all locally nilpotent A-derivations of $R[X,Y]$ has a graded A-module structure.

In Section 2 we set up notations and quote the results which we use. In Section 3 we investigate in detail the case of noetherian normal domains and in Section 4 we discuss results over general noetherian domains.

2. Preliminaries

In this section we first set up the notations, define some of the terms used in the paper and recall their well-known properties. Finally we quote the results which will be used in the paper.

Notations. Throughout the paper we will assume our rings to be commutative. For a ring R, R^* will denote the multiplicative group of units of R. For a prime ideal P of R, $k(P)$ denotes the field R_P/PR_P. The notation $A = R^{[n]}$ will mean that A is a polynomial ring in n variables over R. For an element $F \in R[X_1, \cdots, X_n] (= R^{[n]})$, F_{X_i} denotes the partial derivative of F with respect to X_i.

Definitions. Let R be an integral domain with quotient field K. An element $F \in R[X_1, \cdots, X_n]$ is said to be a *generic variable* if $K[X_1, \cdots, X_n] = K[F]^{[n-1]}$ and a *residual variable* if for every prime ideal P of R, $k(P)[X_1, \cdots, X_n] = k(P)[\overline{F_P}]^{[n-1]}$, where $\overline{F_P}$ denotes the image of F in $k(P)[X_1, \cdots, X_n]$.

Let B be an integral domain. A derivation D of B is said to be *locally nilpotent* if for each $b \in B$, there exists a positive integer n (n may depend on b) such that $D^n(b) = 0$. The derivation D on B is said to be *irreducible* if the only principal ideal of B containing $D(B)$ is B, or, equivalently, $D = bD'$ (with $b \in B$ and D' a derivation on B) implies that $b \in B^*$.

A subring A of a domain B is said to be an *inert* subring of B if for any pair of non-zero elements $x, y \in B$, the product $xy \in A$ if and only if both $x, y \in A$.

A routine verification shows that an inert subring of a U.F.D. is a U.F.D.. If A is an inert subring of B, then A is algebraically closed in B; further if S is a multiplicatively closed subset in A then $S^{-1}A$ is an inert subring of $S^{-1}B$.

Let B be an integral domain containing a field of characteristic zero. It is well-known ([D-F], 1.1(1)) that *the kernel A of a locally nilpotent derivation D on B is an inert subring of B.* Also it is easy to see that for any multiplicatively closed subset S of $A \setminus \{0\}$, D extends to a locally nilpotent derivation of $S^{-1}B$ with kernel $S^{-1}A$ and $B \cap S^{-1}A = A$. Moreover if D is non-zero, then the transcendence degree of B over A is one ([D-F], 1.1(4)).

For an ideal I in a noetherian domain R, let $Ass_R\ (R/I) = \{P_1, \cdots, P_r\}$ and $S = R \setminus (\bigcup_{1 \le i \le r} P_i)$. Then the *$n$-th symbolic power* $I^{(n)}$ of I is defined to be the ideal $I^{(n)} = R \cap I^n(S^{-1}R)$.

We now quote the results to be used.

Lemma 2.1. *For a non-zero element a in a noetherian domain R and a multiplicatively closed subset T of R the n-th symbolic power of the ideal $I = R \cap aT^{-1}R$ is given by $I^{(n)} = R \cap a^nT^{-1}R$.*

Proof. We may assume I is a proper ideal of R. Let $aR = (N_1 \cap \cdots \cap N_r) \cap (N_{r+1} \cap \cdots \cap N_s)$ be a primary decomposition, with N_i being P_i-primary, such that $P_i \cap T = \emptyset$ precisely for $1 \le i \le r$. Then $I = (N_1 \cap \cdots N_r)$ and $aT^{-1}R = IT^{-1}R$. Moreover, $T \subseteq S$, where $S = R \setminus (\bigcup_{1 \le i \le r} P_i)$. Therefore

$$R \cap a^nT^{-1}R = R \cap I^nT^{-1}R \subseteq R \cap I^nS^{-1}R = I^{(n)}.$$

Conversely let $x \in I^{(n)}$. Then there exists $s \in S$ such that $sx \in I^n$ ($\subseteq R \cap a^nT^{-1}R$). Since $Ass_R\ (R/a^nR) = Ass_R\ (R/aR)$, we have

$$Ass_{T^{-1}R}\ (T^{-1}R/a^nT^{-1}R) = Ass_{T^{-1}R}\ (T^{-1}R/aT^{-1}R) = \{T^{-1}P_1, \cdots, T^{-1}P_r\}.$$

Now let $a^nT^{-1}R = Q_1 \cap \cdots \cap Q_r$ be a primary decomposition with Q_i being $T^{-1}P_i$-primary. Now $sx \in Q_i\ \forall i$ but $s \notin T^{-1}P_i$ for any $1 \le i \le r$. Therefore $x \in Q_i$ for every $1 \le i \le r$, and hence $x \in R \cap a^nT^{-1}R$. Hence the result. $\square$

Lemma 2.2. *Let R be a noetherian local ring such that depth $R \ge 2$ and I an ideal in R such that depth $(R/I) \ge 1$. Then depth$_R$ $I \ge 2$.*

Proof. Let k be the residue field of R. By our assumptions $Hom\ (k, R)$, $Ext^1\ (k, R)$ and $Hom\ (k, R/I)$ are all zero modules. Hence from the exact sequence

$$0 \to Hom(k, I) \to Hom(k, R) \to Hom(k, R/I) \to Ext^1(k, I) \to Ext^1(k, R)$$

it follows that $Hom\ (k, I)$ and $Ext^1\ (k, I)$ are both zero and therefore $depth\ I \geq 2$. $\square$

We now state a standard result ([B-H], 1.4.1, pg.19) which gives a criterion for a module to be reflexive.

Proposition 2.3. *Let R be a noetherian ring and M a finite R-module. Then M is reflexive if and only if*

(i) *M_P is reflexive for all $P \in Spec\ R$ with depth $R_P \leq 1$, and*
(ii) *depth $M_P \geq 2$ for all $P \in Spec\ R$ with depth $R_P \geq 2$.*

The following result on finite generation is due to Onoda ([O], 2.14, 2.20).

Theorem 2.4. *Let R be a noetherian domain and A an overdomain of R such that $A[1/x]$ is finitely generated over R for some $x(\neq 0) \in A$. Then the following statements hold:*

(i) *If S is a multiplicatively closed subset of R such that $S^{-1}A$ is finitely generated over $S^{-1}R$, then there exists $s \in S$ such that $A[1/s]$ is finitely generated over R.*
(ii) *A is finitely generated over R if and only if A_M is finitely generated over R_M for all maximal ideals M in R.*

As a consequence by ([G], 2.1) we have

Corollary 2.5. *Let R be a noetherian domain and A an R-subalgebra of a finitely generated overdomain of R. Then the statements (i) and (ii) in Theorem (2.4) hold.*

We now quote a relevant portion of a result of Rentschler ([Rn]). An alternative proof is also given in ([D-F], 1.2).

