ON THE LIMITING POWER FUNCTION OF THE FREQUENCY CHI-SQUARE TEST[1]

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1. Introduction. Several authors have recently investigated the power function of the frequency χ^2-test. Eisenhart [1] and Patnaik [2] have obtained large sample expressions for the power of the simple goodness of fit χ^2-test (i.e. where the class probabilities are completely specified by the null hypothesis). The more complicated case, in which the parameters occurring in the expression for class probabilities require to be estimated, has not received a unified treatment, although the problem has been treated in a number of specific situations by different authors, including, Patnaik [3], Sillito [4], Stevens [5], Pearson and Merrington [6], Poti [7], Chiang [8] and Taylor [9].

Due to difficulties in obtaining the power function of the frequency χ^2-test in the usual manner, Cochran, in an expository article [10] has suggested the derivation of its Pitman limiting power [11], and he illustrated it in the case of the simple goodness of fit test. The concept of asymptotic power suggested by Pitman has also been extensively used in various other areas like nonparametric inference (see e.g. Hoeffding and Rosenblatt [12]) and seems to be a useful tool for comparing alternative consistent tests or alternative designs for experimentation, with regard to their performance in the immediate neighbourhood of the null hypothesis.

The consistency of the frequency χ^2-test has already been established by Neyman [13]. The object of the present paper is to obtain the Pitman limiting power of this test when the unknown parameters occurring in the specification of class probabilities are estimated from the sample by an asymptotically efficient method like the method of maximum likelihood, minimum χ^2 etc. In section 5, we discuss a few applications of the Pitman limiting power for frequency χ^2-tests.

2. Pitman's concept of limiting power [11]. Let H_0 be a certain hypothesis and $\mathfrak{I}$ a test-procedure for testing H_0, which determines the critical region w_n in R_{N_n} (the sample space of N_n dimensions), for $n = 1, 2, \cdots$, ad. inf. Let us assume further that

$$N_{n+1} > N_n \qquad \text{for all } n, \tag{2.1}$$

$$0 < \lim_{n\to\infty} \text{Prob}\,\{w_n \mid H_0\} = \alpha < 1, \tag{2.2}$$

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and for any alternative H

(2.3) $$\lim_{n\to\infty} \text{Prob}\ \{w_n \mid H\} = 1$$

Let $\{H_{0n}\}$ be a family of alternative hypotheses such that

(2.4) $$\lim_{n\to\infty} \text{Prob}\ \{w_n \mid H_{0n}\} = \beta(\mathfrak{I}, \{H_{0n}\})$$

exists and $0 < \beta(\mathfrak{I}, \{H_{0n}\}) < 1$.

We call $\beta(\mathfrak{I}, \{H_{0n}\})$ the limiting power of $\mathfrak{I}$ with respect to the family of alternatives $\{H_{0n}\}$.

This concept of limiting power derives its usefulness from the fact that, if $\mathfrak{I}'$ is any other test procedure, which suggests critical regions w_n', instead of w_n, with w_n' satisfying (2.2) and (2.3), and if

$$\beta(\mathfrak{I}, \{H_{0n}\}) \lesseqgtr \beta(\mathfrak{I}', \{H_{0n}\})$$

then for n sufficiently large

$$\text{Prob}\ \{w_n \mid H_{0n}\} \lesseqgtr \text{Prob}\ \{w_n' \mid H_{0n}\}.$$

3. A theorem in frequency chi-square. Suppose that we have $R = \sum_{i=1}^{q} r_i$ functions $p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s)$, $(i = 1, 2, \cdots, q; j = 1, 2, \cdots, r_i)$, of $s < R - q$ parameters $\alpha_1, \alpha_2, \cdots, \alpha_s$ such that for all points of a non-degenerate interval A in the s-dimensional space of the α_k's the p_{ij} satisfy the following conditions

(a) $\sum_{j=1}^{r_i} p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s) = 1$ for $i = 1, 2, \cdots, q$,

(b) $p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s) > c^2 > 0$ for all ij,

(c) Every p_{ij} has continuous derivatives $\dfrac{\partial p_{ij}}{\partial \alpha_k}$ and $\dfrac{\partial^2 p_{ij}}{\partial \alpha_k \partial \alpha_t}$,

(d) The matrix $D = \left\{\dfrac{\partial p_{ij}}{\partial \alpha_k}\right\}_{R\times s}$ is of rank s.