Theorem 2.6. *If K is a field of characteristic zero and D a non-zero, locally nilpotent K-derivation of $K[X, Y]$, then there exist F, G such that $K[X, Y] = K[F, G]$ and ker $D = K[F]$. Moreover there exists $\alpha \in K[F]$ such that $D = \alpha\Delta_F$, where Δ_F is the derivation defined by $\Delta_F(X) = -F_Y$ and $\Delta_F(Y) = F_X$.*

The following result has been proved by Abhyankar-Eakin-Heinzer ([A-E-H], 4.1) and Russell-Sathaye ([R-S], 3.4).

Theorem 2.7. *If $R \hookrightarrow A \hookrightarrow R^{[m]}$ are U.F.D.s such that the transcendence degree of A over R is one, then $A = R^{[1]}$.*

Corollary 2.8. *Let R be a U.F.D. and A an R-algebra which is an inert subring of $R[X_1, \cdots, X_m]$ $(= R^{[m]})$ of transcendence degree one over R. Then $A = R^{[1]}$.*

The next theorem is due to Bass-Connell-Wright ([B-C-W], 4.4).

Theorem 2.9. *Let A be a finitely presented R-algebra. Suppose that for all maximal ideals P of R, the R_P-algebra A_P is R_P-isomorphic to the symmetric algebra of some R_P-module. Then A is R-isomorphic to the symmetric algebra $Sym_R\ (M)$ of a finitely presented R-module M.*

Corollary 2.10. *Let R be a locally factorial noetherian domain and A an R-algebra which is an inert subring of $R[X_1, \cdots, X_m]$ $(= R^{[m]})$ of transcendence degree one over R. Then A is R-isomorphic to the symmetric algebra of an invertible ideal of R.*

Proof. For every maximal ideal P of R, $A_P = {R_P}^{[1]}$ by (2.8). Now by (2.5), A is finitely generated over R and hence, applying 2.9, $A \cong Sym_R\ (M)$, where M is obviously a finitely generated projective R-module of rank one and hence isomorphic to an invertible ideal of R. □

The following criterion of Russell-Sathaye follows from ([R-S], 2.3.1).

Theorem 2.11. *Let R be an integral domain and A a finitely generated overdomain of R. Suppose that there exists an element π in R which is prime in A such that $\pi A \cap R = \pi R$, $A_\pi = {R_\pi}^{[1]}$ and the image of $R/\pi R$ is algebraically closed in $A/\pi A$. Then $A = R^{[1]}$.*

The following result occurs in ([B-D], 3.2).

Theorem 2.12. *Let R be a noetherian domain such that either R contains a field of characteristic zero or R is seminormal. Then an element F in $R[X,Y]$ $(= R^{[2]})$ is a variable if and only if it is a residual variable.*

3. Locally Nilpotent Derivations over Normal Noetherian Domains

In this section we investigate in detail the kernel of locally nilpotent R-derivations of $R[X,Y]$ where R is a noetherian normal domain containing a field of characteristic zero. The main result is Theorem 3.5, where we give a precise description of the structure of the kernel of such derivations.

We first prove a patching lemma.

Lemma 3.1. *Let R be a noetherian domain and A an overdomain of R such that $JA \cap R = J$ for every ideal J of R. Suppose that there exist non-zero elements $x, y \in R$ satisfying the conditions:*

(i) *x and y form an R-sequence.*
(ii) *$A_x = {R_x}^{[1]}$.*
(iii) *$A_y = {R_y}^{[1]}$.*
(iv) *$A = A_x \cap A_y$.*

Then A has a graded ring structure $\bigoplus_{n\geq 0} A_n$ with $A_0 = R$ and A_n a finite type reflexive R-module for all n. In fact A is isomorphic as a graded R-algebra to the symbolic Rees algebra $\bigoplus_{n\geq 0} I^{(n)}T^n$ of a reflexive ideal I in R of height one.

Remark 3.2. Let $R \subseteq A \subseteq B$ be integral domains such that B is faithfully flat over R. Then $JA \cap R = J$ for every ideal J of R.

Proof of 3.1. Let $A_x = R_x[U]$ and $A_y = R_y[V]$ where $U, V \in A$. Since $A_{xy} = R_{xy}[U] = R_{xy}[V]$, it is easy to see that

$$V = (aU + c)/x^m \text{ for some } c \in R,\ a \in R \cap {R_{xy}}^*,\ m \in \mathbf{Z}^+.$$

Now $c = x^m V - aU \in (x^m, a)A \cap R = (x^m, a)R$, i.e., $c = x^m v - au$ for some $u, v \in R$. Let $F = U - u$ and $G = V - v$. Then

$$A_x = R_x[F], A_y = R_y[G] \text{ with } G = \lambda F \text{ where } \lambda = a/x^m \in {R_{xy}}^*.$$

Now

$$A_x = \bigoplus_{n\geq 0} R_x F^n \text{ and } A_y = \bigoplus_{n\geq 0} R_y G^n.$$

Since

$$A_{xy} = \bigoplus_{n\geq 0} (R_x F^n)_y = \bigoplus_{n\geq 0} (R_y G^n)_x$$

where

$$(R_x F^n)_y = R_{xy} F^n = R_{xy} G^n = (R_y G^n)_x,$$

it follows that

$$A = A_x \cap A_y = \bigoplus_{n\geq 0} A_n \text{ where } A_n = R_x F^n \cap R_y G^n.$$

Thus A is a graded R-algebra with $A_0 = R_x \cap R_y = R$ by (i).

We will now show that A_1 is R-isomorphic to an ideal I of R such that $A_n \cong I^{(n)}$ as R-modules for every n, which will show in particular that each A_n is a finite R-module.

If $a \in {R_y}^*$, then putting $H = G/a = F/x^m$, it is easy to see that $A = A_x \cap A_y = R[H]$ $(= R^{[1]})$, and we are through. So we may assume that aR_y is a proper ideal and let

$$I = R \cap aR_y.$$

Then $I_y = aR_y$, and hence I is an ideal of height one.

We now show that for every n, $A_n = I^{(n)} \cdot (G^n/a^n)$ as R-modules. Let $f \in A_n$. Then

$$f = bG^n = (ba^n/x^{mn})F^n = cF^n/x^{mn} = c(G^n/a^n)$$

where

$$b \in R_y \text{ and } c = a^n b \in a^n R_y \cap R_x = a^n R_y \cap R = I^{(n)}$$

by (2.1). Conversely if $c \in I^{(n)}$, then by (2.1), $c = a^n b$ for some $b \in R_y$. Thus $c(G^n/a^n) = bG^n \in R_y G^n$; on the other hand $c(G^n/a^n) = (c/x^{mn})F^n \in R_x F^n$, so that $cG^n/a^n \in R_y G^n \cap R_x F^n = A_n$. Thus $A_n = I^{(n)} \cdot (G^n/a^n)$ and the map $\phi_n : c(G^n/a^n) \to c$ gives an R-isomorphism of A_n onto $I^{(n)}$.

Using the above isomorphism we now check that A_n is a reflexive R-module for every n. Since $A_0 = R$ is obviously reflexive we assume that $n \geq 1$. Let $P \in \mathit{Spec}\, R$.

If at least one of the elements x, y does not belong to P then by condition (ii) or (iii) it follows easily that A_{nP} is a free R_P-module; in particular, A_{nP} is reflexive and $\mathit{depth}\ A_{nP} = \mathit{depth}\ R_P$.

If both $x, y \in P$, then by hypothesis, they form an R-sequence so that $\mathit{depth}\ R_P \geq 2$. It follows that in this situation PR_P cannot be an associated prime ideal of $R_P/a^n R_P$ and hence $PR_P \notin \mathit{Ass}_{R_P}\,(R_P/{I^{(n)}}_P)$, which shows that $\mathit{depth}\ (R_P/{I^{(n)}}_P) \geq 1$. Now since $A_n \cong I^{(n)}$, by (2.2), it follows that $\mathit{depth}\ A_{nP} \geq 2$.

Thus we see that $\mathit{depth}\ A_{nP} \geq 2$ whenever $\mathit{depth}\ R_P \geq 2$ (irrespective of whether $x, y \in P$ or not). On the other hand if $\mathit{depth}\ R_P \leq 1$, then at least one of x, y does not belong to P, and in this case A_{nP} is seen to be reflexive. Therefore by (2.3), A_n is reflexive.

Now consider the injective map $A_y (= R_y[G]) \hookrightarrow R_y[T] (= {R_y}^{[1]})$ defined by $G \to aT$. Then since $A_n = I^{(n)} \cdot (G^n/a^n)$, the image of $A (= A_x \cap A_y = \bigoplus_{n\geq 0} A_n)$ in $R_y[T]$ is precisely the symbolic Rees algebra $\bigoplus_{n\geq 0} I^{(n)} T^n$ $(\hookrightarrow R[T])$. $\square$

As a consequence of Lemma 3.1, we now give a description of the graded R-algebra structure of an inert subring of the polynomial ring $R[X_1, \cdots, X_m]$ of transcendence degree one over R, when R is a noetherian normal domain.

Proposition 3.3. *Let R be a noetherian normal domain and A an R-algebra which is an inert subring of $R[X_1, \cdots, X_m]$ $(= R^{[m]})$ of transcendence degree one over R. Then A has the structure of a graded ring $\bigoplus_{n\geq 0} A_n$ with $A_0 = R$ and A_n a finite reflexive R-module of rank one for every n. In fact, when R is not a field, there exists an ideal I of unmixed height one in R such that A is isomorphic to the symbolic Rees algebra $\bigoplus_{n\geq 0} I^{(n)}T^n$ as a graded R-algebra.*

Proof. If R is a field or a Dedekind domain then the result follows immediately from (2.8) or (2.10) respectively.

So we assume $dim\ R \geq 2$. Since $A \subseteq R[X_1, \cdots, X_m]$, $JA \cap R = J$ for every ideal J of R. Therefore by (3.1), it is enough to show that there exist elements $x, y \in R$ which form an R-sequence such that $A_x = {R_x}^{[1]}$, $A_y = {R_y}^{[1]}$ and $A = A_x \cap A_y$.

Let $T = R \setminus \{0\}$ and let K denote the quotient field of R. Then clearly $T^{-1}A$ is an inert subring of $K[X_1, \cdots, X_m]$ of transcendence degree one over K, and hence by (2.8), $T^{-1}A = K^{[1]}$. Now since $A \subseteq R[X_1, \cdots, X_m]$, by (2.5), there exists an element $t \in T$ such that $A[1/t]$ is finitely generated. From this and the fact that $T^{-1}A = K^{[1]}$ it follows easily that there exists $x \in T$ such that

$$A_x = {R_x}^{[1]}.$$

If $x \in R^*$, then clearly $A = R^{[1]}$ and all the statements in the theorem follow trivially. So hence onward we assume that xR is a proper ideal in R. Let $Ass_R\ (R/xR) = \{P_1, \cdots, P_r\}$. Since R is a noetherian normal domain, $ht(P_i) = 1$ for all $1 \leq i \leq r$. Let $S = R \setminus (\bigcup_{1\leq i\leq r} P_i)$. Then clearly $S^{-1}R$ is a P.I.D. and $S^{-1}A$ is an inert subring of $(S^{-1}R)[X_1, \cdots, X_m]$ of transcendence degree one over $S^{-1}R$. Therefore, by (2.8), $S^{-1}A = (S^{-1}R)^{[1]}$. By (2.5) there exists $s \in S$ such that $A[1/s]$ is finitely generated over R. Hence from the above equation it follows easily that there exists $y \in S$ such that

$$A_y = {R_y}^{[1]}.$$

By construction the pair x, y form a regular sequence in R. We now show that $A = A_x \cap A_y$. Let

$$c = a/x^j = b/y^\ell \in A_x \cap A_y;\ a, b \in A;\ j, \ell \in \mathbf{Z}^+.$$

As x, y form a regular sequence in R and hence in $R[X_1, \cdots, X_m]$, the equation $y^\ell a = x^j b$ yields that $a \in x^j R[X_1, \cdots, X_m]$. Now since A is an inert subring of $R[X_1, \cdots, X_m]$, it follows that $A \cap x^j R[X_1, \cdots, X_m] = x^j A$. Therefore $c = a/x^j \in A$, showing that

$$A = A_x \cap A_y.$$

This completes the proof. $\square$

Remark 3.4. Let I be an ideal of unmixed height one in a noetherian normal domain R. Then I has a primary decomposition of the form $I = P_1^{(n_1)} \cap \cdots \cap P_r^{(n_r)}$. Let

$[I]$ denote the element $\sum_{1\le i\le r} n_i[P_i]$ in $Cl\ (R)$. Now a routine verification shows that for any two unmixed ideals I and J of height one in R, the following statements are equivalent.

(i) $I \cong J$ as R-modules.
(ii) $\bigoplus_{n\ge 0} I^{(n)}T^n \cong \bigoplus_{n\ge 0} J^{(n)}T^n$ as graded R-algebras.
(iii) $[I]=[J]$ in $Cl\ (R)$.

Thus in Proposition 3.3, given any inert subring A, the choice of I is unique up to its image in the class group (in the above sense). Since for a U.F.D., $Cl\ (R)=0$, the above observation gives another way of explaining the result (2.8) and hence the result ([D-F], 2.1).

We now prove our main theorem on the structure of the kernel of a non-zero locally nilpotent R-derivation of $R[X,Y]$ over a normal domain R containing a field of characteristic zero.

Theorem 3.5. *Let R be a noetherian normal domain containing a field of characteristic zero and let D be a non-zero locally nilpotent R-derivation of the polynomial ring $R[X,Y]$ with kernel A. Then A has the structure of a graded ring $\bigoplus_{n\ge 0} A_n$ with $A_0=R$ and A_n a finite reflexive R-module of rank one for every n. In fact, when R is not a field, there exists an ideal I of unmixed height one in R such that A is isomorphic to the symbolic Rees algebra $\bigoplus_{n\ge 0} I^{(n)}T^n$ as a graded R-algebra.*

Conversely let R be as above and let I be an unmixed ideal of height one in R. Let B be the symbolic Rees algebra $\bigoplus_{n\ge 0} I^{(n)}T^n$. Then there exists a locally nilpotent R-derivation D of $R[X,Y]$ whose kernel is isomorphic to B as a graded R-algebra. In particular B can be embedded as an inert subring of $R[X,Y]$.