(We shall assume that the index pairs (i, j), indicating the rows of the above matrix or of any such matrix we define in future, are arranged in the lexicographic order.)

For $n = 1, 2, \cdots$, ad. inf., let $(N_1^{(n)}, N_2^{(n)}, \cdots, N_q^{(n)})$ be a sequence of row vectors such that for $i = 1, 2, \cdots, q$, and every n, (i) $N_i^{(n)}$ is a natural number, (ii) $N_i^{(n+1)} > N_i^{(n)}$, (iii) if $N_n = \sum_{i=1}^{q} N_i^{(n)}$, then $N_i^{(n)}/N_n = Q_i$ independent of n.

Let $\boldsymbol{\alpha}_0' = (\alpha_1^0, \alpha_2^0, \cdots, \alpha_s^0)$ be an inner point of A and let

$$c_{ij}(i = 1, 2, \cdots, q; j = 1, 2, \cdots, r_i)$$

be a given set of numbers such that

(3.1) $$\sum_{j=1}^{r_i} c_{ij} = 0, \qquad \text{for } i = 1, 2, \cdots, q.$$

Put

$$p^0_{ij} = p_{ij}(\alpha^0_1, \alpha^0_2, \cdots, \alpha^0_s) \tag{3.2}$$

and

$$p_{ijn} = p^0_{ij} + \frac{c_{ij}}{\sqrt{N_n}}. \tag{3.3}$$

Let n_0 be a positive integer such that for $n \geqq n_0$

$$p_{ijn} > 0 \qquad \text{for all } i, j.$$

For $n = n_0, n_0 + 1, \cdots$, ad. inf., let $\{v_{ijn}\}(i = 1, 2, \cdots, q, j = 1, 2, \cdots, r_i)$ be a sequence of R-dimensional random variables such that

$$\text{Prob}\ \{v_{ijn}\} = \prod_{i=1}^{q} \frac{N_i^{(n)}!}{\prod_{j=1}^{r_i} v_{ijn}!} \prod_{j=1}^{r_i} p_{ijn}^{v_{ijn}}, \tag{3.4}$$

if v_{ijn} are any set of non-negative integers (some of which might be zero) and

$$\sum_{j=1}^{r_i} v_{ijn} = N_i^{(n)}, i = 1, 2, \cdots q,$$

$$= 0, \text{ otherwise.}$$

Consider the system of equations:

$$\sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{v_{ijn} - N_n Q_i p_{ij}}{p_{ij}} \frac{\partial p_{ij}}{\partial \alpha_k} = 0, \qquad k = 1, 2, \cdots, s. \tag{3.5}$$

We shall prove

THEOREM 3.1.

(i) *The system of equations* (3.5) *have exactly one system of solutions*

$$\hat{\boldsymbol{\alpha}}'_n = (\hat{\alpha}_{1n}, \hat{\alpha}_{2n}, \cdots, \hat{\alpha}_{sn})$$

such that $\hat{\alpha}'_n$ *converges in probability to* $\hat{\boldsymbol{\alpha}}'_0$ *as* $n \to \infty$ (or, in symbols, $\hat{\boldsymbol{\alpha}}_n \xrightarrow[p]{} \boldsymbol{\alpha}_0$ as $n \to \infty$).

(ii) *The value of* χ^2 *obtained by inserting* $\alpha_k = \hat{\alpha}_{kn}$ *in*

$$\chi^2 = \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{(v_{ijn} - N_n Q_i p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s))^2}{N_n Q_i p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s)} \tag{3.6}$$

is, in the limit as $n \to \infty$, distributed in a non-central χ^2-distribution ([2], [14]), with $R - s - q$ degrees of freedom and non-centrality parameter

$$\lambda = \boldsymbol{\delta}'[I - B(B'B)^{-1}B']\boldsymbol{\delta},$$

where

$$\boldsymbol{\delta} = \left\{\frac{c_{ij}\sqrt{Q_i}}{\sqrt{p^0_{ij}}}\right\}_{R\times 1},$$

and

$$B = \left\{ \frac{\sqrt{Q_i}}{\sqrt{p_{ij}^0}} \left(\frac{\partial p_{ij}}{\partial \alpha_k} \right) \alpha' = \alpha_0' \right\}_{R \times s}$$

Proof of (i). We observe that for $\eta > d = \max_{i,j} Q_i \mid c_{ij} \mid$,

$$\mid v_{ijn} - N_n Q_i p_{ij}^0 \mid \geqq \eta \sqrt{N_n} \Rightarrow \mid v_{ijn} - N_n Q_i p_{ijn} \mid \geqq (\eta - Q_i \mid c_{ij} \mid) \sqrt{N_n}.$$

Hence, using Chebyshev's inequality, we get

$$\text{Prob} \{ \mid v_{ijn} - N_n Q_i p_{ij}^0 \mid \geqq \eta \sqrt{N_n} \} \leqq \frac{p_{ijn}(1 - p_{ijn}) Q_i}{(\eta - Q_i \mid c_{ij} \mid)^2} < \frac{Q_i p_{ijn}}{(\eta - d)^2}$$

Consequently, the probability that we have $\mid v_{ijn} - N_n Q_i p_{ijn} \mid \geqq \eta \sqrt{N_n}$ for at least one subscript (i, j), is smaller than $(\eta - d)^{-2} \sum_i Q_i \sum_j p_{ijn} = (\eta - d)^{-2}$. Thus with a probability greater than $1 - (\eta - d)^{-2}$, we have

$$\mid v_{ijn} - N_n Q_i p_{ij}^0 \mid < \eta \sqrt{N_n} \qquad \text{for all } (i, j).$$

If we put

$$x_{ijn} = \frac{v_{ijn} - N_n Q_i p_{ij}^0}{\sqrt{N_n Q_i p_{ij}^0}}$$

and $a^2 = \min Q_i$, this will imply that with a probability greater than

$$1 - (\eta - d)^{-2},$$

we have

$$\mid x_{ijn} \mid < \frac{\eta}{ac} \qquad \text{for all } (i, j). \tag{3.7}$$

The proof of Theorem 3.1 (i) can now be completed using (3.7), as well as assumptions (a), (b), (c), and (d) and following Cramér's argument ([15], section 30.3).

Proof of (ii). We put

$$y_{ijn} = \frac{v_{ijn} - N_n Q_i p_{ij}(\hat{\alpha}_{1n}, \hat{\alpha}_{2n}, \cdots, \hat{\alpha}_{sn})}{\sqrt{N_n Q_i p_{ij} (\hat{\alpha}_{1n}, \hat{\alpha}_{2n}, \cdots, \hat{\alpha}_{sn})}}$$

$$X_{(n)} = \{x_{ijn}\}_{R \times 1}$$

$$Y_{(n)} = \{y_{ijn}\}_{R \times 1}$$

$$Z_{(n)} = \{z_{ijn}\}_{R \times 1} = Y_{(n)} - [I - B(B'B)^{-1} B'] X_{(n)}$$

The proof of Theorem 3.1 (ii) requires the following results.

Lemma 3.1.

$$z_{ijn} \xrightarrow[p]{} 0 \quad \text{as } n \to \infty$$

Lemma 3.1 can be proved in a manner similar to the proof given in section 30.3 of Cramér's book [15].