Proof. Since the kernel A is an inert subring of $R[X,Y]$ of transcendence degree one over R, the first part of the theorem follows from Proposition 3.3.

We now prove the converse statement. Let $Ass_R\ (R/I)=\{P_1,\cdots,P_r\}$ and $S=R\setminus(\bigcup_{1\le i\le r} P_i)$. Then $S^{-1}R$ is a P.I.D. and hence $IS^{-1}R$ is principal. Therefore we can choose an element $y\in S$ such that IR_y is principal, say, $IR_y=aR_y$ for some $a\in I$. By (2.1) it would follow that $I^{(n)}R_y=a^nR_y\ \forall n\ge 0$. Therefore $B_y=R_y[g]$, where $g=aT$. If $y\in R^*$, then I is principal, so that $B=R^{[1]}$ and we are through (for instance we may consider the R-derivation D defined by $DX=Y$ and $DY=0$). So we assume $y\notin R^*$.

Let $Ass_R\ (R/yR)=\{Q_1,\cdots,Q_\ell\}$. Since $ht\ Q_j=1\ \forall 1\le j\le \ell$ and $y\notin P_i$ for any i, $P_i\not\subseteq Q_j$ for any i,j and hence $I\not\subseteq(\bigcup_{1\le j\le \ell} Q_j)$. Choose $x\in I\setminus(\bigcup_{1\le j\le \ell} Q_j)$. By construction x and y form an R-sequence, and since $IR_x=R_x=xR_x$, $B_x=R_x[f]$ where $f=xT$. Clearly $g=(a/x)f$ and $a\in R^*_{xy}$. Moreover, since $x\in I$ and $I_y=aR_y$, we have

$$x=(u/y^m)a \text{ for some } u\in R\cap(R_{xy})^*,\ m\in \mathbf{Z}^+.$$

We now show that $B = B_x \cap B_y$. Let $b = x^p c = y^q d \in x^p B \cap y^q B$, where $p, q \in \mathbf{Z}^+$ and $c, d \in B$. Let $c = \sum_j c_j T^j$ and $d = \sum_j d_j T^j$, where $c_j, d_j \in I^{(j)}$ $\forall j \geq 0$. Then $x^p c_j = y^q d_j$ $\forall j$. Since x, y form a regular sequence in R and y is a non-zero divisor in R/I, it follows that $c_j \in y^q R \cap I^{(j)} = y^q I^{(j)}$ $\forall j$, showing that $b \in x^p y^q B$. Thus $x^p B \cap y^q B = x^p y^q B$ and hence $B = B_x \cap B_y$.

Now define an R-derivation D on $R[X,Y]$ by

$$DX = -u, \ DY = x.$$

Then $D^2 X = 0 = D^2 Y$, so that D is locally nilpotent. Let $A = Ker\ (D)$. We shall show that the graded R-algebras B and A are isomorphic, which will complete the proof.

Note that D induces locally nilpotent derivations D_x (resp. D_y) on $R_x[X,Y]$ (resp. $R_y[X,Y]$) with kernels A_x (resp. A_y). Now let

$$F = xX + uY, \ G = aX + y^m Y.$$

Then $F, G \in A$ and $G = aF/x$ (since $xy^m = au$). Since

$$R_x[F] \subseteq A_x \subseteq R_x[X,Y] \ (= R_x[F]^{[1]})$$

and the transcendence degree of $R_x[X,Y]$ over A_x is one, it follows that $A_x = R_x[F]$. Similarly $A_y = R_y[G]$. Also note that since x, y form a regular R-sequence and A is an inert subring of $R[X,Y]$, we have (as in the proof of Proposition 3.3) $A = A_x \cap A_y$.

Now let $\phi : B_x (= R_x[f]) \xrightarrow{\cong} A_x (= R_x[F])$ be the R_x-isomorphism defined by $f \to F$ and $\psi : B_y (= R_y[g]) \xrightarrow{\cong} A_y (= R_y[G])$ be the R_y-isomorphism defined by $g \to G$. In A_{xy}, we have $\phi(g) = \phi(af/x) = aF/x = G = \psi(g)$. Thus both ϕ and ψ induce the same isomorphism $B_{xy} \xrightarrow{\cong} A_{xy}$. Hence their restrictions induce an isomorphism $B (= B_x \cap B_y) \xrightarrow{\cong} A (= A_x \cap A_y)$. Hence the result. $\square$

Note that if R is a locally factorial domain (for instance if R is regular) containing a field of characteristic zero, then by (2.10), the kernel A of a locally nilpotent R-derivation of $R[X,Y]$ is finitely generated over R. However, from Theorem 3.5 it follows that in general the kernel need not be finitely generated (even when R is normal), as the following example illustrates.

Example 3.6. Let C be a non-singular elliptic curve in $\mathbf{P}^2_{\mathbf{C}}$ defined by a homogeneous irreducible polynomial F in $\mathbf{C}[X,Y,Z]$ (for instance, take $F = (Y^2 Z - X^3 + XZ^2)$). Let $R = \mathbf{C}[X,Y,Z]_{(X,Y,Z)}/(F)$. Then R is a two-dimensional normal local domain whose class group is not torsion. Therefore there exists a prime ideal P in R of height one such that $P^{(n)}$ is not a principal ideal (for each $n \geq 1$). Hence the symbolic Rees algebra $B = \bigoplus_{n \geq 0} P^{(n)} T^n$ is not finitely generated over R (for the proof see [R] or [C]). Now by Theorem 3.5 there exists a locally nilpotent R-derivation of $R^{[2]}$ whose kernel is isomorphic to B and hence is not finitely generated over R. $\square$

However, in the situation of Theorem 3.5, if the group $Cl\ (R)/Pic\ (R)$ is torsion, then for any unmixed height one ideal I of R, $I^{(\ell)}$ would be an invertible ideal for some ℓ. Then it is easy to see that $I^{(t\ell)} = (I^{(l)})^t$ and hence $I^{(m+tl)} = I^{(m)}(I^{(l)})^t$ $\forall\ 0 \leq m \leq l-1$. Thus $\bigoplus_{n \geq 0} I^{(n)}$ is finitely generated. Therefore by Theorem 3.5 we have

Corollary 3.7. *Let R be a noetherian normal domain containing a field of characteristic zero such that the group $Cl\ (R)/Pic\ (R)$ is torsion. Then the kernel of any locally nilpotent R-derivation of $R[X,Y]$ is finitely generated over R.*

Using Theorem 3.5, we give below an example to show that the above condition is not necessary for the kernel of every locally nilpotent R-derivation of $R[X,Y]$ to be finitely generated over R.

Example 3.8. Let $R = \mathbf{C}[X,Y,Z,W]/(XY - ZW)$, $P = (\overline{X}, \overline{Z})R$ and $Q = (\overline{X}, \overline{W})R$. Then it is well known that $Pic\ (R) = 0$ and $Cl\ (R) = \mathbf{Z}$ and is generated by $[P](= -[Q])$.