LEMMA 3.2. *The limiting distribution of $X'_{(n)}$ is multivariate normal with mean.* $\boldsymbol{\delta}'$ *and covariance matrix*

$$\Lambda_X = I - PP'$$

where

$$P = \{p^0_{ij}\delta_{il}\}_{R\times q}, \qquad (i = 1, 2, \cdots, q, \;\; j = 1, 2, \cdots, n_i), (l = 1, 2, \cdots, q)$$

and δ_{il} is the Kronecker's symbol.

A proof of Lemma 3.2 could be constructed again, on lines similar to that in [15] section 30.1 (see also [16] p. 118.)

LEMMA 3.3. (Cramér's proposition 22.6 [15]). *Suppose that we have for $v = 1, 2, \cdots$*

$$y_v = Ax_v + z_v ,$$

where x_v, y_v and z_v are n-dimensional random variables, while A is a matrix of order $n \cdot n$ with constant elements. Suppose further that, as $v \to \infty$, the distribution of x_v tends to a certain limiting distribution, while z_v converges in probability to zero. Then y_v has the limiting distribution defined by the linear transformation $y = Ax$, where x has the limiting distribution of the x_v.

LEMMA 3.4. *The limiting distribution of $Y'_{(n)}$ is multivariate normal with mean*

$$\boldsymbol{\delta}'[I - B(B'B)^{-1}B']$$

and covariance matrix

$$\begin{aligned}\Lambda_Y &= [I - B(B'B)^{-1}B'][I - PP'][I - B(B'B)^{-1}B'] \\ &= I - B(B'B)^{-1}B' - PP' \text{ (since } B'P = 0 \text{ as may be verified).}\end{aligned}$$

Lemma 3.4 is a direct consequence of the previous lemmas.

LEMMA 3.5. *There exists an orthogonal matrix L of order $R \cdot R$ such that*

$$L'(I - B(B'B)^{-1}B - PP')L = \begin{array}{c c} & \begin{array}{cc} s+q & R-s-q \end{array} \\ \begin{array}{c} s+q \\ R-s-q \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \end{array}$$

To prove Lemma 3.5, we write

$$M(R \times 2R) = [B(B'B)^{-1}B' \,\vdots\, PP']$$

and observe that

$$B(B'B)^{-1}B' + PP' = MM'$$

Since Rank $[B(B'B)^{-1}B] = s$, Rank $[PP'] = q$ and $B'P = 0$ it follows that Rank $[M] = s + q$. Hence Rank $[B(B'B)^{-1}B + PP'] = s + q$. But

$$B(B'B)^{-1}B + PP'$$

is an idempotent matrix. Hence its only nonzero latent root is 1, which is thus of multiplicity $s + q$. Therefore, since $B(B'B)^{-1}B' + PP'$ is a symmetric matrix, there exists an orthogonal matrix

$$L = [\underset{s+q}{L_1} \vdots \underset{R-s-q}{L_2}]R$$

such that

$$L'(B(B'B)^{-1}B' + PP')L = \begin{array}{c c} & \begin{array}{cc} s+q & R-s-q \end{array} \\ \begin{array}{c} s+q \\ R-s-q \end{array} & \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

The same matrix L satisfies Lemma 3.5.

If we now make an orthogonal transformation

$$W'_{(n)} = (w_{1,n}w_{2,n}, \cdots, w_{R,n}) = Y'_{(n)}L$$

it will then follow that the limiting distribution of $W'_{(n)}$ is multivariate normal with mean

$$\boldsymbol{\theta}' = \boldsymbol{\delta}'[I - B(B'B)^{-1}B']L$$

and covariance matrix

$$\Lambda_w = \begin{array}{c c} & \begin{array}{cc} s+q & R-s-q \end{array} \\ \begin{array}{c} s+q \\ R-s-q \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \end{array}$$

But

$$B(B'B)^{-1}B + PP' = [L_1\ L_2]\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{bmatrix} L_1' \\ L_2' \end{bmatrix} = L_1 L_1'$$

Therefore

$$\begin{aligned} I - B(B'B)^{-1}B' &= I - L_1L_1' + PP' \\ &= L_2L_2' + PP', \quad \text{since} \quad LL' = L_1L_1' + L_2L_2' = I \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\theta}' &= \boldsymbol{\delta}'[I - B(B'B)^{-1}B']L \\ &= \boldsymbol{\delta}'[L_2L_2' + PP'][L_1 \vdots L_2] \\ &= \boldsymbol{\delta}'[PP'L_1 \vdots L_2 + PP'L_2] \\ &= \boldsymbol{\delta}'[0 \vdots L_2], \quad \text{since} \quad \boldsymbol{\delta}'P = 0 \end{aligned}$$

Thus as $n \to \infty$,

$$w_{i,n} \underset{p}{\to} 0, \text{ for } i = 1, 2, \cdots, \overline{s + q},$$

and $w_{s+q+1,n}, w_{s+q+2,n}, \cdots, W_{R,n}$ are asymptotically distributed as independent normal variates with unit variance and means given by

$$\lim_{n\to\infty} E(w_{s+q+1,n}, w_{s+q+2,n}, \cdots, w_{R,n}) = \boldsymbol{\delta}' L_2$$

Hence

$$\sum_{i=1}^{q} \sum_{j=1}^{ri} \frac{(v_{ijn} - N_n Q_i p_{ij}(\hat{\alpha}_{1n}, \hat{\alpha}_{2n}, \cdots, \hat{\alpha}_{sn}))^2}{N_n Q_i p_{ij}(\hat{\alpha}_{1n}, \hat{\alpha}_{2n}, \cdots, \hat{\alpha}_{sn})} = Y'_{(n)} Y_{(n)} = W'_{(n)} W_{(n)} = \sum_{i=1}^{R} w_{i,n}^2$$

is, in the limit as $n \to \infty$, distributed as non-central χ^2 with $R - s - q$ degrees of freedom and noncentrality parameter

$$\begin{aligned}\lambda &= \boldsymbol{\delta}' L_2 L_2' \boldsymbol{\delta} \\ &= \boldsymbol{\delta}'(PP' + L_2 L_2')\boldsymbol{\delta} \\ &= \boldsymbol{\delta}'(I - B(B'B)^{-1}B')\boldsymbol{\delta}\end{aligned}$$

This completes the proof of Theorem 3.1 (ii). It will be seen that the proof of Theorem 3.1 given here, follows reasoning similar to that in Cramér ([15], section 30.3). An alternative proof is also possible on the lines of Wald's derivation (Theorem IX [17]) of the large sample distribution of the likelihood ratio criterion, with suitable modifications.

4. The limiting power of the frequency χ^2-test. Neyman [13] considers the following problem:

Consider q sequences of independent trials and let $N_{(i)}$ denote the number of trials in the ith sequence. Each trial of the ith sequence is capable of producing one of the r_i mutually exclusive results, say

$$\rho_{i,1}, \quad \rho_{i,2}, \quad \cdots, \quad \rho_{i,r_i}$$

with unknown probabilities

$$\overset{*}{p}_{i1}, \quad \overset{*}{p}_{i2}, \quad \cdots, \quad \overset{*}{p}_{ir_i}$$

where

$$\sum_{i=1}^{ri} \overset{*}{p}_{ij} = 1$$

Denote by v_{ij} the number of occurences of $\rho_{i,j}$ in the course of the $N_{(i)}$ trials forming the ith sequence.