Let B be the associated graded ring $\bigoplus_{n\geq 0} (P^n/P^{n+1})$. Consider the $\mathbf{C}[Y,W]$-algebra epimorphism $\phi : \mathbf{C}[Y,W][U,V]/(YU - WV) \to B$ defined by $U \to \overline{X}(mod\ P^2)$ and $V \to \overline{Z}(mod\ P^2)$. We first prove that ϕ is an isomorphism, which will show in particular that B is an integral domain. Let J be the ideal $\bigoplus_{n\geq 1} (P^n/P^{n+1})$ in B and let S be the multiplicatively closed subset $R/P \setminus \{0\}$ in B. Then it is easy to see that $S^{-1}B = \mathbf{C}(Y,W)^{[1]}$ with $S^{-1}J$ as a maximal ideal, showing that $ht\ J = 1$. Now since $dim\ (R/P) = 2$, it follows that $dim\ B = 3$, and hence ϕ is an isomorphism.

Since B is an integral domain, a routine induction argument would show that P^n is P-primary, i.e., $P^{(n)} = P^n$. Similarly one can check that $Q^{(n)} = Q^n$. Thus P^n and Q^n are unmixed ideals of height one in R with $[P^n] = n[P]$ and $[Q^n] = n[Q]$ in $Cl\ (R)$.

Now let I be an unmixed ideal of height one in R. Then $[I] = n[P]$ or $n[Q]$ for some non-negative integer n, say $[I] = n[P](= [P^n])$. Therefore $I \cong P^n$. Hence there exists $f \in K^*$ such that $If = P^n$. Therefore $I^m f^m = P^{nm} = (P^n)^{(m)} = I^{(m)} f^m$, so that $I^m = I^{(m)}\ \forall m$. Thus by Theorem 3.5 the kernel of any non-zero locally nilpotent R-derivation of $R^{[2]}$ is R-isomorphic to the Rees algebra of an ideal I (of unmixed height one) in R, and hence is finitely generated over R. $\square$

Remark 3.9. Let R be a noetherian normal domain. The group $Cl\ (R)/Pic\ (R)$ is torsion if and only if $Cl\ (R_M)$ is torsion for all maximal ideals M of R.

Remark 3.10. In the above example $dim\ R = 3$. However if R is a noetherian normal domain of dim 2, then by a result of Cowsik ([C]), Theorem 3.5 and 2.5 it would follow that the condition that $Cl\ (R)/Pic\ (R)$ is torsion is indeed necessary for the kernel of every locally nilpotent R-derivation of $R[X,Y]$ to be finitely generated.

We now give an example to show that Proposition 3.3 and Theorem 3.5 are not valid if R is not normal. Note that if R is a one-dimensional noetherian domain, then the symbolic power $I^{(n)}$ of a non-zero ideal I in R obviously coincides with I^n, and hence the symbolic Rees algebra $\bigoplus_{n\geq 0} I^{(n)}T^n$ is finitely generated over R. However the following example shows that when R is a one-dimensional noetherian domain (containing $\mathbf{Q}$) then the kernel of a non-zero locally nilpotent R-derivation of $R[X,Y]$ need not be finitely generated if R is not normal; in particular, it need not be of the form $\bigoplus_{n\geq 0} I^{(n)}T^n$.

Example 3.11. Let $R = \mathbf{R} + (t)\mathbf{C}[[t]]$. R is a noetherian local domain with maximal ideal $M = (t)\mathbf{C}[[t]]$. Let $\overline{R}$ be the normalisation $\mathbf{C}[[t]]$ of R. Define a locally nilpotent $\overline{R}$-derivation $\overline{D}$ of $\overline{R}[X, Y]$ by

$$\overline{D}(X) = it \text{ and } \overline{D}(Y) = -t.$$

It is easy to see that *ker* $(\overline{D}) = \overline{R}[X + iY]$. Now $\overline{D}$ restricts to a locally nilpotent R-derivation D of $R[X, Y]$ with kernel A, and it is easy to see that

$$A = \mathbf{R} + M\overline{R}[X + iY].$$

We now show that A is not even noetherian, and hence is not finitely generated over R. Since $\overline{R}[X+iY]/M\overline{R}[X+iY]$ $(= \mathbf{C}^{[1]})$ is not finitely generated as a module over $A/M\overline{R}[X + iY]$ $(= \mathbf{R})$, clearly $\overline{R}[X + iY]$ cannot be a finite A-module, and hence for any $f(\neq 0) \in M$, $f\overline{R}[X + iY]$ $(\subseteq A)$ is not finitely generated as an ideal in A. Thus A is not noetherian. □

In the next section we shall show (Proposition 4.11) that all locally nilpotent R-derivations of $R[X, Y]$ over a normal domain R (containing a field of characteristic zero) with a fixed kernel A together with the zero derivation has a graded A-module structure.

4. Locally Nilpotent Derivations over General Noetherian Domains

In this section we shall give a necessary and sufficient condition for the kernel of an irreducible locally nilpotent R-derivation of the polynomial ring $R[X, Y]$ to be $R^{[1]}$ (Theorem 4.7). The crucial step in the proof is Proposition 4.5. Before that we prove some lemmas.

To avoid the tedium of repetition we shall hence onwards assume that R *denotes a noetherian domain containing a field of characteristic zero and* K *denotes the quotient field of* R.

Lemma 4.1. *For any* $F \in K[X, Y] \setminus K$, $(F_X, F_Y)K[X, Y] \cap K[F] \neq (0)$.

Proof. Let $S = K[F] \setminus \{0\}$, $L = S^{-1}K[F]$ $(= K(F))$ and $C = S^{-1}K[X, Y]$. We have the exact sequence

$$\Omega_{L/K} \otimes_L C \xrightarrow{\sigma} \Omega_{C/K} \to \Omega_{C/L} \to 0.$$

Since L is a perfect field, C is smooth and hence $\Omega_{C/L}$ is a projective C-module of rank one ([A-K], pp. 159-162). Hence the exact sequence

$$0 \to \text{ Im } (\sigma) \to \Omega_{C/K} \to \Omega_{C/L} \to 0$$

splits. Now as $\Omega_{C/K}$ is a free C-module of rank two with basis dX and dY and Im (σ) is generated by $F_X dX + F_Y dY$, the elements F_X and F_Y are comaximal in C and hence $(F_X, F_Y)K[X, Y] \cap K[F] \neq (0)$. □

Lemma 4.2. *For any* $F \in K[X, Y] \setminus K$, *if* $K[F]$ *is an inert subring of* $K[X, Y]$, *then the ideal* $(F_X, F_Y)K[X, Y]$ *is not contained in any proper principal ideal of* $K[X, Y]$.

Proof. If possible let $(F_X, F_Y)K[X, Y] \subseteq pK[X, Y]$ where p is a prime element of $K[X, Y]$. By (4.1), $pK[X, Y] \cap K[F] \neq (0)$ and hence it is generated by a non-zero irreducble element $\phi(F)$ of $K[F]$. The inertness condition and the irreducibility of $\phi(F)$ imply a relation $p = u\phi(F)$ for some $u \in K^*$. But that would imply that F_X

and F_Y both belong to the ideal $\phi(F)K[X,Y]$, which is absurd by simple degree considerations. Hence the result. $\square$

Lemma 4.3. *Let $F \in (X,Y)R[X,Y]$ and I be an ideal of R. Then $F \in IR[X,Y]$ if and only if both F_X and F_Y are in $IR[X,Y]$.*

Proof. Let $F = \sum_{i,j} a_{ij}X^iY^j$, $a_{ij} \in R$. Note that $a_{00} = 0$. Since R contains $\mathbf{Q}$, F_X $(= \sum_i (\sum_j a_{ij}Y^j)iX^{i-1})$ is an element of $IR[X,Y]$ if and only if $a_{ij} \in I$ for all (i,j) for which $i > 0$. Similarly $F_Y \in IR[X,Y]$ if and only if $a_{ij} \in I$ for all (i,j) for which $j > 0$. Hence the result. $\square$

Proposition 4.4. *Let F be a generic variable in $R[X,Y]$. Then $R[X,Y] = R[F]^{[1]}$ if F_X, F_Y are comaximal in $R[X,Y]$.*

Proof. By (2.12) it is enough to show that F is a residual variable in $R[X,Y]$. Let $P \in \mathit{Spec}\ R$ and $\overline{F}$ the image of F in $k(P)[X,Y]$. Without loss of generality we may assume that R is local with maximal ideal P and residue field k $(= R/P)$, and prove that $k[X,Y] = k[\overline{F}]^{[1]}$ by induction on $ht\ P$ $(= dim\ R) = d$, say. If $d = 0$, there is nothing to prove.

Let $d = 1$. Now since R is a one-dimensional noetherian local domain, by the Krull-Akizuki theorem ([N], p.115) it is easy to see that there exists a discrete valuation ring (C, π) such that $R \subseteq C \subseteq K$ and the residue field $L = C/\pi$ is finite over k. We first show that F is a variable in $C[X,Y]$. Since F is a generic variable in $C[X,Y]$, by (2.11) it is enough to show that $L[\overline{F}]$ is algebraically closed in $L[X,Y]$. But the algebraic closure of $L[\overline{F}]$ in $L[X,Y]$ is clearly of the form $L[G]$, and by the comaximality assumption of F_X, F_Y it follows easily that $\overline{F}$ is linear in G, showing that $L[\overline{F}] = L[G]$ is algebraically closed in $L[X,Y]$. Thus $C[X,Y] = C[F]^{[1]}$, and hence $L[X,Y] = L[\overline{F}]^{[1]}$. Now L being finite separable over k, it follows that $k[X,Y] = k[\overline{F}]^{[1]}$.

Now the case $d \geq 2$ follows by an easy induction argument. $\square$

Proposition 4.5. *For an element F in $R[X,Y] \setminus R$, the following statements are equivalent :*

(i) *$R[F]$ is an inert subring of $R[X,Y]$.*

(ii) *$K[F]$ is an inert subring of $K[X,Y]$ and the ideal $(F_X, F_Y)R[X,Y]$ is either the unit ideal or has grade 2.*

Proof. (i)⇒(ii). Suppose (i) holds. Then a routine verification shows that $K[F]$ is an inert subring of $K[X,Y]$.

Without loss of generality we may assume that $F \in (X,Y)R[X,Y]$. If $F_X \in R^*$, then we are through. If $F_X = 0$, then $F \in YR[X,Y]$, and hence by the inertness condition it follows easily that $F = uY$ for some $u \in R^*$, i.e., $F_Y \in R^*$, and we are through. Thus we may assume that F_X is a non-zero non-unit element of $R[X,Y]$ and show that F_Y is a non-zero divisor in $R[X,Y]/F_XR[X,Y]$. Let $Q \in Ass_{R[X,Y]}(R[X,Y]/F_XR[X,Y])$. Then $depth\ (R[X,Y]_Q) = 1$. It suffices to show that $F_Y \notin Q$.

Let $P = Q \cap R$. If $P = (0)$, then $QK[X,Y] \in Ass_{K[X,Y]}(K[X,Y]/F_XK[X,Y])$, and hence is a prime ideal of height one and therefore a principal prime ideal. Hence by (4.2), $F_Y \notin QK[X,Y]$, and we are through.

Now suppose that $P \neq (0)$ and let a be a non-zero element of P. Since *depth* $(R[X,Y]_Q) = 1$, $Q \in Ass_{R[X,Y]}(R[X,Y]/aR[X,Y])$; and since $a \in R$, Q is extended from R. Thus $PR[X,Y] = Q \in Ass_{R[X,Y]}(R[X,Y]/aR[X,Y])$. Hence $P \in Ass_R(R/aR)$, i.e., there exists $b \in R \setminus aR$ such that $bP \subseteq aR$, and therefore

$$bQ \subseteq aR[X,Y],\ b \in R \setminus aR.$$

Now if $F \in Q$, then by the above relation it would follow that $bF = aG$ for some $G \in R[X,Y]$. Since $bF \in R[F]$, condition (i) would imply that $G \in R[F]$ and hence $G = cF$ for some $c \in R$. But this relation would imply $b(= ac) \in aR$, contradicting the choice of b. Thus $F \notin Q$, and hence by (4.3), $F_Y \notin Q$.

(ii) $\Rightarrow$ (i) Assume (ii) holds. Since $K[F]$ is an inert subring of $K[X,Y]$, it is enough to prove that $R[X,Y] \cap K[F] = R[F]$, i.e., to show that $cR[X,Y] \cap R[F] = cR[F]\ \forall c \in R$. Thus the proof will be complete if we prove the following claim.

Claim. If $(F_X, F_Y)R[X,Y]$ is either the unit ideal or has grade 2, then $cR[X,Y] \cap R[F] = cR[F]\ \forall c \in R$.

Proof of the Claim. Let $G \in R[X,Y]$ and $\phi(F) = \sum_{0 \le i \le n} a_i F^i$, $a_i \in R$, be such that $cG = \phi(F)$. To prove the claim it is enough to show that $a_i \in cR\ \forall i$.

We first show that the first derivative $\phi'(F) \in cR[X,Y]$. Now

$$(*) \qquad cG_X = \phi'(F)F_X \text{ and } cG_Y = \phi'(F)F_Y.$$

Let $cR[X,Y] = \bigcap N_j$ be a primary decomposition of $cR[X,Y]$ and let P_j be the associated prime ideal of $R[X,Y]/N_j$. Now, $P_jR_{P_j}$ being the associated prime ideal of $R[X,Y]_{P_j}/cR[X,Y]_{P_j}$, *depth* $(R[X,Y]_{P_j}) = 1$. Hence from the given conditions on F_X and F_Y, at least one of them becomes unit in $R[X,Y]_{P_j}$, i.e., at least one of them does not belong to P_j. Hence by $(*)$ we have $\phi'(F) \in N_j$. Since this would hold for every j, we have $\phi'(F) \in cR[X,Y]$.

The above argument shows, by induction, that the m-th derivative $\phi^{(m)}(F) \in cR[X,Y]\ \forall m,\ 1 \le m \le n$. In particular c divides $\phi^{(n)}(F)$ and as $\mathbf{Q} \hookrightarrow R$, it shows that $a_n \in cR$.