On the basis of these observations $\{v_{ij}\}$ it is desired to test the hypothesis that these unknown probabilities p^*_{ij} satisfy certain known functional relations, e.g.

$$H: p^*_{ij} = p_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_s)$$

where the p_{ij}'s are certain functions satisfying the conditions described in section 3, and $(\alpha_1, \alpha_2, \cdots, \alpha_s)$ is an unknown parameter point. Let $\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_s$ be a suitably chosen solution of

$$\sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{v_{ij} - N_{(i)} p_{ij}}{p_{ij}} \frac{\partial p_{ij}}{\partial \alpha_k} = 0, \qquad k = 1, 2, \cdots, s. \tag{4.1}$$

and let $\chi^2_{1-\alpha}(u)$ be the upper α percent point of the χ^2-distribution with u degrees of freedom.

For testing H we compute

$$\chi^2_H = \sum_{i=1}^{q} \sum_{j=1}^{r_i} \frac{(v_{ij} - N_{(i)} p_{ij} (\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_s))^2}{N_{(i)} p_{ij} (\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_s)^2}$$

We reject H if $\chi^2_H > \chi^2_{1-\alpha}(R - s - q)$, and accept otherwise. Put $N = \sum_{i=1}^{q} N_{(i)}$ and $Q_i = N_{(i)}/N$. Let $\{c_{ij}\}$, $\boldsymbol{\delta}$ and B be as defined in section 3. Let $F(\chi^2, u, \lambda)$ be the distribution function of the non-central χ^2 with u degrees of freedom and non-centrality parameter λ. Define the hypothesis

$$H_N : p^*_{ij} = p_{ij} (\alpha_1^0, \alpha_2^0, \cdots, \alpha_s^0) + \frac{c_{ij}}{\sqrt{N}} = p_{ijN} \text{ (say)},$$

where as before $(\alpha_1^0, \alpha_2^0, \cdots, \alpha_s^0)$ is an inner point of A.

From Theorem 3.1, we obtain the limiting power of the χ^2_H-test

$$\beta(\chi^2_H, \{H_N\}) = 1 - F(\chi^2_{1-\alpha}(R - s - q), R - s - q, \lambda)$$

where $\lambda = \boldsymbol{\delta}'(I - B(B'B)^{-1}B)\boldsymbol{\delta}$.

Let $\mathbf{d}' = (d_1, d_2, \cdots, d_s)$ be any vector of real numbers. When

$$\{c_{ij}\}_{R \times 1} = D\mathbf{d},$$

it is easily seen that

$$p_{ij} (\alpha_1^0, \alpha_2^0, \ldots, \alpha_s^0) + \frac{c_{ij}}{\sqrt{N}} = p_{ij}(\alpha_{1N}, \alpha_{2N}, \ldots, \alpha_{sN}) + o\left(\frac{1}{\sqrt{N}}\right)$$

where $\alpha_{kN} = \alpha_k^0 + d_k/N^{\frac{1}{2}}(k = 1, 2, \cdots, s)$. In this case $\boldsymbol{\delta}$ is of the form

$$\boldsymbol{\delta} = B \cdot \mathbf{e}$$

where $\mathbf{e}' = (e_1, e_2, \cdots, e_s)$ is another real vector. We have

$$\begin{aligned} \lambda &= \mathbf{e}'B'(I - B(B'B)^{-1}B')B\mathbf{e} \\ &= 0 \end{aligned}$$

and $\beta(\chi^2_H, \{H_N\}) = \alpha$, as we might expect.

5. Applications. (1) *Planning of experiments for comparing two distribution functions.*

To test the hypothesis that two random variables x_1 and x_2 have identical probability distributions, the test procedure commonly adopted consists in making a sequence of N_i independent observations on the random variable x_i $(i = 1, 2)$. At each observation we observe the numerical value assumed by the random variable and according to this classify the results of each sequence into r measurable mutually exclusive and exhaustive groups (same for both the sequences).

Let v_{ij} denote the number of observations of the ith sequence belonging to the jth group $(i = 1, 2, j = 1, 2, \cdots, r)$, so that $\sum_{j=1}^{r} v_{ij} = N_i(i = 1, 2)$. The hypothesis desired to be tested is equivalent to the hypothesis H^* that there are r positive constants $p_1, p_2, \cdots, p_r$ with $\sum_{j=1}^{r} p_j = 1$ such that the probability of a random observation belonging to the jth group is equal to p_j for both the sequences. (We assume that the groups are so chosen that each of them has a positive probability measure at least w.r.t. one of the distributions.)

If this hypothesis H^* is true, the maximum likelihood estimates of p_j will be given by $\hat{p}_j = v_{.j}/N$, where $v_{.j} = v_{1j} + v_{2j}$ and $N = N_1 + N_2$. Hence for testing the hypothesis we compute

$$\chi^2_{H^*} = \sum_{i=1}^{2} \sum_{j=1}^{r} \frac{(v_{ij} - Q_i v_{.j})^2}{Q_i v_{,j}} \tag{5.1}$$

We reject the hypothesis if

$$\chi^2_{H^*} > \chi^2_{1-\alpha}(r - 1),$$

and accept it otherwise.

Let us now assume that it costs C_i dollars to make an observation on x_i $(i = 1, 2)$. Since both N_1 and N_2 are at our disposal it seems now natural to inquire how best we could allocate our total sampling budget of S dollars to the two populations, or, more precisely, could we determine the ratio $N_1 / (N_1 + N_2) = Q_1$ which will maximize the power of the above test with respect to all alternatives violating the hypothesis H^*, and at the same time ensure that the sampling cost does not exceed S dollars. Due to reasons already stated earlier in this paper, we cannot provide an answer to this question with our existing knowledge. However, if we agree to accept the limiting power function as our criterion for choosing 'the best', we might seek if the best possible sampling plan exists in the sense of maximizing the limiting power.

Let c_{ij} $(i = 1, 2, j = 1, 2, \cdots, r)$ be any given set of deviation parameters such that

$$\sum_{j=1}^{r} c_{ij} = 0, \qquad i = 1, 2, \text{ and for at least one } j,$$

$$c_{ij} \neq c_{2j}.$$

Let us denote by H_s^* the hypothesis

$$H_s^* : p_{ij}(S) = p_j^0 + \frac{c_{ij}}{\sqrt{S}}.$$

If we decide to take N_1 and N_2 in the ratio $Q_1 : (1 - Q_1)$ then the total sample size will be given by

$$N = \frac{S}{C_1 Q_1 + C_2 Q_2} \qquad \text{where} \quad Q_2 = 1 - Q_1 .$$

Hence H_s^* may be rewritten as

$$H_s^* : p_{ij}(S) = p_j^0 + \frac{c_{ij}}{\sqrt{C_1 Q_1 + C_2 Q_2}\sqrt{N}}.$$

From Theorem 3.1, we obtain the limiting power of the $\chi_{H^*}^2$-test

$$\beta(\chi_{H^*}^2, \{H_s^*\}) = 1 - F(\chi_{1-\alpha}^2(r - 1), (r - 1), \chi_{H^*})$$

where

$$\lambda_{H^*} = \boldsymbol{\delta}'(I - B(B'B)^{-1}B')\boldsymbol{\delta}.$$

After some simplification λ_{H^*} reduces to

$$\frac{Q_1 Q_2}{C_1 Q_1 + C_2 Q_2} \sum_{j=1}^{r} (c_{1j} - c_{2j})^2 / p_j^0$$

Since for given x and u, $F(x, u, \lambda)$ is a strictly monotonic decreasing function of λ, the maximum limiting power is attained when λ_{H^*} is maximum, that is when

$$Q_1 = \frac{\sqrt{C_2}}{\sqrt{C_1} + \sqrt{C_2}}$$

Thus to maximize the limiting power the best possible sampling plan, at the specified budget, is given by

$$N_1 = \left[\frac{S}{\sqrt{C_1}\,(\sqrt{C_1} + \sqrt{C_2})}\right]$$

and

$$N_2 = \left[\frac{S}{\sqrt{C_2}\,(\sqrt{C_1} + \sqrt{C_2})}\right]$$

where $[x]$ denotes the largest integer less than x.