Let $\phi_r(F) = \sum_{0 \le i \le n-r} a_i F^i$. By an easy inductive argument as above it follows that c divides a_r, the leading coefficient of ϕ_r for every r, $0 \le r \le n$. Hence the result. $\square$

Corollary 4.6. *An inert subring A of $R[X,Y]$ of transcendence degree one over R is isomorphic to $R^{[1]}$ if and only if there exists an element $F \in A$ such that F_X and F_Y either form a sequence or are comaximal in $R[X,Y]$.*

We now prove the main result of this section.

Theorem 4.7. *Let D be a locally nilpotent R-derivation of $R[X,Y]$ and let A denote the kernel of D. Then the following statements are equivalent:*

(i) *D is irreducible and $A = R^{[1]}$.*

(ii) *DX and DY either form an $R[X,Y]$-sequence or are comaximal in $R[X,Y]$.*

Moreover if DX and DY are comaximal in $R[X,Y]$, then $R[X,Y] = A^{[1]}$.

Proof. (i) $\Rightarrow$ (ii). Let $A = R[F]$. Since $R[F]$ is an inert subring of $R[X,Y]$, by (4.5), F_X and F_Y either form a sequence or are comaximal. Hence from the equation

$$F_X DX + F_Y DY = DF = 0$$

we conclude that

$$DX = uF_Y \text{ and } DY = -uF_X \text{ for some } u \in R[X,Y].$$

Since D is irreducible it follows that $u \in R^*$. Hence DX and DY either form a sequence or are comaximal (since F_X and F_Y have the same property).

(ii) $\Rightarrow$ (i). The irreducibility of D is obvious.

Now D induces a non-zero locally nilpotent derivation on $K[X,Y]$ with kernel $S^{-1}A$, where $S = R \setminus \{0\}$. By (2.6), $S^{-1}A = K[H]$, where H is a variable in $K[X,Y]$. We may choose the above H to be in the ideal $(X,Y)R[X,Y]$. From the equation

$$H_X DX + H_Y DY = DH = 0$$

and from condition (ii), it follows that

$$H_X = gDY \text{ and } H_Y = -gDX \text{ for some } g \in R[X,Y].$$

Since H is a variable in $K[X,Y]$, it follows that

$$g \in K^* \cap R[X,Y] = R \setminus \{0\}.$$

Hence by (4.3), it is easy to see that

$$H = gF, \text{ where } F \in A \text{ and } DX = -F_Y, DY = F_X.$$

Thus $K[F]$ $(= K[H])$ is an inert subring of $K[X,Y]$ with the property that F_X and F_Y either form a sequence or are comaximal. Hence by (4.5), $R[F]$ is an inert subring of $R[X,Y]$. Now since A is contained in the quotient field of $R[F]$, it follows easily that $A = R[F]$.

If DX, DY (and hence F_X, F_Y) are comaximal in $R[X,Y]$, then since F is a generic variable, by (4.4) it follows that $R[X,Y] = R[F]^{[1]} = A^{[1]}$. $\square$

Remark 4.8. Note that when R is a U.F.D., any irreducible locally nilpotent R-derivation of $R[X,Y]$ obviously satisfies condition (ii) in Theorem 4.7 and any locally nilpotent derivation is a multiple of an irreducible locally nilpotent derivation. Therefore when R is noetherian the result ([D-F], 2.1) follows from Theorem 4.7. The above theorem also shows that the result ([D-F], 2.5), which was proved for a U.F.D., is true when R is any noetherian domain (containing a field of characteristic zero).

Corollary 4.9. *The following statements are equivalent.*

(i) *D is a locally nilpotent derivation of $R[X,Y]$ with kernel $R[F](= R^{[1]})$.*

(ii) *$D = \alpha\Delta_F$, where F is a generic variable in $R[X,Y]$ such that F_X, F_Y either form a sequence or are comaximal in $R[X,Y]$, $\alpha \in R[F] \setminus \{0\}$ and Δ_F is the derivation defined by $\Delta_F(X) = -F_Y$ and $\Delta_F(Y) = F_X$.*

The sum of two locally nilpotent derivations need not be locally nilpotent. For instance, define D_1, D_2 on $R[X,Y]$ by $D_1(X) = 0, D_1(Y) = X$ and $D_2(X) = Y, D_2(Y) = 0$. Then $(D_1 + D_2)^n\ (X+Y) = X + Y\ \forall n$. However we can make the following observation.

Lemma 4.10. *Let A be the kernel of a non-zero locally nilpotent R-derivation of $R[X,Y]$. Then any non-zero locally nilpotent A-derivation of $R[X,Y]$ has kernel A. Moreover the set of all locally nilpotent A-derivations of $R[X,Y]$ has an A-module structure.*

Proof. Let M be the set of all locally nilpotent A-derivations of $R[X,Y]$. Let $D(\neq 0) \in M$. Now $A \subseteq ker\ D \subseteq R[X,Y]$. Since $R[X,Y]$ has the same transcendence degree over both $ker\ D$ and A, $ker\ D$ is algebraic over A. But A is algebraically closed in $R[X,Y]$. Hence $ker\ D = A$.

Let $S = R \setminus \{0\}$. Now any non-zero $D \in M$ extends to a locally nilpotent K-derivation $\tilde{D}$ of $K[X,Y]$ with kernel $S^{-1}A$ and, by (2.6), $S^{-1}A = K[F]$ for some element F in $R[X,Y]$. Thus again by (2.6), $\tilde{D} = \alpha\Delta_F$ for some non-zero $\alpha \in K[F](= S^{-1}A)$. From this description it clearly follows that if $D_1, D_2 \in M$, then $D_1 + D_2$ is locally nilpotent and hence belongs to M. Also it is easy to see that if $D \in M$ and $a \in A$, then $aD \in M$. Thus M is an A-module. □

If the kernel A of some non-zero locally nilpotent R-derivation of $R[X,Y]$ is $R^{[1]}$ (for instance if R is a U.F.D.), then by (4.9) and (4.10), all locally nilpotent A-derivations of $R[X,Y]$ has an A-module structure isomorphic to any principal ideal of A. We now give a description of the A-module structure of all locally nilpotent A-derivations of $R[X,Y]$ when R is normal.

Proposition 4.11. *Let R be a noetherian normal domain and let A be the kernel of a non-zero locally nilpotent R-derivation of $R[X,Y]$. Then A is a graded R-algebra $\bigoplus_{n\geq 0} A_n$ isomorphic to the symbolic Rees algebra of an unmixed height one ideal of R. Let M be the set of all locally nilpotent A-derivations of $R[X,Y]$. Then M has a graded A-module structure. Moreover M is isomorphic to $\bigoplus_{n\geq 1} A_n$ as a graded A-module.*

Proof. The graded R-algebra structure of A has been deduced in (3.5). Also, by (4.10) M has an A-module structure and the kernel of any non-zero element of M is A.

Now as in the proof of (3.3) there exist elements $x, y \in R$ such that $A_x = {R_x}^{[1]}$, $A_y = {R_y}^{[1]}$ and $A = A_x \cap A_y$. Again as in the proof of (3.3) there exist an element $a \in R \cap {R_{xy}}^*$ and elements $F, G \in A$ such that $A_x = R_x[F]$, $A_y = R_y[G]$ and $G = \lambda F$, where $\lambda = a/x^m$ for some $m \in \mathbf{Z}^+$. As before, let $I = R \cap aR_y$.

Now let $D \in M$. Then D induces a locally nilpotent R_x-derivation D_x (resp. R_y-derivation D_y) on $R_x[X,Y]$ (resp. $R_y[X,Y]$) with kernel $A_x = R_x[F]$ (resp. $A_y = R_y[G]$). By (4.9), $D_x = \alpha\Delta_F$ and $D_y = \beta\Delta_G$ for some $\alpha \in A_x, \beta \in A_y$. Now on A_{xy}, the two patch up. Therefore $\alpha\Delta_F = \beta\Delta_G = (\beta a/x^m)\Delta_F$, and hence $\beta a = x^m\alpha \in A_x \cap A_y = A$.

Conversely it is easy to see that for any $\beta \in A_y$ such that $\beta a \in A$, $\beta\Delta_G \in M$.

Hence $D \in M$ if and only if $D = \beta\Delta_G$ for some $\beta \in A_y$ satisfying $\beta a \in A$. Since G and a are fixed, we can therefore give an A-module isomorphism $M \to A \cap aA_y$ by $D(= \beta\Delta_G) \to a\beta$. Now by (3.3) the proof will be complete if we prove the following claim.

Claim. $A \cap aA_y \cong \bigoplus_{n\geq 1} I^{(n)}T^n$.

Proof of the Claim. Consider the inclusion $A_y(= R_y[G]) \hookrightarrow R_y[T]$, where $T = G/a$. As in the proof of (3.1), the image of A in $R_y[T]$ is $\bigoplus_{n\geq 0} I^{(n)}T^n$. On the other hand, the image of aA_y is clearly $\bigoplus_{n\geq 0} a^{n+1}R_yT^n$. Now by (2.1) $I^{(n)} \cap a^{n+1}R_y = R \cap a^{n+1}R_y = I^{(n+1)}$. Therefore the image of $A \cap aA_y$ in $R_y[T]$ is $\bigoplus_{n\geq 0} I^{(n+1)}T^n \cong \bigoplus_{n\geq 1} I^{(n)}T^n$. □

We now give an example to illustrate Proposition 4.11.

Example 4.12. Let $R = \mathbf{C}[X,Y,Z,W]/(XY - ZW)$. Let x, y, z, w denote the images of X, Y, Z, W respectively in R. Let $P = (x, z)$. Let F, G be elements of $R[U, V] = R^{[2]}$ defined as follows:

$$F = xU + wV, \quad G = zU + yV.$$

Let Δ_F and Δ_G be two R-derivations of $R[U, V]$ defined as follows:

$$\Delta_F(U) = w, \Delta_F(V) = -x \ , \ \Delta_G(U) = y, \Delta_G(V) = -z.$$

Then Δ_F and Δ_G are two irreducible locally nilpotent derivations such that $ker\ \Delta_F$ $=ker\ \Delta_G$, which we denote by A. Moreover, $A_x = R_x[F]$ and $A_y = R_y[G]$. Since $P^{(n)} = P^n$ (see Example 3.8), it is easy to see that $A = R[F, G]$ (the R-subalgebra of $R[U, V]$ generated by F and G). Moreover, $A = \bigoplus_{n\geq 0} A_n$, where $A_n = \sum_{i+j=n} RF^iG^j$.

Since $z\Delta_F = x\Delta_G$, again using the fact $P^{(n)} = P^n$ it is easy to see that the A-module M of all locally nilpotent A-derivations of $R[U, V]$ is generated by Δ_F and Δ_G.

Let $I = \bigoplus_{n\geq 1} A_n$. Then I is an ideal of A generated by F and G. Since $zF = xG$, for elements a, b in A, $aF + bG = 0$ if and only if $xa + zb = 0$. Since $\Delta_F(V) = -x$ and $\Delta_G(V) = -z$, $a\Delta_F + b\Delta_G = 0$ if and only if $xa + zb = 0$.

The above discussion shows that $F \mapsto \Delta_F$ and $G \mapsto \Delta_G$ is a well defined isomorphism of the A-modules I and M. □

We conclude the paper with the following observation.

The triangulability criterion ([D-F], 2.8) formulated for U.F.D. is true in general in the following form :

Remark 4.13. Let D be an irreducible locally nilpotent R-derivation on $R[X, Y]$ with kernel $A = R^{[1]}$. Then D is triangulable over R if and only if there exists a variable G in $R[X, Y]$ such that $K[X, Y] = (A \otimes_R K)[G]$.

References

[A-E-H] S.S. Abhyankar, P. Eakin and W. Heinzer, On the uniqueness of the coefficient ring in a polynomial ring, *J. Algebra* **23** (1972), 310-342. MR **46:**5300

[A-K] A. Altman and S. Kleiman, *Introduction to Grothendieck duality theory*, Lecture Notes in Mathematics, No. **146**, Springer-Verlag, 1970. MR **43:**224

[B-C-W] H. Bass, E.H. Connell and D.L. Wright, Locally polynomial algebras are symmetric algebras, *Invent. Math.* **38** (1977), 279-299. MR **55:**5613

[B-D] S.M. Bhatwadekar and A.K. Dutta, On residual variables and stably polynomial algebras, *Comm. Algebra* **21(2)** (1993), 635-645. MR **93k:**13028

[B-H] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1993. MR **95h:**13020

[C] R.C. Cowsik, Symbolic powers and the number of defining equations, *Algebra and its Applications*, Lecture Notes in Pure and Applied Mathematics, No. **91**, Marcel Dekker, New York, 1984, 13-14. MR **85g:**00023

[D-F] D. Daigle and G. Freudenburg, Locally nilpotent derivations over a UFD and an application to rank two locally nilpotent derivations of $k[X_1, \cdots, X_n]$, Preprint.

[G] J.M. Giral, Krull dimension, transcendence degree and subalgebras of finitely generated algebras, *Arch. Math.* **36** (1981), 305-312. MR **82h:**13008

[N] M. Nagata, *Local rings*, Interscience, 1962. MR **27:**5790

[O] N. Onoda, Subrings of finitely generated rings over a pseudogeometric ring, *Japan J. Math.* **10(1)** (1984), 29-53. MR **88d:**13024

[R] D. Rees, On a problem of Zariski, *Illinois J. Math.* **2** (1958), 145-149. MR **20:**2341

[Rn] R. Rentschler, Opérations du groupe additif sur le plan affine, *C.R. Aca. Sc. Paris Sér. A* **267** (1968), 384-387. MR **38:**1093

[R-S] P. Russell and A. Sathaye, On finding and cancelling variables in $k[X,Y,Z]$, *J. Algebra* **57** (1979), 151-166. MR **80j:**14030

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay-400 005, India

E-mail address: smb@tifrvax.tifr.res.in

Stat - Math Unit, Indian Statistical Institute, 203, B.T. Road, Calcutta-700 035, India

E-mail address: amartya@isical.ernet.in