(2) *Planning of experiments to detect shifts in response.*

Consider the following problem discussed by McNemar [18] who was interested in ascertaining the effectiveness of an interpolated experience like a movie or a lecture in shifting individual responses to certain stimuli. Let us take the simple situation in which every individual responds to the stimuli in one of two different ways (say, '0' or '1'). Let π_{ij} denote the proportion of individuals in the population, who give response 'i' before the interpolated experience and response 'j' after it ($i = 0, 1; j = 0, 1$).

Write

$$\pi_{i.} = \pi_{i0} + \pi_{i1} \qquad (i = 0, 1)$$

$$\pi_{.j} = \pi_{0j} + \pi_{1j} \qquad (j = 0, 1)$$

We shall say there is no shift in response if

$$H_0 : \pi_{1.} = \pi_{.1}$$

is true.

To test this hypothesis one can conceive of at least two alternative ways of experimentation:

(a) two samples, each of size n are selected independently, one from the pre-experience group and the other from the post-experience group. The test for the equality of proportions then, is easily seen to be a particular case of the test given earlier in this section under Application (1). Let us denote the chisquare obtained for this test by χ_a^2.

(b) the same set of individuals, n, in number selected from the pre-experience group is again examined after the experience, and the results classified in a 2×2 table as follows:

	Post experience response		
	0	1	total
Pre-experience response			
0	n_{00}	n_{01}	$n_{0.}$
1	n_{10}	n_{11}	$n_{1.}$
total	$n_{.0}$	$n_{.1}$	n

Under procedure (b), to test H_0, we compute

$$\chi_b^2 = \frac{(n_{10} - n_{01})^2}{n_{01} + n_{10}}$$

and reject H_0, only if, $\chi_b^2 > \chi_{1-\alpha}^2$ (1). Let us denote by H_{0n} the hypothesis

$$H_{0n} : \pi_{ij} = \overset{0}{\pi}_{ij} + \frac{c_{ij}}{\sqrt{n}},$$

where $\Sigma\overset{0}{\pi}_{ij} = 1$, $\overset{0}{\pi}_{01} = \overset{0}{\pi}_{10} = \pi^0$ (say), $\Sigma c_{ij} = 0$, and $c_{01} \neq c_{10}$. From Theorem 3.1, we obtain after certain algebraic simplification:

$$\beta(\chi_a^2, \{H_{0n}\}) = 1 - F(\chi_{1-\alpha}^2(1), 1, \lambda_a)$$

and

$$\beta(\chi_b^2, \{H_{0n}\}) = 1 - F(\chi_{1-\alpha}^2(1), 1, \lambda_b),$$

where

$$\lambda_a = \frac{(c_{10} - c_{01})^2}{2\{(\pi_{11}^0 + \pi^0)(\pi_{00}^0 + \pi^0)\}}$$

and

$$\lambda_b = \frac{(c_{10} - c_{01})^2}{2\pi^0}.$$

The denominator in λ_a can be rewritten as $2\ \{\pi^0 - \pi^{0^2} + \pi_{00}^0\pi_{11}^0\}$. Hence, $\lambda_b >, <$ or $= \lambda_a$, according as $(\pi_{00}^0\pi_{11}^0 - \pi_{01}^0\pi_{10}^0) >, <$ or $= 0$ respectively. This shows that at least from the point of view of maximising limiting power, procedure (b) would be superior to procedure (a) when the association between the two response types, as measured by $\pi_{00}^0\pi_{11}^0 - \pi_{01}^0\pi_{10}^0$, is positive; inferior to (a) when it is negative; and equivalent to (a) when it is zero.

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